Introduction to the Calculus of Variations
Problem Sheet 1

Swarnendu Sil

Spring 2021, IISc

1. Prove Fermat's theorem that every local extrema of a C^1 function is a critical point. In fact, the theorem is true without the C^1 assumption if by critical point we mean not just stationary points, but also all points where the function is not differentiable. Give an example of a convex function which has an unique local minima which is also global at a point where the function is not differentiable.

2. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is C^1 and $x_0 \in \mathbb{R}^n$ is a critical point of F. If F is C^2 in a neighbourhood of x_0, then prove that

   (a) x_0 is a local minima implies $\nabla^2 F(x_0) \geq 0$.

   (b) Conversely, $\nabla^2 F(x_0) > 0$ implies x_0 is a local minima.

   (c) Let $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

   $$F(x, y) = x^2 - y^2.$$

   Sketch its graph. Convince yourself that the origin $(0, 0)$ is a non-degenrate critical point (i.e. a ciritcal point where the Hessian is nonsingular) where the Hessian is indefinite and F have neither a local maxima nor a local minima at the origin. A critical point which is not a local extrema is called a **saddle point**. The graph of this function is an example of a **hyperbolic paraboloid**. Prove that the intersection of the graph with either the xz or the yz-plane passing through the origin are graphs of two functions from $\mathbb{R}$ to $\mathbb{R}$ both of which have a local extrema at $0 \in \mathbb{R}$, one of them being a minima and the other a maxima.

   (d) Let $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

   $$F(x, y) = x^2 + y^3.$$

   Investigate its critical point at the origin. Is it nondegenerate? Is it a local extrema or a saddle? Consider the two functions from $\mathbb{R}$ to $\mathbb{R}$ whose graphs are obtained by intersecting the graph of F with axial planes passing through the origin as before. Do they have an extrema at the origin this time?

(e) Let $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(x, y) = x^4 - y^4.$$

Investigate its critical point at the origin. Is it nondegenerate? Is it a local extrema or a saddle? Consider the two functions from $\mathbb{R}$ to $\mathbb{R}$ whose graphs are obtained by intersecting the graph of F with axial planes passing through the origin as before. Do they have an extrema at the origin this time?

(f) Let $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(x, y) = x^3 - 3xy^2.$$

Investigate its critical point at the origin. Is it nondegenerate? Is it a local extrema or a saddle? The graph is called a **monkey saddle**. Consider the two functions from $\mathbb{R}$ to $\mathbb{R}$ whose graphs are obtained by intersecting the graph of F with axial planes passing through the origin as before. Do they have an extrema at the origin this time? What kind of critical points do they have? Take any vector $v \in \mathbb{R}^2$ in the xy-plane and consider the vertical (z-axis) plane passing through that vector. Intersect the graph of F by this plane to obtain the graph of a function from $\mathbb{R}$ to $\mathbb{R}$ as before. Study what kind of critical points would this function have and how would it depend on your choice of v.

3. Let $x_0 \in \mathbb{R}^n$ be a critical point of $F : \mathbb{R}^n \to \mathbb{R}$, which is **convex** and C^1.

(a) Prove that x_0 is a local minima.

(b) Show that it must also be a global minima.

(c) Give examples (with sketches of graphs) of $F : \mathbb{R} \to \mathbb{R}$ to illustrate that neither conclusion is valid if the convexity assumption is dropped.

(d) Give an example (with sketches of graphs) of a convex, smooth function $F : \mathbb{R} \to \mathbb{R}$ with multiple minima.

(e) Show that if $F : \mathbb{R}^n \to \mathbb{R}$ is **strictly convex**, i.e. satisfies

$$F\left(tx + (1-t)y\right) < tF(x) + (1-t)F(y),$$

for every $x, y \in \mathbb{R}^n$ and for every $t \in (0, 1)$, then F has at most one local minima. Of course, if there exists one, it is global.

(f) Give an example (with sketches of graphs) of a strictly convex, smooth function $F : \mathbb{R} \to \mathbb{R}$ without a minima.

(g) A function $F : \mathbb{R}^n \to \mathbb{R}$ is called **strongly convex** if there exists a constant $c > 0$ such that

$$x \mapsto f(x) - c\left\|x\right\|^2$$

is convex.

i. Show that if F is strongly convex, F has an unique local minima which is also global.

ii. Show that strong convexity implies strict convexity. Can you think of a function $F : \mathbb{R} \to \mathbb{R}$ which is strictly convex but not strongly convex?

4. Prove that any convex function $F : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz. In fact, the same result is true if $F : \mathbb{R}^n \to \mathbb{R}$ is only **separately convex**, i.e. convex in each variable separately. In particular, a convex function is lower semicontinuous. (Do not get discouraged if you do not solve it. This one is not quite trivial.)

5. **(Coercivity conditions)**

   (a) Give example of a function $f : \mathbb{R} \to \mathbb{R}$ which is norm-coercive but nor coercive. (*This is the only place where we would distinguish between coercivity and norm-coercivity. Henceforth, coercive would mean norm-coercive*).

   (b) Show that a function with superlinear growth at infinity is coercive. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ satisfies α-growth condition at infinity, i.e.

   $$\lim_{\|x\| \to \infty} \frac{|F(x)|}{\|x\|^{\alpha}} = +\infty,$$

   for some $\alpha \in \mathbb{R}$. What condition on α is sufficient for coercivity of F? Give examples to show the optimality of the condition.

   (c) Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous.

   i. Show that F is coercive if and only if for every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : \|F(x)\| \leq \alpha\}$ is compact.

   ii. Show that if F is coercive, then F either has at least one global minima or is unbounded below. Give an example to show this is no longer true without the continuity hypothesis on F. Can you think of a weaker hypothesis?

   (d) Show that if $F : \mathbb{R}^n \to \mathbb{R}$ is strongly convex and C^1, then F is both coercive and bounded below.