Analysis III
aka
Advanced functional analysis and PDE

Swarnendu Sil*

Department of Mathematics
Indian Institute of Science

Spring Semester 2022

1 Course Objective

To provide a thorough introduction to the functional analytic methods employed to analyze the existence, uniqueness and regularity questions for weak solutions of Partial Differential Equations.

2 Prerequisites

- **Analysis** A course in Real analysis and a course in Measure and Integration is crucial. These would be used quite freely. L^p spaces would be used constantly.
- **Functional analysis** The methods we would discuss in this course are primarily functional analytic in nature. So basic concepts and results that are covered in a first course in functional analysis would be used freely. These include
  1. The notion of a normed linear space, Banach and Hilbert spaces,
  2. Topological dual spaces and their usual properties,
  3. strong (norm), weak and weak star topologies
  4. The Banch-Mazur lemma, the Banach-Alaoglu, Bananch-Steinhauss, Open mapping, Closed graph theorems etc.

  We would also discuss locally convex topological vector spaces, seminorms, Frechet spaces and their duals briefly. But knowledge of these topics will not be assumed.

*swarnendusil@iisc.ac.in

3 Course contents and outline

Our goal is to cover the following topics.

- **Part 1: Distributions and Fourier transform**
  - **Chapter 1: Introduction**
    - The role of functional analysis and function spaces in PDEs,
    - Different types of function spaces, examples
    - Brief discussion of locally convex topological vector spaces
    - Quick recap of normed linear spaces, Banach and Hilbert spaces and their duals, strong, weak and weak star topologies
    - Seminorms and Frechet spaces, duals of Frechet spaces.
  - **Chapter 2: Distributions**
    - Topology on the space of test functions,
    - Distributions, basic properties of distributions, distributional derivatives, convolution and regularization,
    - Order of a distribution, locally integrable function and radon measures as distributions,
    - Support and singular support of distributions.
  - **Chapter 3: Tempered distributions and Fourier transform**
    - Schwartz Space, tempered distributions,
    - Fourier transform of tempered distributions, basic properties of Fourier transforms, Fourier transform as an isometry, Fourier inversion formula,
    - Paley-Wiener theorems,
    - Application of Fourier transform to PDEs:
      - Fundamental solutions of linear constant coefficient PDEs,
      - Riesz and Bessel potentials, Sobolev spaces on $\mathbb{R}^n$.
- **Part 2: Sobolev spaces**
  - **Chapter 4: Weak derivatives and Sobolev spaces**
    - Motivation and idea of weak derivatives, relation with distributional derivatives,
    - Sobolev spaces, definition and basic properties,
    - Approximation and extensions,
    - Duals of Sobolev spaces, fractional order Sobolev spaces,
    - Traces,
    - Sobolev inequalities and Sobolev embeddings
      - Gagliardo-Nirenberg-Sobolev inequalities,
      - Poincaré and Poincaré-Sobolev inequalities,

      - · Morrey's inequality,
      - · Sobolev embeddings,
      - · Rellich-Kondrachov compact embeddings.
  - **Chapter 5: Sobolev spaces involving time**
    - ∗ Bochner integral,
    - ∗ Sobolev spaces involving time, definitions and basic properties,
    - ∗ Embeddings.

- **Part 3: Application to PDE**
  - **Chapter 6: Elliptic boundary value problems**
    - ∗ Weak formulation of elliptic boundary value problems,
    - ∗ Lax-Milgram lemma and existence of weak solutions,
    - ∗ Energy and uniqueness of weak solutions,
    - ∗ Regularity of weak solutions, Caccioppoli inequality, interior and boundary L^2 regularity estimates.
    - ∗ Calculus of variations, Dirichlet principle.
    - ∗ Maximum principle.
  - **Chapter 7: Linear evolution equations**
    - ∗ Weak formulation,
    - ∗ Existence of weak solutions,
    - ∗ Uniqueness and Energy methods.

4 Main references

The basic references for this course would be the Lecture notes. For the most part, we would be following Brezis [1] and Evans [2]. We would sometimes follow some other references for particular materials as well.

Suggested books

[1] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[2] Evans, L. C. *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 1998.