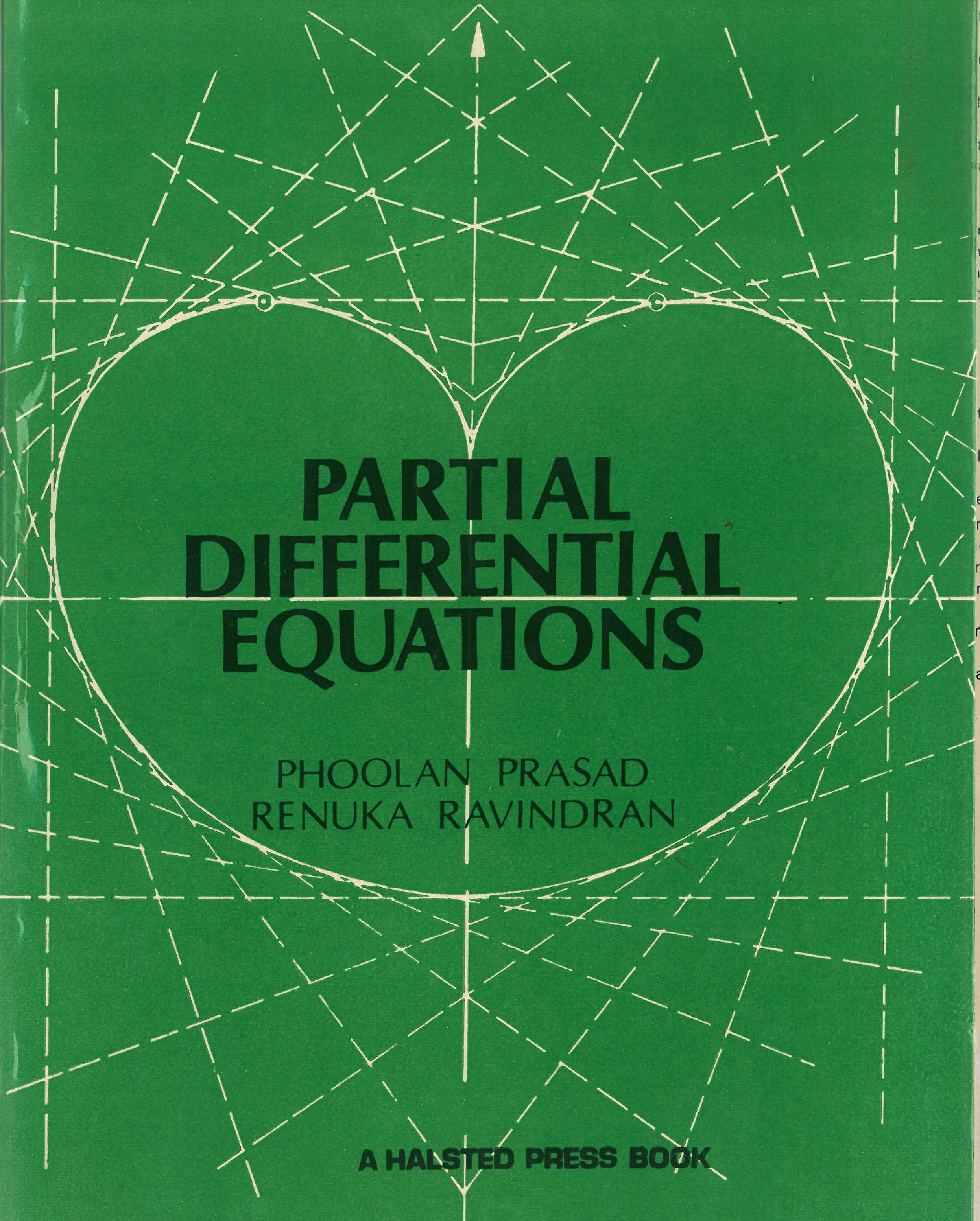

PARTIAL DIFFERENTIAL EQUATIONS

PHOOLAN PRASAD
RENUKA RAVINDRAN

A HALSTED PRESS BOOK

Partial Differential Equations

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To

My Brother Chandan Prasad

P.P.

Foreword

The distinguished authors of this book on 'Partial Differential Equations' have written a most valuable account of the subject, which gives a really comprehensive introduction to all those parts of the theory of partial differential equations that are needed in practical applications of that theory, whether in the physical sciences or in the different branches of engineering. The book is also set out excellently as a body of mathematical analysis of wide general interest. All the essential ideas of the subject are explained with great clarity. We can particularly admire the way in which ideas are first introduced in relatively simple cases and then gradually extended to more complicated cases and to more advanced applications.

There has long been a deeply felt need for a high-class textbook at this level giving a comprehensive introduction to the theory of partial differential equations. That significant gap in the existing p.d.e. literature is now admirably filled by the excellent work which Dr. Prasad and Dr. Ravindran have jointly produced.

SIR JAMES LIGHTHILL

Preface

The theory of partial differential equations is a subject that has grown beyond all expectations. It has found its way into all branches of science and engineering due to its wide range of applications.

It is difficult to choose suitable material for an introductory course on partial differential equations. This is due-not only to the extensive material available on the subject, but also to the fact that there is more than one approach to the study of the subject. On the one hand we have a classical treatment of wellknown equations, including those arising in physics and mechanics. On the other hand we have new and powerful methods, such as Fourier analysis and distribution theory, to deal with more general linear equations.

The people who study partial differential equations are also a varied group-students of M. Sc. and M. Phil. degrees in Mathematics, M. Tech. and M.E. in the engineering disciplines, research scientists and engineers and teachers in universities. A large proportion of this group is interested in a basic introductory course, in which theory and application are interrelated and develop side by side. This requires not only rigorous, but also constructive proofs, emphasising the structure and properties of solutions. With this in mind, we have had to omit the more general approach through the study of linear operators. We have preferred to bring out the effect of nonlinear terms in the equations from the very beginning of the book as the study of nonlinear phenomena is fast gaining in importance.

This book has grown out of our experience of teaching partial differential equations at the Indian Institute of Science for the last fourteen years to students and research workers in mathematics, science and engineering. Our aim has been to present in this book not only a rigorous introduction to the theory of partial differential equations, but also the material useful for applications. The book can be covered in a two semester course on partial differential equations. It could also be used for a one semester course, if the starred sections are omitted.

The authors are grateful to Prof. V.G. Tikekar who has rendered most valuable help through constructive criticism of the original manuscript. The authors thank Dr. B.J. Venkatachala for providing Fig. 8.2 in Chapter 3.

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Contents

CHAPTER 1

Single First Order Partial Differential Equations

§ 1 MEANING OF A PARTIAL DIFFERENTIAL EQUATION

A partial differential equation for a function $u(x_\alpha)$ of m independent variables x_α ($\alpha = 1, 2, \ldots, m$) is a relationship between the function and its partial derivatives $u_{x_\alpha}, u_{x_\alpha x_\beta}, \cdots$. We represent this relationship in the form

$$F(x_1, \cdots, x_m; u; u_{x_1}, \cdots, u_{x_m}; u_{x_1 x_1}; u_{x_1 x_2} \cdots) = 0 \tag{1.1}$$

or briefly

$$F(x_\alpha, u, u_{x_\alpha}, u_{x_\alpha x_\beta}, \cdots) = 0$$

where only a finite number of derivatives occur on the left hand side and the function F is defined over a domain D_3 in the space of the variables appearing in its arguments.* The *order* of the partial differential equation is the order of the highest derivatives appearing in the function F.

A classical (*or genuine*) *solution* of the partial differential equation is a function $u = u(x_\alpha)$ defined over a domain D of (x_α)-space such that all partial derivatives of u appearing in the equation exist and are continuous in D,

$$(x_\alpha, u(x_\alpha), u_{x_\beta}(x_\alpha), u_{x_\beta x_\gamma}(x_\alpha), \ldots) \in D_3$$

when $x_\alpha \in D$ and

$$F(x_\alpha, u(x_\alpha), u_{x_\beta}(x_\alpha), u_{x_\beta x_\gamma}(x_\alpha), \ldots) = 0$$

for all $(x_\alpha) \in D$. We also say that the function $u(x_\alpha)$ satisfies equation (1.1). We shall refer to a genuine solution simply as a solution.

In the discussion of partial differential equations, we shall assume that all functions are real-valued functions with real arguments unless otherwise stated.

The simplest partial differential equations to study are those of the first order for the determination of just one unknown function. Apart from the

*D denotes a domain in (x_α)-space where a solution is defined, D_1 is a domain where the coefficients of a linear equation are defined, D_2 is a domain in (x_α, u)-space and D_3 is a domain in $(x_\alpha, u, u_{x_\alpha}, u_{x_\alpha x_\beta}, \ldots)$ space.

fact that they form the basis of the study of a class of higher order equations, called hyperbolic equations (see Chapter 2 § 1, and Chapter 3), they are the simplest kind of equations for which methods of solution are available and for which the existence, uniqueness, and stability can be discussed in great detail. In this chapter, we shall present some basic results concerning first order partial differential equation.

§ 1.1 First Order Partial Differential Equations in Two Independent Variables and the Cauchy Problem

In this chapter, while dealing with the partial differential equations in two independent variables, we shall denote the independent variables by x and y. A first order partial differential equation in two unknowns in its most general form is given by

$$F(x, y, u, u_x, u_y)=0 \tag{1.2}$$

where F is a known function of its arguments. When the function F is not a linear expression in u_x and u_y, the equation (1.2) is said to be a *nonlinear equation*. When F is a linear expression in u_x and u_y, but not necessarily linear in u the equation is

$$a(x, y, u)u_x+b(x, y, u)u_y=c(x, y, u) \tag{1.3}$$

where a and b depend on u also. This equation is called a *quasilinear equation*. A first order *semilinear equation* is an equation of the form

$$a(x, y)u_x+b(x, y)u_y=c(x, y, u) \tag{1.4}$$

where the coefficients of u_x and u_y do not depend on u and the nonlinearity in the equation is present only in the inhomogeneous term on the right hand side of (1.4). A *linear* first order equation is of the form

$$a(x, y)u_x+b(x, y)u_y=c_1(x, y)u+c_2(x, y) \tag{1.5}$$

where the dependent variable u and its partial derivatives u_x, u_y all appear linearly with a, b, c_1 and c_2 as functions of x and y only.

The solution $u=u(x, y)$ represents a surface in (x, y, u) space. This surface is called an *integral surface* of the partial differential equation.

While dealing with partial differential equations appearing in science and engineering, we rarely try to find out or discuss properties of a solution in its most general form. Almost always we deal with those solutions of differential equations which satisfy certain conditions. In the case of first order partial differential equations, the search for these specific solutions can be formulated as a Cauchy problem.

The Cauchy Problem. Consider an interval I on the real line and three arbitrary functions $x_0(\eta)$, $y_0(\eta)$ and $u_0(\eta)$ of a single variable $\eta \in I$ such that the derivatives $x_0'(\eta)$ and $y_0'(\eta)$ are piecewise continuous and $(x_0')^2+(y_0')^2 \neq 0$. A Cauchy problem for a first order equation (1.2) is to find a domain D in (x, y) plane containing $(x_0(\eta), y_0(\eta))$ for all $\eta \in I$ and a solution $u=u(x, y)$

of the equation such that

$$u(x_0(\eta), y_0(\eta)) = u_0(\eta) \tag{1.6}$$

for all values of $\eta \in I$.

Geometrically, $x = x_0(\eta)$, $y = y_0(\eta)$ represent a curve γ in (x, y) plane. We call this curve a *datum curve*. The Cauchy problem is to determine a solution of $F(x, y, u, u_x, u_y) = 0$ in a neighbourhood of γ such that u takes prescribed values $u_0(\eta)$ on γ.

The solution of the Cauchy problem also involves such questions as the conditions on the functions F, $x_0(\eta)$, $y_0(\eta)$ and $u_0(\eta)$ under which a solution exists and is unique.

§ 2 SEMILINEAR AND QUASILINEAR EQUATIONS IN TWO INDEPENDENT VARIABLES

We start with a semilinear equation instead of a linear equation as the theory of the former does not require any special treatment as compared to that of the latter.

§ 2.1 Semilinear Equations

Consider a single semilinear first order equation in two independent variables (x, y) for a single unknown quantity:

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u). \tag{2.1}$$

We assume that a, b, c, are continuously differentiable functions of their arguments and a and b are not simultaneously zero. a, $b \in C^1(D_1)$ and $c \in C^1(D_2)$, where D_1 and D_2 are domains in (x, y)-plane and (x, y, u)-space respectively, such that whenever $(x, y, u) \in D_2$, $(x, y) \in D_1$.

At a given point $(x, y) \in D_1$ the left hand side of (2.1) represents a derivative of $u(x, y)$ in the direction of the vector $(a(x, y), b(x, y))$. Therefore, if we consider a one parameter family of curves whose tangent at each point is in the above direction i.e., the family of curves defined by the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)} \tag{2.2}$$

the variation of u along these curves is given by $\dfrac{du}{dx} = u_x + \dfrac{dy}{dx} u_y = \dfrac{au_x + bu_y}{a}$,

which with the help of (2.1) gives

$$\frac{du}{dx} = \frac{c(x, y, u)}{a(x, y)} \cdot \tag{2.3}$$

Consider a curve represented by a solution of equation (2.2). We can choose a variable σ such that this curve has a parametric representation $x = x(\sigma)$, $y = y(\sigma)$ and $x(\sigma)$ and $y(\sigma)$ satisfy a pair of ordinary differential equations

$$\frac{dx}{d\sigma} = a(x, y), \quad \frac{dy}{d\sigma} = b(x, y). \tag{2.4}$$

Then the variation of u along the curve is given by

$$\frac{du}{d\sigma}=c(x, y, u). \tag{2.5}$$

Equations (2.2) or (2.4) are called *characteristic equations*. The solution of (2.2) can be written in the form

$$f(x, y, C)=0 \tag{2.6}$$

where C is a constant of integration. This equation represents a one-parameter family of curves with C as the parameter. We call these curves the *characteristic curves* of the partial differential equation. In the domain D_1 consider another curve $x=x^{(0)}(\eta)$, $y=y^{(0)}(\eta)$ such that it is nowhere tangential to the characteristic curves. Solving (2.4) with the condition $x=x^{(0)}(\eta)$, $y=y^{(0)}(\eta)$ at $\sigma=0$, we get a solution of the form

$$x=x(\sigma, \eta), \quad y=y(\sigma, \eta). \tag{2.7}$$

Because of the equivalence of (2.2) and (2.4), the equation (2.7) also represents the one-parameter family of characteristic curves of equation (2.1). In the parametric representation (2.7), σ varies along a characteristic curve. η remains constant along a characteristic curve and different values of η determine different characteristic curves. The equation (2.3) or (2.5) is called *compatibility condition* along a characteristic curve.

Suppose that $u(x, y)$ is assigned an initial value u_0 at a point (x_0, y_0) in (x, y)-plane. Since $a(x, y)$, $b(x, y)$ and $c(x, y, u)$ are C^1 functions of their arguments, the initial value problem for the ordinary differential equations (2.4) and (2.5) with initial values, x_0, y_0, u_0 has a unique solution. Therefore, through the point (x_0, y_0) there passes a unique characteristic curve given by

$$x=x(x_0, y_0, \sigma), \quad y=y(x_0, y_0, \sigma) \tag{2.8}$$

and along this curve

$$u=u(x_0, y_0, u_0, \sigma) \tag{2.9}$$

is uniquely determined by the equation (2.5). This shows that, if u is given at any point, it is uniquely determined everywhere along the characteristic curve (denoted by C_c) passing through the point as long as it does not pass through a *singular point** and as long as (x, y, u) remains in D_2, where $c(x, y, u)$ is defined. This suggests the following method of solution of the Cauchy problem stated in §1.1.

We take an arbitrary point $P_0(x_0(\eta), y_0(\eta))$ on the datum curve γ. The value of u at P_0 is $u_0(\eta)$. Solving the characteristic equations and the compatibility condition with initial values $x=x_0(\eta)$, $y=y_0(\eta)$, $u=u_0(\eta)$ at $\sigma=0$ we get

$$x=x(x_0(\eta), y_0(\eta), \sigma), \; y=y(x_0(\eta), y_0(\eta), \sigma) \tag{2.10}$$

and

$$u=u(x_0(\eta), y_0(\eta), u_0(\eta), \sigma). \tag{2.11}$$

*In the present case, when $a, b \in C^1(D_1)$, a singular point corresponds to a point where both a and b vanish simultaneously.

Solving the pair of equations (2.10) for σ and η in terms of x, y and substituting in (2.11) we get a solution of the Cauchy problem in a neighbourhood of the curve γ.

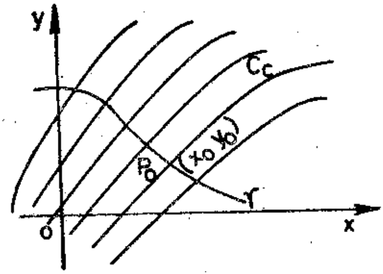

Fig. 2.1. Solution of a cauchy problem with the help of characteristic Curves C_c.

As shown in Theorem 2.1 on page 10, the method fails if the curve γ coincides with a characteristic curve. From the compatibility condition (2.5) we also note that if γ is a characteristic curve, the variation of the Cauchy data $u_0(\eta)$ on γ is constrained by the relation (2.5) and so cannot be arbitrarily prescribed on it.

Example 2.1 Consider a Cauchy problem for the partial differential equation

$$2u_x + 3u_y = 1 \tag{2.12}$$

with Cauchy data prescribed on the straight line $\gamma : \beta x - \alpha y = 0$, where α and β are constants. A parametric representation of the Cauchy data is

$$x = \alpha\eta, \quad y = \beta\eta, \quad u(\alpha\eta, \beta\eta) = f(\eta) \tag{2.13}$$

where $f(\eta)$ is a given function.

The characteristic curve passing through a point $(\alpha\eta, \beta\eta)$ is obtained by solving the equations (2.4) with $a = 2$ and $b = 3$ in the form

$$x = 2\sigma + \alpha\eta, \quad y = 3\sigma + \beta\eta. \tag{2.14}$$

The compatibility condition giving the rate of change of the variable u along a characteristic is (2.5) with $c = 1$. Integrating it with the initial condition $u = f(\eta)$ at $\sigma = 0$, we get

$$u = \sigma + f(\eta).$$

Solving σ and η from (2.14) and substituting in the above we get

$$u = \frac{\alpha y - \beta x}{3\alpha - 2\beta} + f\left(\frac{3x - 2y}{3\alpha - 2\beta}\right) \tag{2.15}$$

provided we assume that

$$3\alpha - 2\beta \neq 0. \tag{2.16}$$

Equation (2.15) represents a genuine solution of the equation (2.12) if the given function $f(\eta)$ is continuously differentiable. Then u_x and u_y are C^1 functions in the entire (x, y)-plane and satisfy the equation (2.12).

When the constants α and β are such that $3\alpha-2\beta=0$ the above method of finding the solution breaks down. In this case the straight line γ is itself a characteristic curve. Along a characteristic curve $\frac{dx}{d\sigma}=2$. Comparing with the first equation in (2.13), we can take the variable σ to be the same as $\frac{\alpha}{2}\eta$. The compatibility condition (2.5) shows that the function $f(\eta)$ in the above Cauchy problem cannot be arbitrarily prescribed but must satisfy the relation

$$\frac{df(\eta)}{d\eta}=\frac{\alpha}{2}. \tag{2.17}$$

This condition completely determines the function $f(\eta)$ except for a constant of integration:

$$f(\eta)=\frac{\alpha}{2}\eta. \tag{2.18}$$

It is simple to check that this "characteristic Cauchy problem" with the Cauchy data

$$x=\alpha\eta, \quad y=\frac{3}{2}\alpha\eta, \quad u=\frac{\alpha}{2}\eta$$

has solutions of the form

$$u=\frac{1}{2}x+g(3x-2y) \tag{2.19}$$

where $g(\xi)$ is an arbitrary C^1 function of ξ and satisfies

$$g(0)=0.$$

This example verifies a general property, namely, the solution of a characteristic Cauchy problem when it exists, is nonunique in that it involves an arbitrary function.

EXERCISE 2.1

1. Show that a characteristic of the equation $u_x-u_y=0$ touches the branch of the hyperbola $xy=1$ in the first quadrant of the (x, y)-plane at the point $P(1, 1)$. Verify that the point P divides the hyperbola into two portions such that the Cauchy data prescribed on one portion determines the value of u on the other portion.
2. Find the characteristics of the equations
   (i) $yu_x-xu_y=0$,
   (ii) $2xyu_x-(x^2+y^2)u_y=0$,
   (iii) $(x^2-y^2+1)u_x+2xyu_y=0$.

3. Draw the characteristics of 2(i) and hence show that every solution of the equation is an even function of both variables x and y.
4. Find the solution of the equation 2(i) given that
$$u(x, 0) = x^2, \qquad -\infty < x < \infty.$$
5. Show that if u is prescribed on the interval $0 \leqslant y \leqslant 1$ of the y-axis, the solution of 2(iii) is completely determined in the first quadrant of the (x, y)-plane.
6. Find the solution of the partial differential equation
$$(x+1)^2 u_x + (y-1)^2 u_y = (x+y)u$$
satisfying the condition $u(x, 0) = -1-x$ for $-1 < x < \infty$.
Discuss the nature of the characteristics starting from the various points of the x-axis and show that the solution is determined in the domain bounded by the curves $xy = -1$, $x = -1$ and $y = 1$.
7. Find the solutions of the following Cauchy problems and the domains in which they are determined in the (x, y)-plane:
   (i) $yu_x + xu_y = 2u$ with $u(x, 0) = f(x)$ for $x > 0$,
   (ii) $yu_x + xu_y = 2u$ with $u(0, y) = g(y)$ for $y > 0$,
   (iii) $u_x + u_y = u^2$ with $u(x, 0) = 1$ for $-\infty < x < \infty$.

§2.2 Quasilinear Equations

Now, we pass on to the general quasilinear equation of the first order
$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{2.20}$$
where the coefficients a and b depend on the dependent variable u also. We assume that a, b, c are C^1 functions in a domain D_2 of (x, y, u)-space. We recall here the geometrical interpretation of a solution $u = u(x, y)$ as a surface in (x, y, u)-space, called *integral surface*. The direction ratios of the normal to the surface are $(u_x, u_y, -1)$, so that writing (2.20) as
$$(a, b, c)\cdot(u_x, u_y, -1) = 0 \tag{2.21}$$
where the left hand side is the scalar product of two vectors, we can interpret the equation as being equivalent to a condition that the integral surface at each point has the property that the vector (a, b, c) is tangential to the surface.

At any point (x, y, u) in D_2, the vector $(a(x, y, u), b(x, y, u), c(x, y, u))$ defines a direction, called *Monge direction*.* Therefore, the coefficients in the equation (2.20) define a direction field, i.e. the field of Monge directions in the domain D_2 of (x, y, u)-space. A surface $u = u(x, y)$ is an integral surface,

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*In most of the books the Monge direction and Monge curve are called characteristic direction and characteristic curve; these terms we reserve for the projections of the Monge direction and the Monge curve on the (x, y)-plane respectively. Similarly, instead of calling all the three equations given by (2.23) and (2.24) as characteristic equations we shall call only the two equations (2.23) characteristic equations and the equation (2.24) the compatibility condition.

if and only if, at each point of the surface the tangent plane contains the Monge direction at that point. Thus at a given point (x, y, u) the tangent plane of the integral surface has one degree of freedom, i.e. it can rotate about the Monge direction. A space curve whose tangent at every point coincides with the Monge direction is called a *Monge curve* and is determined by the equations

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}. \tag{2.22}$$

In terms of a parameter σ, such that $d\sigma$ is the common value of the three ratios in (2.22), we can write the characteristic equations and the compatibility condition respectively as

$$\frac{dx}{d\sigma} = a(x, y, u), \quad \frac{dy}{d\sigma} = b(x, y, u) \tag{2.23}$$

and

$$\frac{du}{d\sigma} = c(x, y, u). \tag{2.24}$$

As in §2.1, we consider a surface in D_2 given by $x = x^{(0)}(\eta_1, \eta_2)$, $y = y^{(0)}(\eta_1, \eta_2)$, $u = u^0(\eta_1, \eta_2)$, such that it nowhere touches a Monge curve. Solving the system of equations (2.23) and (2.24), with the condition $x = x^{(0)}(\eta_1, \eta_2)$ $y = y^0(\eta_1, \eta_2)$, $u = u^0(\eta_1, \eta_2)$ at $\sigma = 0$, we get a representation of the Monge curves, in the form

$$x = x(\sigma, \eta_1, \eta_2), \; y = y(\sigma, \eta_1, \eta_2), \; u = u(\sigma, \eta_1, \eta_2). \tag{2.25}$$

The totality of Monge curves form a two-parameter family of curves with parameters η_1 and η_2.

The projection of a Monge curve on (x, y)-plane is called a *characteristic curve* of (2.20). We note here that the characteristic equations (2.4) of the semilinear equation (2.1) are not coupled with the compatibility condition (2.5) and hence can be integrated independently. Thus the one parameter family of characteristic curves of a semilinear equation can be drawn once for all without any reference to the compatibility condition. For the quasilinear equation (2.20), the characteristic equations and compatibility condition are coupled. Therefore, to determine the characteristics in the case of a quasilinear equation, we have to draw them by solving the three equations (2.23) and (2.24) together. The totality of all the characteristic curves (in (x, y)-plane) of a quasilinear equation form a two parameter family of curves. For a given solution, u is a known function of x and y, and the equation (2.23) for the characteristics can be solved without any reference to the compatibility condition (2.24), as in the case of semilinear equations. In this case through any point (x, y), there is only one characteristic curve and the set of all characteristic curves form a one-parameter family of curves in the (x, y)-plane.

Example 2.2 Consider the partial differential equation

$$uu_x + u_y = 0.$$

The Monge curve through the point (x_0, y_0, u_0) is the straight line given by the equations

$$x-x_0=u_0(y-y_0), \quad u=u_0.$$

The characteristic curves through an arbitrary point (x_0, y_0) in (x, y)-plane is the one parameter family of straight lines passing through the point and depending on the parameter u_0.

Consider a surface generated by a one parameter sub-family of Monge curves. The tangent plane at a point of the surface contains the Monge direction at that point. Therefore, *every surface generated by a one parameter sub-family of Monge curves is an integral surface of* (2.20). The converse of this statement is also true. Let $u=u(x, y)$ be an integral surface S. Let $x=x_0(\eta)$, $y=y_0(\eta)$, $u=u_0(\eta) \equiv u(x_0(\eta), y_0(\eta))$ be a space curve lying on S and suppose that the functions $x_0(\eta), y_0(\eta)$ are so prescribed that this curve is not a Monge curve. Consider the solution of

$$\frac{dx}{d\sigma}=a(x, y, u(x, y)), \frac{dy}{d\sigma}=b(x, y, u(x, y)) \tag{2.26}$$

with $x=x_0(\eta)$, $y=y_0(\eta)$ at $\sigma=0$ in the form $x=x(\sigma, \eta)$, $y=y(\sigma, \eta)$. In (2.26) u is a known function of x, y from the equation of the integral surface S. Then along the one parameter family of curves

$$x=x(\sigma, \eta), y=y(\sigma, \eta), u=u(x(\sigma, \eta), y(\sigma, \eta)); \tag{2.27}$$

with η as parameter

lying on S, we have

$$\frac{du}{d\sigma}=\frac{dx}{d\sigma}u_x+\frac{dy}{d\sigma}u_y=au_x+bu_y=c(x, y, u). \tag{2.28}$$

In view of (2.26) and (2.28), we infer that the curves (2.27) are Monge curves. These Monge curves generate the integral surface S as η varies. We have shown that starting from a non-Monge curve on an integral surface, we can determine a one-parameter sub-family of Monge curves that generate the surface. Thus *any integral surface S is generated by a family of Monge curves depending on a single parameter* η.

Now we have also proved that through an arbitrary point of an integral surface there passes a Monge curve which lies entirely on the integral surface. This with the uniqueness theorem of the solution of an initial value problem of the ordinary differential equations (2.23) and (2.24) implies that *if a Monge curve is tangential to an integral surface at any point, it lies entirely on the integral surface.*

We can now present a method for the solution of a Cauchy problem for the quasilinear equation (2.20). We first note that geometrically $x=x_0(\eta)$, $y=y_0(\eta)$, $u=u_0(\eta)$ represents a curve Γ in (x, y, u)-space. We call this curve an *initial curve*. The datum curve γ, on which the Cauchy data is prescribed, is the projection of Γ on the (x, y)-plane. A geometrical interpretation of a

Cauchy problem for a first order partial differential equation is to find an integral surface of the equation passing through the initial curve Γ. The results of the last two paragraphs show that in order to solve a Cauchy problem we just have to find the surface generated by the one parameter family of Monge curves, starting from the points $(x_0(\eta), y_0(\eta), u_0(\eta))$, in the form

$$x = x(\sigma, \eta), \ y = y(\sigma, \eta), \ u = u(\sigma, \eta). \tag{2.29}$$

This is a parametric representation of the equation of the required integral surface. We shall again have to exclude datum curves which are tangential to the characteristic curves. We present here a precise formulation in the following theorem.

Theorem 2.1 Let $x_0(\eta)$, $y_0(\eta)$ and $u_0(\eta)$ be continuously differentiable functions of η in a closed interval, say [0, 1] and a, b, c be functions of x, y, u having continuous first order partial derivatives with respect to their arguments in some domain D_2 of (x, y, u)-space containing the initial curve

$$\Gamma : x = x_0(\eta), \ y = y_0(\eta), \ u = u_0(\eta); \ 0 \leqslant \eta \leqslant 1 \tag{2.30}$$

and satisfying the condition

$$\frac{dy_0(\eta)}{d\eta} a(x_0(\eta), y_0(\eta), u_0(\eta)) - \frac{dx_0(\eta)}{d\eta} b(x_0(\eta), y_0(\eta), u_0(\eta)) \neq 0. \tag{2.31}$$

Then there exists a solution $u = u(x, y)$ of the quasilinear equation (2.20) in the neighbourhood of the datum curve $\gamma : x = x_0(\eta)$, $y = y_0(\eta)$ and satisfying the condition (1.6), namely

$$u_0(\eta) = \eta(x_0(\eta), y_0(\eta)), \ 0 \leqslant \eta \leqslant 1 \tag{2.32}$$

and the solution is unique.

Note: The condition (2.31) excludes the possibility that γ could be a characteristic curve.

Proof Since a, b, c have continuous partial derivatives with respect to x, y, u; the ordinary differential equations (2.23) and (2.24) have a unique continuously differentiable solution (see Coddington and Levinson, 1955, Chapter 1, §5) of the form (2.29) satisfying the initial condition

$$x(0, \eta) = x_0(\eta), \ y(0, \eta) = y_0(\eta), \ u(0, \eta) = u_0(\eta). \tag{2.33}$$

As $x_0(\eta)$, $y_0(\eta)$, $u_0(\eta)$ are continuously differentiable, the solution (2.29) is continuously differentiable with respect to η also (see Coddington and Levinson, 1955, Chapter 1, §7). In view of our assumption (2.31) the Jacobian

$$\frac{\partial(x, y)}{\partial(\sigma, \eta)} \equiv \begin{vmatrix} x_\sigma & x_\eta \\ y_\sigma & y_\eta \end{vmatrix} = (ay_\eta - bx_\eta) \tag{2.34}$$

does not vanish at $\sigma=0$ for $0 \leqslant \eta \leqslant 1$. Therefore*, in a neighbourhood of $\sigma=0$, we can uniquely solve for σ and η in terms of x and y from the first two relations in (2.29) and substitute in the third relation to get u as a function of x and y:

$$u(x, y)=u(\sigma(x, y), \eta(x, y)). \tag{2.35}$$

It also follows that $u \in C^1$.

At any point of the datum curve, $u(x_0(\eta), y_0(\eta))=u(0, \eta)=u_0(\eta)$, which shows that the initial condition (2.33) is satisfied.

From (2.24), i.e. $u_\sigma=c$, we have $u_x x_\sigma+u_y y_\sigma=c$ or $au_x+bu_y=c$ showing that the function $u(x, y)$ given by (2.35) satisfies the equation (2.20).

To prove the uniqueness of the solution we first note that if a Monge curve is tangential to an integral surface at any point, it lies entirely on the surface. Let us assume now that there are two integral surfaces S and S' passing through an initial curve Γ, given by (2.30). Then for an arbitrary given value of η, the Monge curve (2.29) starting from the point $(x_0(\eta), y_0(\eta), u_0(\eta))$ lies entirely on both surfaces S and S'. Hence S and S' are generated by the same subfamily of Monge curves which implies that the two integral surfaces are the same.

Example 2.3 Consider the equation

$$uu_x+u_y=0 \tag{2.36}$$

with the Cauchy data

$$u(x, 0)=x,\ 0 \leqslant x \leqslant 1.$$

prescribed only on a portion of the x-axis. The Cauchy data can be put in the form (2.30):

$$x=\eta,\ y=0,\ u=\eta,\ 0 \leqslant \eta \leqslant 1. \tag{2.37}$$

Solving the characteristic equations and the compatibility condition

$$\frac{dx}{d\sigma}=u,\ \frac{dy}{d\sigma}=1,\ \frac{du}{d\sigma}=0$$

with initial data (2.37) we get

$$x=\eta(\sigma+1),\ y=\sigma,\ u=\eta. \tag{2.38}$$

The characteristic curve passing through a point $x=\eta$ on the x-axis is a straight line $x=\eta(y+1)$. These characteristics for all admissible but fixed values of η(i.e. $0 \leqslant \eta \leqslant 1$) pass through the same point $(0, -1)$ and cover

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*The notation $u(\sigma, \eta)$ implies that u is a function of σ and η, while $u(x, y)$ signifies that it is a function of x and y. The functional form is not necessarily the same in each case, in fact, $u(x, y)=u(\sigma(x, y), \eta(x, y))$ and $u(\sigma, \eta)=u(x(\sigma, \eta), y(\sigma, \eta))$.

Theorem: If $X(\sigma, \eta)$ and $Y(\sigma, \eta) \in C^1$ in a neighbourhood of a point (σ^*, η^*) and $X(\sigma^*, \eta^*)=x^*$, $Y(\sigma^*, \eta^*)=y^*$ and the Jacobian $\dfrac{\partial(X, Y)}{\partial(\sigma, \eta)} \neq 0$ at (σ^*, η^*), then there exists a neighbourhood $N(x^*, y^*)$ of the point (x^*, y^*) and a unique pair of functions $\sigma=\Sigma(x, y)$, $\eta=\mathcal{H}(x, y)$ such that

(i) $x=X(\Sigma(x, y), \mathcal{H}(x, y))$, $y=Y(\Sigma(x, y), \mathcal{H}(x, y))$

(ii) $\sigma^*=\Sigma(x^*, y^*)$, $\eta^*=\mathcal{H}(x^*, y^*)$

(iii) $\Sigma(x, y), \mathcal{H}(x, y) \in C^1(N(x^*, y^*))$.

This is called inverse function theorem. It is a particular case of implicit function theorem.

the wedged shaped portion D of the (x, y)-plane bounded by two extreme characteristics $x=0$ and $x=y+1$. $u=\eta$ in (2.38) shows that u is constant on those characteristics, being equal to the abscissa of the point where the characteristic intersects the x-axis. The solution is determined in the wedged shaped region D as shown in the Fig. 2.2.

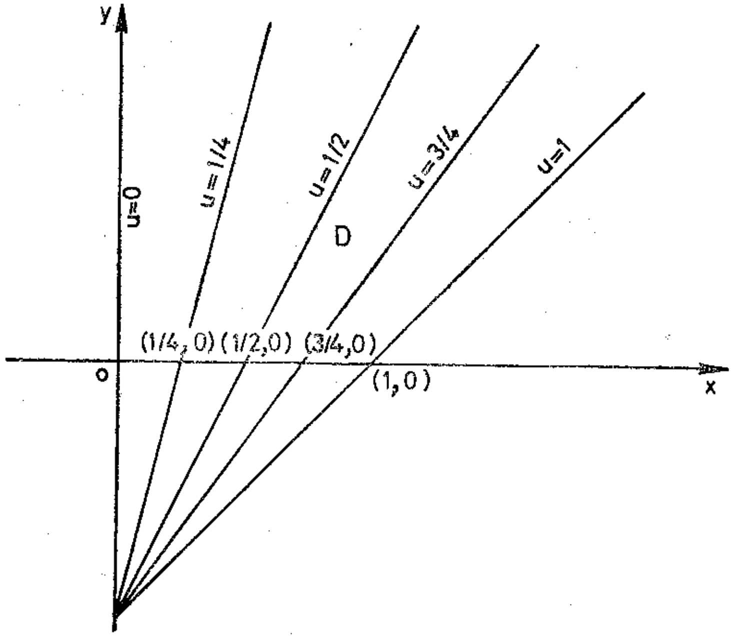

Fig. 2.2. The solution is determined in the wedge shaped region D of the (x, y)-plane.

We note two very important aspects of the quasilinear equations from this example.

(i) The domain D in the (x, y)-plane in which the solution is determined depends on the data prescribed in the Cauchy problem. Had we prescribed $u(x, 0)=\text{constant}=\frac{1}{2}$, say, for $0 \leqslant x \leqslant 1$, the characteristics would have been a family of parallel straight lines $y-2x=-2\eta$ and the domain D would have been the infinite strip bounded by the extreme characteristics $y-2x=0$ and $y-2x=-2$ as shown in the Fig. 2.3.

(ii) Even though the coefficients in the equation (2.36) and the Cauchy data (2.37) are regular, the solution develops a singularity at the point $(0, -1)$. Geometrically, this is evident from the fact that the characteristics which carry different values of u all intersect at $(0, -1)$. Analytically, this is clear from the explicit form of the solution obtained from (2.38) after eliminating σ and η:

$$u=\frac{x}{y+1}. \tag{2.39}$$

The appearance of a singularity in the solution of a Cauchy problem for certain Cauchy data is a property associated with nonlinear differential equations.

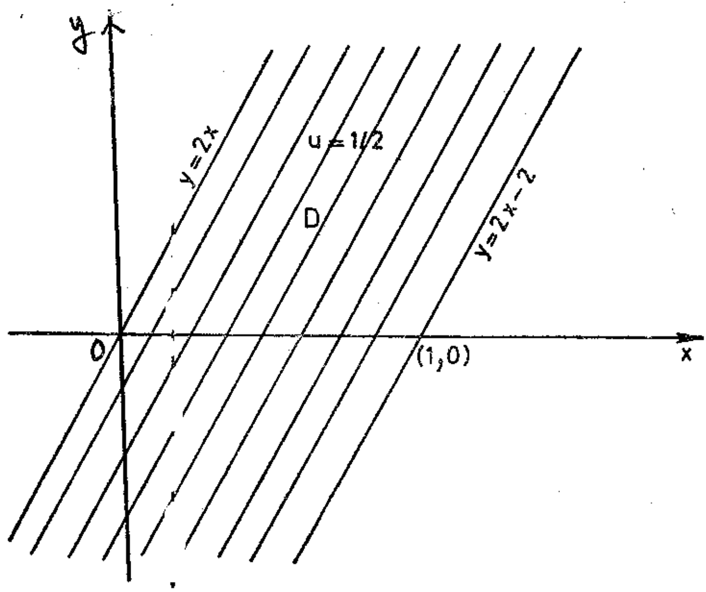

Fig. 2.3. The domain D when the Cauchy data is $u(x, 0)=1/2$ for $0 \leqslant x \leqslant 1$.

§2.3 The Characteristic Cauchy Problem

We have just seen that if the datum curve γ is such that the Cauchy data satisfies (2.31), then a unique solution of the Cauchy problem exists in a neighbourhood of the curve. Now suppose that

$$\frac{dy_0(\eta)}{d\eta}\,a(x_0(\eta), y_0(\eta), u_0(\eta)) - \frac{dx_0(\eta)}{d\eta}\,b(x_0(\eta), y_0(\eta), u_0(\eta)) = 0 \qquad (2.40)$$

everywhere along the curve γ, i.e. γ is a characteristic curve for a possible solution. Let us suppose further that a solution: $u=u(x, y)$, of the Cauchy problem exists. Then from (2.40) and (2.20) it follows that

$$\frac{du_0(\eta)}{d\eta} = \frac{d}{d\eta}\,u(x_0(\eta), y_0(\eta)) = \frac{dx_0}{d\eta}\,u_x(x_0, y_0) + \frac{dy_0}{d\eta}\,u_y(x_0, y_0)$$

must be proportional to $c_0(x_0(\eta), y_0(\eta), u_0(\eta))$. Therefore, the functions $x_0(\eta)$, $y_0(\eta)$, $u_0(\eta)$ satisfy the equations

$$\frac{dx_0}{a(x_0(\eta), y_0(\eta), u_0(\eta))} = \frac{dy_0}{b(x_0(\eta), y_0(\eta), u_0(\eta))} = \frac{du_0}{c(x_0(\eta), y_0(\eta), u_0(\eta))}$$

and the initial curve Γ is necessarily a Monge curve.

Consider now another curve Γ' in (x, y, u)-space which is not a Monge curve and which intersects Γ at some point. Then we can obtain an integral surface S' passing through Γ'. As one point of Γ lies on S', the entire original initial curve Γ will lie on S' and hence S' is an integral surface passing through Γ. Consider now another curve Γ'', which is not a Monge curve and which intersects Γ, but does not lie on S'. Then we get another integral surface S'' containing Γ and different from S'.

Therefore, *the solution of a characteristic initial value problem, if it exists, is nonunique.*

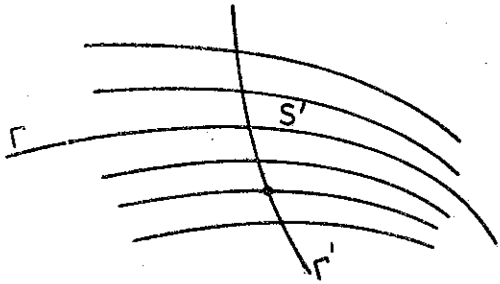

Fig. 2.4. Integral surface containing an initial curve Γ which is a Monge curve

§2.4 General Solution

Uptil now we have discussed only those solutions of a first order differential equation which satisfy certain prescribed conditions (i.e. solution of a Cauchy problem). In general these particular solutions are completely determined. For a single quasilinear equation of first order it is possible to get an explicit form of a *general solution* which is defined to be a solution from which all particular solutions can be obtained.

A relation of the form $f(x, y, u)=C$, where C is a constant, is called a *first integral* of the system of first order ordinary differential equations (2.22) (or (2.23) and (2.24)), if the function $f(x, y, u)$ has a constant value along an integral curve of (2.22) (i.e. along a Monge curve). It follows, therefore, that if $f(x, y, u)=C$ be a first integral of (2.22) and $x=x(\sigma)$, $y=y(\sigma)$, $u=u(\sigma)$ be any solution of these equations, then $f(x(\sigma), y(\sigma), u(\sigma))$ is independent of σ.

The general solution of the ordinary differential equations (2.22) consists of any two independent first integrals

$$\varphi(x, y, u)=C_1 \quad \text{and} \quad \psi(x, y, u)=C_2 \tag{2.41}$$

which together also constitute another representation of the two parameter family of Monge curves of (2.20). *The surface represented by a first integral, say $\varphi(x, y, u)=C_1$, is generated by a one parameter family of Monge curves by varying the parameter C_2 and hence represents an integral surface of (2.20).* Now it follows that each one of the two equations in (2.41) represents a one parameter family of integral surfaces of (2.20). Next we prove a theorem which connects any two independent families of integral surfaces to the general solution of the quasilinear equation.

Theorem 2.2 If $\varphi(x, y, u)=C_1$ and $\psi(x, y, u)=C_2$ be two independent first integrals of the ordinary differential equations (2.22), and $\varphi_u^2+\psi_u^2 \neq 0$ the general solution of the partial differential equation (2.20) is given by

$$h(\varphi(x, y, u), \psi(x, y, u))=0 \tag{2.42}$$

where h is an arbitrary function.

Proof Since the first integral $\varphi(x, y, u)=C_1$ represents an integral surface, the equation (2.20) is satisfied by $u_x=-\varphi_x/\varphi_u$, $u_y=-\varphi_y/\varphi_u$. This gives

$$a\varphi_x+b\varphi_y+c\varphi_u=0. \tag{2.43}$$

Similarly

$$a\psi_x+b\psi_y+c\psi_u=0. \tag{2.44}$$

If $f(x, y, u)=0$ be the equation of any integral surface of (2.20), we also have

$$af_x+bf_y+cf_u=0 \tag{2.45}$$

Since $a^2+b^2+c^2 \not\equiv 0$, it follows from (2.43)—(2.45) that the Jacobian $\dfrac{\partial(f, \varphi, \psi)}{\partial(x, y, u)}\equiv 0$. This implies that $f=h(\varphi, \psi)$ where h is an arbitrary function of its arguments, showing that the equation of any integral surface is given by (2.42).

The two parameter family of Monge curves in (x, y, u)-space is represented by the equations (2.41). The integral surface (2.42) is generated by a one parameter sub-family of the Monge curves, obtained by restricting the values of C_1 and C_2 by the relation

$$h(C_1, C_2)=0. \tag{2.46}$$

For a given Cauchy problem it is simple to determine the one parameter sub-family of the Monge curves which generate the integral surface passing through the initial curve Γ represented by (2.30). The parameters C_1 and C_2 for which the Monge curves intersect the curve Γ, satisfy

$$\varphi(x_0(\eta), y_0(\eta), u_0(\eta))=C_1$$

and

$$\psi(x_0(\eta), y_0(\eta), u_0(\eta))=C_2.$$

Eliminating η from these two, we get a relation of the form (2.46) between C_1 and C_2. This determines the function h. The solution of the Cauchy problem is obtained by solving u in terms of x and y from (2.42).

Example 2.4 Consider the differential equation

$$(y+2ux)u_x-(x+2uy)u_y=\tfrac{1}{2}(x^2-y^2). \tag{2.47}$$

The characteristic equations and the compatibility condition are

$$\frac{dx}{y+2ux}=\frac{dy}{-(x+2uy)}=\frac{du}{\frac{1}{2}(x^2-y^2)}.$$

To get one first integral we derive from these

$$\frac{x\,dx+y\,dy}{2u(x^2-y^2)}=\frac{2du}{x^2-y^2}$$

which immediately leads to

$$\varphi(x, y, u)\equiv x^2+y^2-4u^2=C_1. \tag{2.48}$$

For another independent first integral we derive a second combination

$$\frac{y\,dx+x\,dy}{y^2-x^2}=\frac{2\,du}{x^2-y^2}$$

which leads to

$$\psi(x, y, u)\equiv xy+2u=C_2. \tag{2.49}$$

The general integral of the equation (2.47) is given by

$$\left.\begin{aligned} &h(x^2+y^2-4u^2,\, xy+2u)=0 \\ \text{or}\quad &x^2+y^2-4u^2=f(xy+2u) \end{aligned}\right\} \tag{2.50}$$

where h or f are arbitrary functions of their arguments.

Consider a Cauchy problem in which u is prescribed to be zero on the straight line $x-y=0$. Parametrically, we can write it in the form

$$x=\eta,\ y=\eta,\ u=0.$$

From (2.48) and (2.49) we get $2\eta^2=C_1$ and $\eta^2=C_2$ which gives $C_1=2C_2$. Therefore, the solution of the Cauchy problem is obtained, when we take $h(\varphi, \psi)=\varphi-2\psi$. This gives

$$u=\frac{1}{2}\{\sqrt{(x-y^2)+1}-1\}. \tag{2.51}$$

We note that the solution of the Cauchy problem is determined uniquely at all points in the (x, y)-plane.

EXERCISE 2.2

1. Show that all the characteristic curves of the partial differential equation

$$(2x+u)u_x+(2y+u)u_y=u$$

through the point (1, 1) are given by the same straight line $x-y=0$.

2. Discuss the solution of the differential equation

$$uu_x+u_y=0, \qquad y>0,\ -\infty<x<\infty$$

with Cauchy data

$$u(x, 0)=\begin{cases}\alpha^2-x^2 & \text{for } |x|\leqslant\alpha\\ 0 & \text{for } |x|>\alpha.\end{cases}$$

3. Find the solution of the differential equation

$$\left(1-\frac{m}{r}u\right)u_x - mMu_y = 0$$

satisfying

$$u(0, y) = \frac{My}{\rho - y}$$

where m, r, ρ, M are constants, in a neighbourhood of the point $x=0$, $y=0$.

4. Find the general integral of the equation

$$(2x-y)y^2u_x + 8(y-2x)x^2u_y = 2(4x^2+y^2)u$$

and deduce the solution of the Cauchy problem when $u(x, 0) = \dfrac{1}{2x}$ on a portion of the x-axis.

5. Show that the result of elimination of an arbitrary function $h(\varphi, \psi)$ of two arguments from the relation

$$h(\varphi(x, y, u), \psi(x, y, u)) = 0$$

where φ and ψ are two known functions is a quasilinear equation (2.20). This is the converse of the result contained in the theorem 2.2.

§3 FIRST ORDER NONLINEAR EQUATIONS IN TWO INDEPENDENT VARIABLES

Now we pass on to a discussion of the most general first order equation, i.e. an equation of the form

$$F(x, y, u, p, q) = 0 \tag{3.1}$$

where F is a given function of its arguments and

$$p = u_x, \qquad q = u_y. \tag{3.2}$$

In this section we shall consider a nonlinear partial differential equation, i.e. equation (3.1) where F is not linear in p and q. We assume here that the function F possesses continuous second order partial derivatives over a domain D_3 of (x, y, u, p, q)-space with $F_p^2 + F_q^2 \neq 0$. Let the projection of D_3 on the (x, y, u)-space be denoted by D_2.

§3.1 Monge Strip and Charpit Equations

Let $u = u(x, y)$ represent an integral surface S of (3.1) in (x, y, u)-space, then $(p, q, -1)$ are the direction ratios of the normal to S. The differential equation (3.1) states that at a given point $P(x_0, y_0, u_0)$ on S, there is a relation between p_0 and q_0. This relation $F(x_0, y_0, u_0, p_0, q_0) = 0$ between p_0 and q_0 is not linear. Hence, unlike the case of a quasilinear equation, all tangent planes to possible integral surfaces through P do not pass through a fixed line but form a family of planes enveloping a conical surface, called the

Monge Cone, with P as its vertex. The differential equation thus assigns a Monge cone at every point, i.e. a field of Monge cones in the domain D_2 of (x, y, u)-space. *The problem of solving the differential equation (3.1) is to find surfaces which fit in this field, i.e. surfaces which touch the Monge cone at each point along a generator.*

We note that the Monge cone need not be closed as seen in Problem 1, Exercise 3.1.

Example 3.1 Consider the partial differential equation

$$p^2 - q^2 = 1. \tag{3.3}$$

At every point of the (x, y, u)-space the relation (3.3) can be expressed parametrically as

$$p_0 = \cosh \lambda, \qquad q_0 = \sinh \lambda, \qquad -\infty < \lambda < \infty. \tag{3.4}$$

The equation of the tangent planes at (x_0, y_0, u_0) are

$$(x - x_0) \cosh \lambda + (y - y_0) \sinh \lambda - (u - u_0) = 0. \tag{3.5}$$

The envelope of these planes is λ-eliminant of (3.5) and

$$(x - x_0) \sinh \lambda + (y - y_0) \cosh \lambda = 0 \tag{3.6}$$

which is obtained by differentiating (3.5) partially with respect to λ. Therefore, the Monge cone of (3.3) is

$$(x - x_0)^2 - (y - y_0)^2 - (u - u_0)^2 = 0. \tag{3.7}$$

This is a right circular cone with semi-vertical angle $\pi/4$ and whose axis is the straight line passing through (x_0, y_0, u_0) and parallel to the x-axis.

Since an integral surface is touched by a Monge cone along a generator, we proceed to determine the equations to a generator of the Monge cone of (3.1). At a given point (x_0, y_0, u_0), the relation between p_0 and q_0 can be expressed parametrically in the form

$$p_0 = p_0(x_0, y_0, u_0, \lambda), \qquad q_0 = q_0(x_0, y_0, u_0, \lambda) \tag{3.8}$$

which satisfy

$$F(x_0, y_0, u_0, p_0(x_0, y_0, u_0, \lambda), q_0(x_0, y_0, u_0, \lambda)) = 0 \tag{3.9}$$

for all values of the parameter λ for which p_0 and q_0 in (3.8) are defined.

The equations of the tangent planes for λ and $\lambda + \delta\lambda$ are

$$p_0(x_0, y_0, u_0, \lambda)(x - x_0) + q_0(x_0, y_0, u_0, \lambda)(y - y_0) = u - u_0 \tag{3.10}$$

and

$$p_0(x_0, y_0, u_0, \lambda + \delta\lambda)(x - x_0) + q_0(x_0, y_0, u_0, \lambda + \delta\lambda)(y - y_0) = u - u_0 \tag{3.11}$$

The limiting position of the line of intersection of these planes as $\delta\lambda \to 0$ is a generator of the Monge cone at (x_0, y_0, u_0). Expanding p_0 and q_0 in (3.11) in powers of $\delta\lambda$, using (3.10) and retaining only the first degree terms, we get

$$\frac{dp_0}{d\lambda}(x - x_0) + \frac{dq_0}{d\lambda}(y - y_0) = 0. \tag{3.12}$$

(3.10) and (3.12) are the equations to the generators in terms of the parameter λ. We can eliminate the derivatives $\frac{dp_0}{d\lambda}$ and $\frac{dq_0}{d\lambda}$ with the help of (3.9) which gives

$$(F_p)_0\frac{dp_0}{d\lambda}+(F_q)_0\frac{dq_0}{d\lambda}=0. \tag{3.13}$$

From (3.10), (3.12) and (3.13) we get the following equations of the generators of the Monge cone at (x_0, y_0, u_0)

$$\frac{x-x_0}{(F_p)_0}=\frac{y-y_0}{(F_q)_0}=\frac{u-u_0}{(pF_p+qF_q)_0}, \tag{3.14}$$

If we replace $x-x_0$, $y-y_0$, $u-u_0$ by dx, dy, du, respectively, corresponding to an infinitesimal movement, $x-x_0=dx$, $y-y_0=dy$, $u-u_0=du$, from (x_0, y_0, u_0) along the generator, then (3.14) tends to

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{du}{pF_p+qF_q}. \tag{3.15}$$

We note that, for the quasilinear equation (2.20), equations (3.15) reduce to (2.22) showing that the Monge cone degenerates into the Monge line element.

Suppose we are given an integral surface $S: u=u(x, y)$, where $u(x, y)$ has continuous second order partial derivatives with respect to x and y. At the points of S we know u, p and q as functions of x and y. Also at each point of the surface S, there exists a Monge cone which touches the surface along a generator of the cone. The lines of contact between the tangent planes of S and the corresponding cones, that is, the generators along which the surface is touched, define a direction field on the surface, which we shall call *Monge directions on S* (Fig. 3.1). Monge directions for a quasilinear equation and Monge directions on an integral surface for a nonlinear equation have the common property that they are special directions tangential to the integral surface. However, in the nonlinear case, they have no existence of their own but are defined only when an integral surface is prescribed.

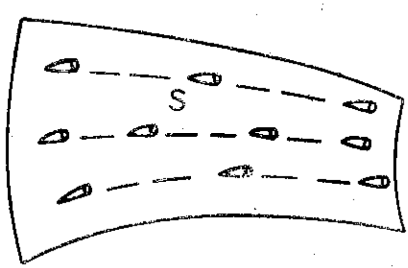

Fig. 3.1. Monge directions on an integral surface S.

The above direction field also defines a one parameter family of curves on S, we call these curves *Monge curves on S*, and these curves generate S. Denoting the ratios in (3.15) by $d\sigma$, we notice that the Monge curves on S can be determined by solving the ordinary differential equations

$$\frac{dx}{d\sigma}=F_p(x, y, u(x, y), u_x(x, y), u_y(x, y)) \tag{3.16}$$

and

$$\frac{dy}{d\sigma}=F_q(x, y, u(x, y), u_x(x, y), u_y(x, y)) \tag{3.17}$$

in the form

$$x=x(\sigma, x_0, y_0), \qquad y=y(\sigma, x_0, y_0) \tag{3.18}$$

and then determining u from

$$u=u(\sigma, x_0, y_0)\equiv u(x(\sigma, x_0, y_0), y(\sigma, x_0, y_0)). \tag{3.19}$$

Here $(x_0, y_0, u(x_0, y_0))$ is a point on the surface S and the Monge curve on S given by (3.18) and (3.19) passes through this point. Since

$$\frac{du}{d\sigma}=u_x\frac{dx}{d\sigma}+u_y\frac{dy}{d\sigma},$$

it follows from (3.16) and (3.17) that along these curves u varies according to

$$\frac{du}{d\sigma}=pF_p+qF_q \tag{3.20}$$

where $u=u(x, y)$ has been substituted in the expression on the right hand side.

Example 3.2 Consider the function

$$u=x\cos\varphi+y\sin\varphi, \qquad \varphi=\text{constant} \tag{3.21}$$

which represents an integral surface of the equation

$$F\equiv p^2+q^2-1=0. \tag{3.22}$$

Then (3.16) and (3.17) give

$$\frac{dx}{d\sigma}=2p=2\cos\varphi$$

$$\frac{dy}{d\sigma}=2q=2\sin\varphi.$$

Therefore, the Monge curves of (3.22) on the integral surface (3.21) are given by

$$x=x_0+2\sigma\cos\varphi, \qquad y=y_0+2\sigma\sin\varphi$$

and

$$u=x_0\cos\varphi+y_0\sin\varphi+2\sigma.$$

Along the Monge curves on S the variations of p and q are known from the expressions $p=u_x(x(\sigma, x_0, y_0), y(\sigma, x_0, y_0))$ and $q=u_y(x(\sigma, x_0, y_0), y(\sigma, x_0\, y_0))$

respectively. Now we shall determine the rates of change of p and q along a Monge curve on S. Since (3.1) is identically satisfied by $u=u(x, y)$, differentiating it with respect to x we get the identity

$$F_x+F_u u_x+F_p u_{xx}+F_q u_{yx}=0 \text{ on } S. \tag{3.23}$$

Along Monge curve on S

$$\frac{du_x}{d\sigma}=u_{xx}\frac{dx}{d\sigma}+u_{xy}\frac{dy}{d\sigma}=u_{xx}F_p+u_{xy}F_q.$$

For a sufficiently smooth solution, $u_{xy}=u_{yx}$ so that from (3.23), we get

$$\frac{dp}{d\sigma}=-(F_x+pF_u). \tag{3.24}$$

Similarly the variation of q along a Monge curve on S is

$$\frac{dq}{d\sigma}=-(F_y+qF_u). \tag{3.25}$$

Given an integral surface, we have shown that there exists a family of Monge curves, which generate the surface and along which x, y, u, p, q vary according to

$$\frac{dx}{d\sigma}=F_p \tag{3.26}$$

$$\frac{dy}{d\sigma}=F_q \tag{3.27}$$

$$\frac{du}{d\sigma}=pF_p+qF_q \tag{3.28}$$

$$\frac{dp}{d\sigma}=-F_x-pF_u \tag{3.29}$$

and

$$\frac{dq}{d\sigma}=-F_y-qF_u. \tag{3.30}$$

In what we have discussed uptil now, Monge curves exist only on a given integral surface. We now reverse the process by disregarding the fact that the system of ordinary differential equations (3.26)-(3.30) was derived with the help of an integral surface. We can do so since, for a given function F, the equations (3.26)-(3.30) form a complete set of five equations. In literature, these equations are called system of characteristic equations. However, we shall call the first two equations (3.26) and (3.27) *characteristic equations*, the last three equations (3.28)-(3.30) *compatibility conditions* and the system formed with all the five equations (3.26)-(3.30), *Charpit's equations*.

A set $(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma))$ of five differentiable functions is said to be a *strip*, if when we consider the curve $x=x(\sigma),\ y=y(\sigma),\ u=u(\sigma)$, the planes with normals given by $(p(\sigma), q(\sigma), -1)$ are tangential to it.

A solution $x=x(\sigma)$, $y=y(\sigma)$, $u=u(\sigma)$, $p=p(\sigma)$ and $q=q(\sigma)$ of the Charpit's equations satisfies the strip condition

$$\frac{du}{d\sigma}=p(\sigma)\frac{dx}{d\sigma}+q(\sigma)\frac{dy}{d\sigma}. \tag{3.31}$$

Note that not every set of five functions can be interpreted as a strip (Fig. 3.2). A strip requires that the planes with normals $(p, q, -1)$ be tangent to the curve, i.e. they must satisfy the strip condition (3.31) and the normals should vary continuously along the curve. For a solution of Charpit's equations (3.26)-(3.30), the strip condition is guaranteed by the first three equations.

Along a solution of the Charpit's equations, we have

$$\frac{dF}{d\sigma}=F_x\frac{dx}{d\sigma}+F_y\frac{dy}{d\sigma}+F_u\frac{du}{d\sigma}+F_p\frac{dp}{d\sigma}+F_q\frac{dq}{d\sigma} \tag{3.32}$$

which becomes identically equal to zero when we use (3.26)-(3.30). Therefore, F remains constant along an integral curve of the Charpit's equations in (x, y, u, p, q)-space. If $F=0$ is satisfied at an *initial point* $\sigma=0$, $F=0$ everywhere along the solution of Charpit's equations.

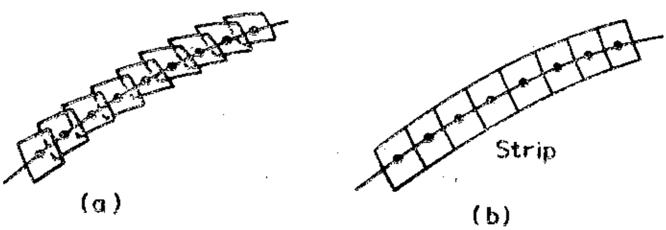

Fig. 3.2. Any set of five functions does not form a strip as in (a). The planes must be tangent to the curve and their normal should vary continuously (b).

The initial values for a solution of Charpit's equations can be prescribed by specifying x, y, u, p and q on a four-dimensional surface in (x, y, u, p, q)-space (where σ can be chosen 0). Therefore, the system of Charpit's equations defines a four parameter family of strips. From this four parameter family we choose a three parameter sub-family of strips by imposing the condition that $F=0$ at $\sigma=0$, which implies $F=0$ along these strips. We call this three parameter sub-family of strips *Monge strips* and the projections on (x, y)-plane of the corresponding space curves in (x, y, u)-space (on neglecting p and q), *characteristic curves.* Thus, the characteristic curves of a nonlinear partial differential equation form a three-parameter family of curves in (x, y)-plane. However, we note that in the case of quasilinear and nonlinear equations, it is meaningless to say without any reference to a solution that a particular curve in the (x, y)-plane is a characteristic curve. Different solutions give different families of characteristic curves.

We shall show that *if a Monge strip, say M, has one element (i.e. the values of $x(\sigma)$, $y(\sigma)$, $u(\sigma)$, $p(\sigma)$, $q(\sigma)$, for some σ, say $\sigma=0$) common with an integral surface $S: u=u(x, y)$, then the strip belongs entirely to the integral surface.* Let us suppose that at the point P, the integral surface S and the strip M have common values of (x, y, u, p, q). Since S is an integral surface, we can find a unique Monge curve on S passing through P. This, together with p and q at points on this curve, gives a Monge strip M' on S. Since both strips M and M' satisfy the Charpit's equations (3.26)-(3.30) with the same initial conditions at P, it follows from the uniqueness theorem of solutions of ordinary differential equations that M and M' are the same. As M' belongs entirely to the integral surface, the result follows.

EXERCISE 3.1

1. Show that the Monge cone of the equation

$$p=q^2$$

is an open cone which is generated by a one parameter family of straight lines whose one end is fixed but the other end moves on a parabola.

2. Consider the partial differential equation

$$F \equiv u(p^2+q^2)-1=0.$$

(i) Show that the general solution of the Charpit equations is a four parameter family of strips represented by

$$x=x_0+\frac{2}{3}u_0(2\sigma)^{3/2}\cos\theta,\ y=y_0+\frac{2}{3}u_0(2\sigma)^{3/2}\sin\theta$$

$$u=2u_0\sigma, \qquad p=\frac{\cos\theta}{\sqrt{2\sigma}}, \qquad q=\frac{\sin\theta}{\sqrt{2\sigma}}$$

where x_0, y_0, u_0 and θ are the parameters.

(ii) Find the three parameter sub-family representing the totality of all Monge strips.

(iii) Show that the characteristic curves consist of all straight lines in the (x, y)-plane.

Problem 3 Find a representation of Monge strips of the equation

$$2pqx^2y^2-px-qy-u=0 \qquad (1)$$

in the form

$$x=(2m_2-m_3e^{\sigma})^{-1},\ y=(2m_1-m_4e^{\sigma})^{-1},$$
$$p=m_1(2m_2-m_3e^{\sigma})^2,\ q=m_2(2m_1-m_4e^{\sigma})^2,$$
$$u=-2m_1m_2+(m_1m_3+m_2m_4)e^{\sigma}, \qquad (2)$$

where one of the arbitrary constants m_1, m_2, m_3, and m_4 can be absorbed in in a choice of σ. Solution on the next page.

Solution Charpit equations of (1) are

$$\frac{dx}{d\sigma} = 2qx^2y^2 - x \quad (3), \qquad \frac{dy}{d\sigma} = 2px^2y^2 - y \quad (4),$$
$$\frac{dp}{d\sigma} = 2pqxy^2 - x \quad (5), \qquad \frac{dq}{d\sigma} = 2pqx^2y - y \quad (6),$$
$$\frac{du}{d\sigma} = 2pqx^2y_2 - px - qy \quad (7).$$

In the results below, m_1, m_2, m_3, and m_4 are arbitrary constants.

$$x\frac{dp}{d\sigma} = -2(pqx^2y^2 - px) = -2(u+qy) \quad (using(1)), \quad (8)$$
$$similarly \;\; p\frac{dx}{d\sigma} = u + qy. \quad (9)$$
$$\Rightarrow \frac{x}{p}\frac{dp}{dx} = -2 \;\; \Rightarrow \;\; px^2 = m_1 \;\; (10), \;\; similarly \;\; qy^2 = m_2. \;\; (11)$$

From (3) using (11)

$$\frac{dx}{d\sigma} = (2m_2x - 1)x \qquad \Rightarrow \;\; x = \frac{1}{2m_2 - m_3e^{\sigma}}, \quad (12)$$
$$similarly \;\; y = \frac{1}{2m_1 - m_4e^{\sigma}}. \quad (13)$$

From (10) and (12), and (11) and (13) we get the expressions for p and q. Using the pde (1),

$$u = 2pqx^2y^2 - px - qy = -2m_1m_2 + 9(m_1m_3 = m_2m_4)e^{\sigma} \quad (14).$$

(2) is not the general solution of the Charpit equations (3)-(7), since we have used (1) at many steps, for example in derivation of (8). Note also that we have not used the equation (7). Instead we have used (1) to derive the expression (14) of u. Thus we have got a general form of equations of a Monge strip, which has four arbitrary constants, one more than what one should have.

One of the four constants m_1, m_2, m_3, m_4 can be absorbed in the choice of the origin of σ. For example, we write (13) in the form $(2m_1y - 1)/y = m_4e^{\sigma} = \exp(\sigma + ln(m_4))$ and set $\sigma' = \sigma + ln(m_4)$. In the derivation of (8), (9), (10), (11) and (12); the only integration with respect to σ appears in (12). This remains same when replace σ by σ'.

Thus we have derived the equations representing the Monge strip in the question.

§3.2 Solution of a Cauchy Problem

We are now in a position to discuss a method of solution of a Cauchy problem. If there exists an integral surfaces passing through a space curve Γ:

$$x=x_0(\eta),\ y=y_0(\eta),\ u=u_0(\eta); \tag{3.33}$$

the first order partial derivatives $p=p_0(\eta)$ and $q=q_0(\eta)$, evaluated from the equation of the integral surface at the points of Γ, satisfy the equation (3.1), i.e.

$$F(x_0(\eta),\ y_0(\eta),\ u_0(\eta),\ p_0(\eta),\ q_0(\eta))=0. \tag{3.34}$$

Moreover since $u_0(\eta)=u(x_0(\eta),\ y_0(\eta))$, differentiating with respect to η, we find that the strip condition with respect to η:

$$u_0'(\eta)=p_0(\eta)\ x_0'(\eta)+q_0(\eta)y_0'(\eta) \tag{3.35}$$

is satisfied at every point of Γ. Therefore, irrespective of the choice of S, we can now solve for $p_0(\eta)$ and $q_0(\eta)$ from (3.34) and (3.35) to get an initial strip

$$x=x_0(\eta),\ y=y_0(\eta),\ u=u_0(\eta),\ p=p_0(\eta),\ q=q_0(\eta). \tag{3.36}$$

We can solve the Charpit's equations (3.26)-(3.30) with initial values of x, y, u, p and q at $\sigma=0$ given by (3.36) and get the Monge strips starting from the various points of Γ. Since p_0, q_0 satisfy the strip condition (3.35) with respect to η, these Monge strips smoothly join to form a surface. Due to (3.34), F is identically zero along each Monge strip, hence the surface thus generated is an integral surface of (3.1) passing through Γ. We note that there can be more than one integral surface passing through Γ, since there can be more than one pair of functions $p_0(\eta)$, $q_0(\eta)$ satisfying the equations (3.34) and (3.35). However, once a set of values of p_0 and q_0 are selected, we expect to get a unique solution of the Cauchy problem. In order that the solution exists and is unique, it will be necessary to impose some restrictions on the initial curve Γ. The precise formulation of the theorem is given below.

Theorem 3.1 Suppose the function $F(x, y, u, p, q) \in C^2(D_3)$ where D_3 is a domain in (x, y, u, p, q)-space. Further suppose that along a datum curve $x=x_0(\eta)$, $y=y_0(\eta)$, on $I=\{\eta: 0 \leqslant \eta \leqslant 1\}$ the initial values $u=u_0(\eta)$ are assigned. Let the functions $x_0(\eta)$, $y_0(\eta)$, $u_0(\eta)$ belong to $C^2(I)$; the functions $p_0(\eta)$, $q_0(\eta)$, satisfying the two equations (3.34) and (3.35), belong to $C^1(I)$ and the set $(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)) \in D_3$ for $\eta \in I$ and satisfies

$$\frac{dx_0}{d\eta}F_q(x_0, y_0, u_0, p_0, q_0)-\frac{dy_0}{d\eta}F_p(x_0, y_0, u_0, p_0, q_0)\neq 0. \tag{3.37}$$

Then we can find a domain D in (x, y)-plane containing the datum curve and a unique solution in D:

$$u=u(x, y) \tag{3.38}$$

such that for $\eta \in I$

$$u(x_0(\eta), y_0(\eta)) = u_0(\eta) \tag{3.39}$$

$$u_x(x_0(\eta), y_0(\eta)) = p_0(\eta) \text{ and } u_y(x_0(\eta), y_0(\eta)) = q_0(\eta). \tag{3.40}$$

Note Unless $x_0(\eta)$, $y_0(\eta)$, $u_0(\eta)$ are assumed to be $C^2(I)$ the solutions p_0, q_0 of (3.34) and (3.35) cannot be $C^1(I)$. Further, the condition (3.37) implies that the datum curve γ is not a characteristic curve of a possible solution.

Proof Since the functions appearing on the right hand side of the Charpit's equations (3.26)-(3.30) belong to $C^1(D_3)$ and $x_0(\eta)$, $y_0(\eta)$, $u_0(\eta)$, $p_0(\eta)$, $q_0(\eta)$ are $C^1(I)$, there exists a unique solution of the Charpit's equations with initial condition $(x, y, u, p, q) = (x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta))$ at $\sigma = 0$:

$$x = X(\sigma, \eta),\ y = Y(\sigma, \eta),\ u = U(\sigma, \eta),\ p = P(\sigma, \eta),\ q = Q(\sigma, \eta) \tag{3.41}$$

whose partial derivatives with respect to σ and η exist and are continuous.

From (3.26), (3.27) and (3.37) it follows that

$$\left.\frac{\partial(X, Y)}{\partial(\eta, \sigma)}\right|_{\text{at } \sigma=0} = \frac{dx_0}{d\eta} F_q(x_0, y_0, u_0, p_0, q_0) - \frac{dy_0}{d\eta} F_p(x_0, y_0, u_0, p_0, q_0) \neq 0. \tag{3.42}$$

Therefore, there exists a neighbourhood $N(x_0, y_0)$ of a point $(x_0(\eta), y_0(\eta))$ on the datum curve in (x, y)-plane (corresponding to $\sigma = 0$), such that in $N(x_0, y_0)$ we can solve the first two equations of (3.41) uniquely in the form

$$\sigma = \sigma(x, y),\ \eta = \eta(x, y). \tag{3.43}$$

Substituting (3.43) in the expressions for u, p and q in (3.41) we get

$$u = U(\sigma(x, y), \eta(x, y)) \equiv u(x, y) \tag{3.44}$$

$$p = P(\sigma(x, y), \eta(x, y)) \equiv p(x, y) \tag{3.45}$$

$$q = Q(\sigma(x, y), \eta(x, y)) \equiv q(x, y) \tag{3.46}$$

which are continuously differentiable functions* of x and y. We shall now show that (3.44) is the solution of the Cauchy problem. It is obvious that on the datum curve $\sigma = 0$, the function (3.44) takes the prescribed value $u_0(\eta)$. Further, on the family of Monge strips (3.41), $F(x, y, u, p, q)$ has a constant value $F(x_0, y_0, u_0, p_0, q_0)$ which is zero, i.e.

$$F(x, y, u(x, y), p(x, y), q(x, y)) = 0 \text{ for } (x, y) \in N(x_0, y_0). \tag{3.47}$$

Therefore, the function $u(x, y)$ in (3.44) is a solution of the differential equation (3.1) provided, we can show that

$$u_x(x, y) = p(x, y),\ u_y(x, y) = q(x, y). \tag{3.48}$$

Consider the function

$$W(\sigma, \eta) = U_\eta - P X_\eta - Q Y_\eta \tag{3.49}$$

---

*From the inverse function theorem, it follows that $\sigma(x, y)$, $\eta(x, y)$ are $C^1(N(x_0, y_0))$. Now, U being a C^1 function of σ and η it follows that $u \in C^1(N(x_0, y_0))$.

whose value, $W(0, \eta)$, on the datum curve is zero. Differentiating (3.49) with respect to σ

$$\frac{\partial W}{\partial \sigma} = U_{\eta\sigma} - PX_{\eta\sigma} - QY_{\eta\sigma} - P_\sigma X_\eta - Q_\sigma Y_\eta$$

$$= \frac{\partial}{\partial \eta}(U_\sigma - PX_\sigma - QY_\sigma) + P_\eta X_\sigma + Q_\eta Y_\sigma - P_\sigma X_\eta - Q_\sigma Y_\eta$$

$$= 0 + P_\eta F_p + Q_\eta F_q + X_\eta(F_x + PF_u) + Y_\eta(F_y + QF_u)$$

where we have used the Charpit's equations in the last result. Adding and subtracting $F_u U_\eta$ we get

$$\frac{\partial W}{\partial \sigma} = (F_x X_\eta + F_y Y_\eta + F_u U_\eta + F_p P_\eta + F_q Q_\eta) - F_u(-PX_\eta - QY_\eta + U_\eta)$$

$$= F_\eta - F_u W.$$

Since F is identically zero along each of the Monge strips (3.41), $F_\eta \equiv 0$. The function W now satisfies the following linear homogeneous ordinary differential equation

$$\frac{\partial W}{\partial \sigma} = -F_u(\sigma, \eta) W \tag{3.50}$$

with solution

$$W = W(0, \eta) \exp\left\{-\int_0^\sigma F_u(\sigma, \eta)\, d\sigma\right\}. \tag{3.51}$$

Since $W(0, \eta) = 0$, $W(\sigma, \eta) = 0$ for all values of (σ, η) such that $(x, y) \in N(x_0, y_0)$. Therefore,

$$U_\eta = PX_\eta + QY_\eta. \tag{3.52}$$

From the Charpit's equations we also have

$$U_\sigma = PX_\sigma + QY_\sigma. \tag{3.53}$$

From (3.44) we get

$$u_x = U_\sigma \sigma_x + U_\eta \eta_x = \sigma_x(PX_\sigma + QY_\sigma) + \eta_x(PX_\eta + QY_\eta)$$

$$= P(X_\sigma \sigma_x + X_\eta \eta_x) + Q(Y_\sigma \sigma_x + Y_\eta \eta_x)$$

$$= P\frac{\partial x}{\partial x} + Q\frac{\partial y}{\partial x} = P \cdot 1 + Q \cdot 0 = P(\sigma, \eta) = p(x, y) \tag{3.54}$$

where we have used the expressions of x and y from the first two equations (3.41). Similarly we can show that

$$u_y = q(x, y). \tag{3.55}$$

Therefore, from (3.47) it follows that $u(x, y)$ given by (3.44) is a solution of the differential equation (3.1), in the domain $N(x_0, y_0)$.

To prove the uniqueness of the solution, let us assume that S' is another integral surface represented by the solution $u = u'(x, y)$ of the Cauchy problem. The surface S' can be covered (or generated) by a family of Monge strips after solving (3.16) and (3.17) with u replaced by u'. These Monge strips satisfy the same initial conditions at their points of intersection with the initial curve Γ, as the strips (3.41). From the uniqueness theorem for a solution of the Charpit's ordinary differential equations, it follows that this family of Monge strips on the integral surface S' must be the same as the strips (3.41). Therefore, the integral surface S coincides with S', i.e. $u=u'$ in $N(x_0, y_0)$.

Example 3.3 Consider the equation

$$p^2+q^2=1 \tag{3.56}$$

and a straight line in (x, y)-plane.

$$x=x_0 \equiv \eta \sin\beta \cos\alpha, \; y=y_0 \equiv \eta \sin\beta \sin\alpha \tag{3.57}$$

on which u is prescribed by

$$u=u_0 \equiv \eta \cos\beta \tag{3.58}$$

where α and β are constants.

The Monge cone at (x_0, y_0, u_0) is the envelope of the planes

$$(x-x_0)\cos\lambda+(y-y_0)\sin\lambda-(u-u_0)=0.$$

The Monge cone is therefore represented by the equation

$$(x-x_0)^2+(y-y_0)^2=(u-u_0)^2$$

which gives a right circular cone with vertex at (x_0, y_0, u_0), axis parallel to u-axis and semi-vertical angle $\pi/4$.

For the initial strip we have to solve the equations

$$p_0^2+q_0^2=1 \tag{3.59}$$

and

$$p_0 \sin\beta\cos\alpha+q_0\sin\beta\sin\alpha=\cos\beta. \tag{3.60}$$

If $\beta < \pi/4$, the equations (3.59) and (3.60) do not possess a real solution for p_0 and q_0 showing that the solution of the Cauchy problem does not exist. This can be explained (see Fig. 3.3, case (a)) from the fact that the

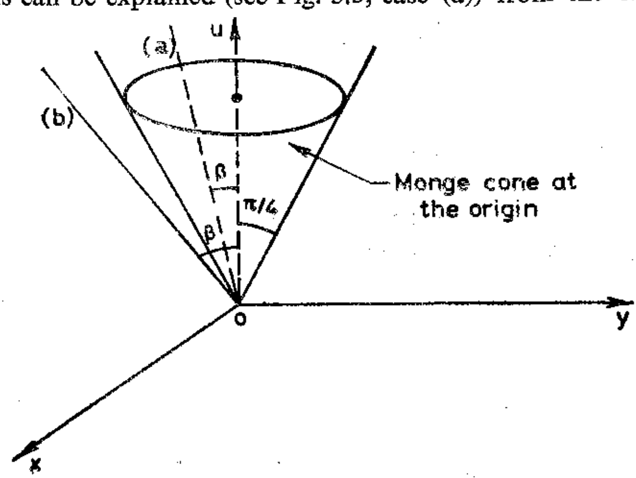

Fig. 3.3. Case (a): When $\beta < \pi/4$ the initial curve Γ is in the interior of the Monge cone.
Case (b): When $\beta > \pi/4$, Γ is outside the Monge cone.

space curve given by (3.57) and (3.58) through which the integral surface should pass, lies in the interior of the Monge cone at the origin. Naturally, it is not possible for an integral surface to touch the Monge cone along a generator of the cone and also to pass through a line within it.

For $\pi/4 < \beta < \pi/2$, we get two sets of values of p_0 and q_0

$$p_0 = \cot\beta\cos\alpha \pm \sin\alpha(1-\cot^2\beta)^{1/2} \tag{3.61}$$

$$q_0 = \cot\beta\sin\alpha \mp \cos\alpha(1-\cot^2\beta)^{1/2} \tag{3.62}$$

which are independent of η.
The Charpit's equations are

$$\frac{dx}{d\sigma} = 2p$$

$$\frac{dy}{d\sigma} = 2q$$

$$\frac{du}{d\sigma} = 2(p^2+q^2) = 2, \text{ (using 3.56)}$$

$$\frac{dp}{d\sigma} = 0$$

and

$$\frac{dq}{d\sigma} = 0.$$

Solving these with the initial values (3.57), (3.58(, (3.61) and (3.62), we get

$$\left.\begin{aligned} &x = 2p_0\sigma + \eta\sin\beta\cos\alpha,\ y = 2q_0\sigma + \eta\sin\beta\sin\alpha \\ &u = 2\sigma + \eta\cos\beta,\ p = p_0,\ q = q_0. \end{aligned}\right\} \tag{3.63}$$

Eliminating σ and η from (3.63) we get the two solutions of the Cauchy problem corresponding to the two sets of values of p_0 and q_0:

$$u = \cot\beta(x\cos\alpha + y\sin\alpha) \pm \sqrt{1-\cot^2\beta}(x\sin\alpha - y\cos\alpha). \tag{3.64}$$

They represent two planes which pass through the initial line Γ and touch the Monge cones along two generators.

§3.3 Solution of a Characteristic Cauchy Problem

We have seen that when the condition (3.37) is satisfied, i.e. when the data is such that the datum curve γ in (x, y)-plane is nowhere tangential to the characteristic curves for a possible solution, the solution of the Cauchy problem exists and is unique. However, when $F_q x_0'(\eta) - F_p y_0'(\eta) = 0$ holds everywhere along γ and the initial manifold M: $(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta))$ belongs to an integral surface S, then following the arguments of §3.1 for the derivation of Charpit equations (3.26)-(3.30) we can show that the strip M must be a Monge strip on S with the parameter σ replaced by η. Hence in the exceptional case, $F_q x_0' - F_p y_0' = 0$, a necessary condition for the existence of a solution of the Cauchy poblem is that the initial strip M is a Monge strip. This condition is also sufficient. In fact, if this condition is satisfied, there exists not only one but an infinite number of solutions of the *characteristic Cauchy problem.*

If $F_q x_0' - F_p y_0' = 0$ and the initial strip is not a Monge strip, then it follows from above that there exists no solution of the Cauchy problem having continuous derivatives up to the second order in a neighbourhood of the datum curve.

EXERCISE 3.2

1. Solve the following Cauchy problems:

(i) $\frac{1}{2}(p^2+q^2)=u$

with Cauchy data prescribed on the circle $x^2+y^2=1$ by

$$u(\cos\theta, \sin\theta)=1,\ 0 \leqslant \theta \leqslant 2\pi$$

(ii) $p^2+q^2+\left(p-\frac{1}{2}x\right)\left(q-\frac{1}{2}y\right)-u=0$

with Cauchy data prescribed on the x-axis by

$$u(x, 0)=0$$

(iii) $2pq-u=0$

with Cauchy data prescribed on the y-axis by

$$u(0, y)=\frac{1}{2}y^2$$

(iv) $2p^2x+qy-u=0$

with Cauchy data

$$u(x, 1)=-\frac{1}{2}x.$$

2. Consider a two parameter family of functions $u=\varphi(x, y, a, b)$, where φ is a known function of its arguments and a, b are parameters. If the rank of the matrix

$$\begin{bmatrix} \varphi_a & \varphi_{xa} & \varphi_{ya} \\ \varphi_b & \varphi_{xb} & \varphi_{yb} \end{bmatrix}$$

is 2, show that the result of the elimination of a and b from the relations $\varphi(x, y, a, b)=u$, $\varphi_x(x, y, a, b)=u_x$, $\varphi_y(x, y, a, b)=u_y$ leads to a first order nonlinear equation

$$F(x, y, u, u_x, u_y)=0.$$

3. Two first order partial differential equations are said to be *compatible*, if they have a common solution. Show that the necessary and sufficient condition for the two equations

$$F(x, y, u, p, q)=0 \text{ and } G(x, y, u, p, q)=0$$

to be compatible is that

$$\frac{\partial(F, G)}{\partial(x, p)}+p\frac{\partial(F, G)}{\partial(u, p)}+\frac{\partial(F, G)}{\partial(y, q)}+q\frac{\partial(F, G)}{\partial(u, q)}=0$$

is satisfied either identically or as a consequence of relations $F=0$ and $G=0$.

§4 COMPLETE INTEGRAL

The method of characteristics developed in previous sections for solving a first order partial differential equation gives a very deep insight into the conditions under which the solution of a Cauchy problem exists and is unique. For the present we shall be concerned with a powerful but a formal technique of solution with the help of a "general solution". In problem 2 of the Exercise 3.2 we saw that the result of elimination of two arbitrary constants a and b from a relation

$$u=\varphi(x, y, a, b) \tag{4.1}$$

leads to a nonlinear equation

$$F(x, y, u, u_x, u_y)=0. \tag{4.2}$$

We note that (4.1) satisfies (4.2) for all values of a and b.

We shall show that a solution of the form (4.1) of (4.2) is sufficiently general in the sense that all other solutions of this equation can be obtained from it merely by simple operations of differentiation and elimination of the constants.

Definition: A two parameter family of solutions (4.1) of the equation (4.2) is called a *complete integral* of the equation if the rank of the matrix

$$\begin{bmatrix} \varphi_a & \varphi_{xa} & \varphi_{ya} \\ \varphi_b & \varphi_{xb} & \varphi_{yb} \end{bmatrix}$$

is two in an appropriate domain of the variables x, y, a, b.

The condition that the above matrix has rank two assures that the function φ depends on two independent parameters* and the elimination of a and b from (4.1) and

$$u_x=\varphi_x(x, y, a, b), \quad u_y=\varphi_y(x, y, a, b) \tag{4.3}$$

leads to the equation (4.2).**

* If a and b be combined into one parameter $c = c(a, b)$, then two rows of the matrix become linearly dependent and its rank rank becomes one.

** If the rank is two, a and b can be solved from (4.3) and these can be substituted in (4.1).

§4.1 Determination of a Complete Integral

It is simple to determine a complete integral for a given partial differential equation (4.2). The problem 3 of Exercise 3.2 gives a condition for the existence of a common solution of two equations $F(x, y, u, u_x, u_y)=0$ and $G(x, y, u, u_x, u_y)=0$. Once these two equations have a common solution, we first solve them simultaneously for u_x and u_y in terms of x, y and u:

$$u_x=h(x, y, u) \text{ and } u_y=k(x, y, u)$$

and then the differential relation

$$h(x, y, u)\, dx+k(x, y, u)\, dy=du \tag{4.4}$$

will possess an integrating factor and can be integrated giving a relation between x, y and u and an arbitrary constant b. Therefore, a complete integral of (4.2) can be determined if we can determine a compatible equation $G(x, y, u, p, q)=0$ containing an arbitrary constant a. But this is simple, since the result of problem 3, of exercise 3.2 shows that any G satisfying the equation:

$$F_p \frac{\partial G}{\partial x}+F_q \frac{\partial G}{\partial y}+(pF_p+qF_q) \frac{\partial G}{\partial u}-(F_x+pF_u)\frac{\partial G}{\partial p}$$
$$-(F_y+qF_u) \frac{\partial G}{\partial q}=0 \tag{4.5}$$

would be a compatible equation.

This is a first order linear homogeneous partial differential equation for G in five independent variables x, y, u, p and q. The theory of this equation (as indicated in the next section, i.e. §5) is similar to that of a linear equation in two independent variables. For the equation (4.5), the characteristic equations and the compatibility conditions are

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{du}{pF_p+qF_q}=\frac{dp}{-(F_x+pF_u)}=\frac{dq}{-(F_y+qF_u)} \tag{4.6}$$

Since the compatibility condition implies that $G=$ constant on the characteristic curves in (x, y, u, p, q)-space, it follows that if we can get any first integral, say $s(x, y, u, p, q)=a$ of the characteristic equations, then $G \equiv s(x, y, u, p, q)-a=0$ is the required equation containing an arbitrary constant a and compatible with $F(x, y, u, u_x, u_y)=0$.

We note here that the characteristic equations of (4.5) are nothing but the Charpit equations (3.26)-(3.30) of the equation (4.2).

Example 4.2: The Charpit equations for the partial differential equation

$$x^2p^2+y^2q^2-4=0 \tag{4.7}$$

are

$$\frac{dx}{2x^2p}=\frac{dy}{2y^2q}=\frac{du}{2(x^2p^2+y^2q^2)}=\frac{dp}{-2xp^2}=\frac{dq}{-2yq^2}.$$

We take the relation $\frac{dx}{2x^2p}=\frac{dp}{-2xp^2}$ which gives

$$G \equiv xp=\text{constant}=a, \text{ say.} \tag{4.8}$$

Taking one set of values of p and q from (4.7) and (4.8) and substituting in (4.4), we get

$$\frac{a}{x}\,dx+\frac{\sqrt{4-a^2}}{y}\,dy=du.$$

Integrating this we get a complete integral

$$u=a\ln x+\sqrt{4-a^2}\ln y+b \tag{4.9}$$

containing two arbitrary constants a and b.

EXERCISE 4.1

1. Show that a complete integral of
   (a) $F(p, q)=0$, where F involves only p and q and $F(p, Q(p))=0$ is $u=ax+Q(a)y+b$.
   (b) $F\equiv f(x, p)-g(y, q)=0$ is obtained by solving p and q from $f(x, p)=a$, $g(y, q)=a$ and integrating $du=pdx+qdy$,
   (c) $F\equiv u-px-qy-f(p, q)=0$ is $u=ax+by+f(a, b)$.
2. If the independent variables x and y do not appear in the equation $F(u, p, q)=0$, then show that the complete integral can be obtained by solving p from $F(u, p, ap)=0$, taking $q=ap$ and integrating $du=p\,dx+q\,dy$.

§4.2 New Solutions from a Complete Integral

Let us try to generate some new solutions from the complete integral. Consider a function which is obtained from (4.1) by replacing the constants a and b by some functions of x and y:

$$u=\varphi(x, y, a(x, y), b(x, y)). \tag{4.10}$$

Then

$$u_x=\varphi_x+\varphi_a a_x+\varphi_b b_x,\ u_y=\varphi_y+\varphi_a a_y+\varphi_b b_y. \tag{4.11}$$

The new function (4.10) will also satisfy the equation (4.2) if and only if u_x and u_y are the same as those given in the relations (4.3). For this to be true, we have

$$\varphi_a a_x+\varphi_b b_x=0,\ \varphi_a a_y+\varphi_b b_y=0. \tag{4.12}$$

(4.12) is automatically satisfied, provided

$$\text{Case A}: a=\text{constant},\quad b=\text{constant} \tag{4.13}$$

$$\text{Case B}: \varphi_a(x, y, a, b)=0,\ \varphi_b(x, y, a, b)=0. \tag{4.14}$$

In the case A, when a and b are both constants, we get back the complete integral (4.1) itself.

In the case B, the two relations in (4.14) can be used to solve a and b in terms of x and y. This, when substituted in (4,10) gives a new solution, called a *singular solution* of (4.2). Geometrically, the singular solution, obtained by the elimination of a and b from (4.1) and (4.14) represents the envelope of the two parameter family of surfaces given by (4.1) (see Goursat §219, 1959). (4.12) can also be satisfied by non-zero values of ϕ_a and ϕ_b, provided

$$\text{Case C}: \frac{\partial(a, b)}{\partial(x, y)} = 0. \tag{4.15}$$

The equation (4.15) implies that there is a relation between a and b of the form

$$b = \omega(a) \tag{4.16}$$

where ω is an arbitrary function of its argument. Substituting $b = \omega(a)$ in (4.12), we get

$$\varphi_a(x, y, a, \omega(a)) + \varphi_b(x, y, a, \omega(a))\omega'(a) = 0. \tag{4.17}$$

For every choice of the function ω in (4.16), we can solve a and b in terms of x and y from (4.16) and (4.17) and substitute in (4.10) to get a new solution. Geometrically, the new solution represents an envelope of the one-parameter subfamily of integral surfaces

$$u = \varphi(x, y, a, \omega(a)) \tag{4.18}$$

obtained by selecting b as a function of a. The solution thus obtained depends on an arbitrary function $\omega(a)$ and is called a *general solution*. We shall show in the next section that given a noncharacteristic Cauchy problem, it is simple to choose the function $\omega(a)$ so that the solution thus obtained solves the Cauchy problem.

The envelope of a two-parameter family or a one-parameter subfamily of a complete integral is always an integral surface, since x, y, u, p and q at any point of the envelope coincides with that of some member, which is itself an integral surface.

Let us view the relations (4.17) and (4.18) in a slightly different way.* Since the function $\omega(a)$ is arbitrary, we can choose a, the value $\omega(a)$ of ω at a and the derivative $\omega'(a)$ as independent of each other. Denoting $\omega(a)$ by b and $\omega'(a)$ by c we find that the two equations

$$u = \varphi(x, y, a, b) \text{ and } \varphi_a(x, y, a, b) + c\varphi_b(x, y, a, b) = 0 \tag{4.19}$$

together represent a three parameter family of curves in (x, y, u)-space, the parameters being now a, b, and c. Consider any three values of a, b and c. Then there exists a function ω, such that $\omega(a) = b$, and $\omega'(a) = c$. For these values of a and b, (4.18) is a member of a family of surfaces, whose envelope is obtained by this choice of ω. This member touches the envelope along a curve of contact given by the equations (4.19). Therefore a member

* **An alternative explanation:** Since the function ω is arbitrary, we can write $\omega(\zeta) = b + c(\zeta - a) + d(\zeta - a)^2 + \cdots$, where $b, c, d, \cdots$ are arbitrary. hence setting $\omega'(a) = c$ in (4.17), we find that the two equations

of the three parameter family of curves (4.19) is the curve of contact of two integral surfaces, one of them being a member of the family (4.18) and, the other being the envelope of the family, and hence must be a Monge curve. Along this Monge curve the variations of p and q, forming the Monge strip, is given by

$$p = \varphi_x(x, y, a, b),\ q = \varphi_y(x, y, a, b). \tag{4.20}$$

Therefore, the four equations (4.19) and (4.20), from which any four of x, y, u, p, q can be expressed in terms of remaining one and the three parameters a, b and c, represent the complete set of the three parameter family of Monge strips of the equation (4.2). The four equations are the four independent first integrals of the Charpit equations. *Thus, we have* proved

Theorem A three-parameter solution of Charpit's ODEs (3.26) and (3.30), representing the set of all Monge strips, can be obtained from a complete integral of the PDE merely by differentiation.

§4.3 Solution of a Cauchy Problem

We shall now show that once we know a complete integral, by simple operations of differentiation and elimination alone, we can fin d the solution of a Cauchy problem. Let us proceed by geometrical arguments.

We are required to construct an integral surface S of (4.2) passing through an initial curve

$$\Gamma:\ x = x_0(\eta),\ y = y_0(\eta),\ u = u_0(\eta). \tag{4.21}$$

If S is either a member of the two parameter family (4.1) or it is the singular integral surface represented by the a, b-eliminant of the equations (4.1) and (4.14), this can be verified by direct substitution. In case it is neither then let us try to see if S can coincide with the envelope E of a one parameter subfamily T given by (4.18) for a suitable choice of the function $\omega(a)$ in (4.16). Given a point P on an envelope E, we can always find a member T_P of the subfamily such that T_p touches E along a curve passing through P. Therefore, if we assume that the envelope E passes through the initial curve Γ and choose the point P on Γ, then T_p which touches E must also touch the curve Γ at P. At the points of intersection of Γ and any member of (4.1) the parameter η satisfies

$$\phi(x_0(\eta), y_0(\eta), a, b) = u_0(\eta) \tag{4.22}$$

For the subfamily T of (4.1), the condition that Γ and T_p touches at P imlies that the equation (4.22) must give two equal roots for η which is the same thing as saying that the equation (4.22) and

$$\frac{\partial}{\partial \eta}\phi(x_0(\eta), y_0(\eta), a, b) = u_0'(\eta) \tag{4.23}$$

must have a common root. Eliminating η from these two equations we get the relation (4.16) between a and b for which the envelope E passes through Γ. This envelope is the required integral surface. Let us explain this method with the help of an example.

Example 4.2 Consider the problem number 1(iv) in the exercise 3.2. The partial differential equation is

$$2p^2x+qy-u=0 \tag{4.24}$$

and the Cauchy data can be put in the form

$$x=x_0(\eta)\equiv\eta,\ y=y_0(\eta)\equiv 1,\ u=u_0(\eta)\equiv -\frac{1}{2}\eta. \tag{4.25}$$

To derive a complete integral, we write the Charpit's equations:

$$\frac{dx}{4px}=\frac{dy}{y}=\frac{du}{4p^2x+qy}=\frac{dp}{-2p^2+p}$$ and $dq=0$

which immediately gives a compatible equation

$$q=a \tag{4.26}$$

containing an arbitrary constant. From (4.24) and (4.26) we get

$$p=\sqrt{\frac{u-ay}{2x}}. \tag{4.27}$$

We rewrite the equation $du=pdx+qdy$ with p and q given by (4.26) and (4.27) in the form

$$\frac{du-a\,dy}{\sqrt{u-ay}}=\frac{dx}{\sqrt{2}\sqrt{x}}$$

which gives a complete integral

$$\sqrt{u-ay}=\frac{1}{\sqrt{2}}\sqrt{x}+b$$

or

$$\left(u-ay-\frac{x}{2}-b\right)^2=2bx. \tag{4.28}$$

Substituting (4.25) in (4.28), we get

$$(\eta+a+b)^2=2b\eta \tag{4.29}$$

which after differentiation with respect to η gives

$$2(\eta+a+b)=2b. \tag{4.30}$$

Eliminating η from (4.29) and (4.30), we get

$$b=-2a. \tag{4.31}$$

Substituting b from (4.31) in (4.28) and forming the envelope of the one parameter family thus obtained, we get the solution of the Cauchy problem:

$$u=\frac{xy}{2(y-2)}. \tag{4.32}$$

EXERCISE 4.2

1. Use the method of complete integrals to solve the following Cauchy problems:

   (i) $2pq-u=0,\ \ u(\eta, 1)=\frac{1}{2}\eta$

   (ii) $p-q=\frac{1}{2}(x^2+y^2),\ u(\eta, \eta)=\frac{1}{2}\eta^2$ for $-\infty<\eta<\infty$

   (iii) $p^2+q^2=u,\ u(\cos\eta, \sin\eta)=1$ for $0\leqslant\eta\leqslant 2\pi$

   (iv) $u=px+qy+p+q-2pq,$

   $$u(\eta, \eta)=2\eta \qquad \text{for } -\infty<\eta<\infty.$$

2. Given any two complete integrals: $u=\varphi(x, y, a, b)$, $u=\psi(x, y, c, d)$ of a first order partial differential equation, show that one complete integral can be derived from the other.
   (Hint: Take an initial curve Γ lying on $u=\psi(x, y, c, d)$ and depending on the two constants c and d. Solve the Couchy problem using the first complete integral.)
3. Find a complete integral of

   $$4(p+q)(u-xp-yq)=1$$

   and use it to find the equations representing the complete set of three parameter family of Monge strips as discussed in the end of the section 4.2.

*§5. FIRST ORDER EQUATIONS IN MORE THAN TWO INDEPENDENT VARIABLES

Before we pass on to the theory of first order partial differential equations in more than two independent variables, we shall discuss here a few concepts in the m dimensional space of variables $x_1, x_2, \ldots, x_m$, collectively denoted by x_α. We shall make a convention that the range of a suffix α, β or γ is $1, 2, \ldots, m$ and that of the suffix r be $1, 2, \ldots, m-1$. We shall also use the summation convention that a repeated suffix in a term will imply sum over the range of the suffix.

*§5.1 Differentiation in Higher Dimensions

Let us consider a direction field $(a_1(x_\alpha), a_2(x_\alpha), \ldots, a_m(x_\alpha))$ defined in a domain D_1 of the m-dimensional space (x_α) and we assume that $a_1^2+\ldots+a_m^2\neq 0$ in D_1. If $u=u(x_\alpha)$ be a C^1 function, then $a_\alpha\frac{\partial u}{\partial x_\alpha}$, where we use the summation convention for the suffix α, represents differentiation of the function u at (x_α) in the direction given by the vector (a_α). Since a_α are continuously differentiable functions, the ordinary differential equations

$$\frac{dx_\alpha}{d\sigma} = a_\alpha(x_\beta) \tag{5.1}$$

can be solved in the form

$$x_\alpha = x_\alpha(\sigma, \eta_r), \quad r = 1, 2, \ldots, m-1 \tag{5.2}$$

where η_r's are $(m-1)$ arbitrary constants. Equation (5.2) represents an $(m-1)$ parameter family of curves, a member of which passes through every point of D_1.

Then

$$\frac{\partial u}{\partial \sigma} = a_\alpha \frac{\partial u}{\partial x_\alpha} \tag{5.3}$$

denotes the *differentiation in the direction of a member of the family of curves* (5.2).

Let us consider an $m-1$ dimensional manifold (hypersurface) S: $\varphi(x_\alpha) = 0$ in (x_α)-space such that φ is C^1 in a neighbourhood of S and $\varphi_{x_\alpha} \varphi_{x_\alpha} \neq 0$ on S. The vector (φ_{x_α}) at the points of S is in the direction of the normal to S. The expression

$$\varphi_{x_\alpha} u_{x_\alpha} \tag{5.4}$$

evaluated at the points of S is a *derivative in the direction of the normal* and

$$\{\varphi_{x_\alpha}/(\varphi_{x_\beta}\varphi_{x_\beta})^{1/2}\} u_{x_\alpha} \tag{5.5}$$

evaluated at the points of S is the *normal derivative* of u. If

$$a_\alpha \varphi_{x_\alpha} = 0 \tag{5.6}$$

at a point P on S, the vector (a_α) is orthogonal to the vector (φ_{x_α}) and hence lies in the tangent plane of S at the point P. When (5.6) is satisfied, $\partial u/\partial \sigma$ given by (5.3) is called a *tangential derivative* on S. On the other hand, when

$$a_\alpha \varphi_{x_\alpha} \neq 0 \tag{5.7}$$

we call $\partial u/\partial \sigma$ a transversal *derivative*. Where $a_\alpha = \lambda \varphi_{x_\alpha}$, where λ is a scalar function $\neq 0$, the outward directional derivative (5.3) is in the direction of the normal.

We can easily verify that the operator

$$\varphi_{x_\alpha} \frac{\partial}{\partial x_\beta} - \varphi_{x_\beta} \frac{\partial}{\partial x_\alpha} \tag{5.8}$$

for each pair of values of α and β, $\alpha \neq \beta$, represents a tangential derivative, i.e. a derivative within the surface $\varphi(x_\alpha) = 0$.

We shall now indicate a procedure, by which we can easily obtain a parametric representation of the equation of a surface S which is initially represented by a relation $\varphi(x_\alpha)=0$. Let $\eta_r(x_\alpha)\in C^1$, $r=1, 2, \ldots, m-1$ be $m-1$ functions such that the Jacobian

$$J=\frac{\partial(\eta_1, \eta_2, \ldots, \eta_{m-1}, \varphi)}{\partial(x_1, x_2, \ldots, x_m)} \tag{5.9}$$

is nonzero and finite at every point of the surface S. Since $J\neq 0$ on S, we can express x_α in terms of η_r and φ. Then the parametric representation of S is given by

$$x_\alpha=x_{\alpha 0}(\eta_1, \eta_2, \ldots, \eta_{m-1})\equiv x_\alpha(\eta_1, \eta_2, \ldots, \eta_{m-1}, \varphi=0). \tag{5.10}$$

When $\eta_1, \eta_2, \ldots, \eta_{m-1}$ vary in a suitable domain, the surfaces S is described. We now prove the following theorem.

Theorem 5.1 The tangential derivatives of u at a point P on the surface S depend only on the distribution of the values of u on the surface itself in the neighbourhood of P.

Proof: The values of u on S can be expressed in the form

$$u=u_0(\eta_r)\equiv u(x_\alpha(\eta_r, \varphi=0)). \tag{5.11}$$

The expression $\partial u_0/\partial\eta_r$ represents the rate of change of the function u as we move along the line of intersection of $m-1$ hyper surfaces $\varphi=0$, $\eta_1=$ constant, $\ldots$, $\eta_{r-1}=$ constant, $\eta_{r+1}=$ constant, $\ldots$, $\eta_{m-1}=$ constant. Therefore, $\partial u_0/\partial\eta_r$, $r=1, \ldots, m-1$ are $m-1$ independent tangential derivatives on S. Further, $\partial u/\partial\varphi$ is an outward derivative (not necessarily a normal derivative). The same can be verified using (5.6) by setting $\dfrac{\partial u_0}{\partial\eta_r}=\dfrac{\partial x_\alpha}{\partial\eta_r}\cdot\dfrac{\partial u_0}{\partial x_\alpha}$, i.e. $a_\alpha=\dfrac{\partial x_\alpha}{\partial\eta_r}$ and using (5.7) by setting $\dfrac{\partial u}{\partial\phi}=\dfrac{\partial x_\alpha}{\partial\varphi}\cdot\dfrac{\partial u}{\partial x_\alpha}$, i.e. $a_\alpha=\dfrac{\partial x_\alpha}{\partial\phi}$. Let (a_α) be a vector satisfying (5.6) then the derivative in its direction is

$$a_\alpha\frac{\partial u}{\partial x_\alpha} \text{ on } S=\left[a_\alpha\left(u_\varphi\frac{\partial\varphi}{\partial x_\alpha}+u_{\eta_r}\frac{\partial\eta_r}{\partial x_\alpha}\right)\right]_{\varphi=0}$$

$$=a_\alpha\frac{\partial\eta_r}{\partial x_\alpha}\frac{\partial u_0}{\partial\eta_r}. \tag{5.12}$$

(5.12) expresses any derivative on the left hand side in terms of $m-1$ inner derivatives $\partial u_0/\partial\eta_r$, which can be calculated from (5.11) expressing the distribution of u on S. This proves the theorem.

We have shown that for an $(m-1)$-dimensional manifold S, there are only $m-1$ independent derivatives and any other derivative can be expressed as a linear combination of these derivatives. Further all first order partial derivatives of a function u on S can be obtained if $m-1$ mutually independent first order derivatives and a derivative such as u_φ are prescribed on S.

Let us explain the concepts introduced above with the help of the geometry of two-dimensional space (x, y). Let $\varphi(x, y)=0$ be a curve in (x, y)-plane. The direction ratios of the normal and tangent to the curve are

$$\text{direction ratios of normal: } (\varphi_x, \varphi_y), \tag{5.13}$$

$$\text{direction ratios of tangent: } (\varphi_y, -\varphi_x). \tag{5.14}$$

Let $\eta(x, y)$ be a function of x, y such that

$$J=\frac{\partial(\eta, \varphi)}{\partial(x, y)} \neq 0. \tag{5.15}$$

Then for transformation from (x, y) to (η, φ) and vice-versa we have the following relations

$$\frac{\partial}{\partial \varphi}=-\frac{1}{J}\left(\eta_y \frac{\partial}{\partial x}-\eta_x \frac{\partial}{\partial y}\right) \tag{5.16}$$

and

$$\frac{\partial}{\partial \eta}=\frac{1}{J}\left(\varphi_y \frac{\partial}{\partial x}-\varphi_x \frac{\partial}{\partial y}\right). \tag{5.17}$$

On the curve $\varphi=0$, we can get $x=x(\eta)$ and $y=y(\eta)$. A derivative in tangent direction a function u is

$$\frac{\partial u}{\partial \eta}=\frac{1}{J}\left(\varphi_y \frac{\partial u}{\partial x}-\varphi_x \frac{\partial u}{\partial y}\right) \tag{5.18}$$

or simply $\varphi_y \frac{\partial u}{\partial x}-\varphi_x \frac{\partial u}{\partial y}$. A transversal derivative is an expression of the form

$$a \frac{\partial u}{\partial x}+b \frac{\partial u}{\partial y} \tag{5.19}$$

where $a \varphi_x+b \varphi_y \neq 0$.

In particular, the derivative in the normal direction

$$\varphi_x \frac{\partial u}{\partial x}+\varphi_y \frac{\partial u}{\partial y}$$

is a transversal derivative. Also $\partial u / \partial \phi$ given by (5.16) is a transversal derivative, since from (5.15),

$$\varphi_x \eta_y-\varphi_y \eta_x \neq 0.$$

Example 5.1 If we take the function φ to be

$$\varphi \equiv x^2-y^2-1 \tag{5.20}$$

then an independent function is

$$\eta=x^2+y^2. \tag{5.21}$$

We can verify that $J=-8 x y$ is not zero in the neighbourhood of the curve $\varphi=0$, i.e. $x^2-y^2=1$. In this case

a in tangent direction derivative is $\dfrac{\partial u}{\partial \eta}=\dfrac{1}{4 x y}\left(y \dfrac{\partial u}{\partial x}+x \dfrac{\partial u}{\partial y}\right)$

a transversal derivative is $\dfrac{\partial u}{\partial \varphi}=-\dfrac{1}{4 x y}\left(x \dfrac{\partial u}{\partial y}-y \dfrac{\partial u}{\partial x}\right)$

and

$$\text{a derivative in normal direction is } x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial y}.$$

We note that the outer derivative $\partial u/\partial \varphi$ is not same as a derivative in the direction of the normal. However, by choosing the function η suitably, say $\eta = 2xy$ we can make $\partial u/\partial \varphi$ to be a derivative in the normal direction. In this case the two families of curves $\varphi =$ constant and $\eta =$ constant intersect orthogonally.

Now we pose the Cauchy problem for a single first order partial differential equation in more than two independent variables. The equation is

$$F(x_\alpha, u, p_\alpha) = 0 \tag{5.22}$$

where

$$p_\alpha = u_{x_\alpha}. \tag{5.23}$$

Cauchy Problem: Given $m+1$ arbitrary functions $x_{\alpha 0}(\eta_r)$, $u_0(\eta_r)$ of $m-1$ variables η_r such that the rank of the $(m-1)\times m$ matrix

$$\Delta = \begin{bmatrix} \frac{\partial x_{10}}{\partial \eta_1} \cdots\cdots \frac{\partial x_{m0}}{\partial \eta_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial x_{10}}{\partial \eta_{m-1}} \cdots\cdots \frac{\partial x_{m0}}{\partial \eta_{m-1}} \end{bmatrix} \tag{5.24}$$

is $m-1$, the Cauchy problem for the first order equation (5.22) is to find a solution $u = u(x_\alpha)$ in a domain D containing $(x_{\alpha 0}(\eta_r))$ and satisfying

$$u_0(\eta_r) = u(x_{\alpha 0}(\eta_r)) \tag{5.25}$$

for all values of η_r for which the functions $x_{\alpha 0}$, u_0 are defined.

We give here a geometrical interpretation of the above statement. We note that

$$x_\alpha = x_{\alpha 0}(\eta_r) \tag{5.26}$$

represents an $(m-1)$-dimensional datum manifold γ in (x_α)-space, on which the values of u are prescribed by

$$u = u_0(\eta_r). \tag{5.27}$$

The condition that the matrix Δ in (5.24) has rank $m-1$ for all values of η_r implies that the datum manifold is free from singular points. We denote the datum manifold by $\varphi(x_\alpha) = 0$.

Together, equations (5.26), (5.27) represent an $(m-1)$-dimensional manifold Γ in $m+1$ dimensional (x_α, u)-space and the Cauchy problem is to find an m-dimensional integral manifold (integral hypersurface) in this space passing through the initial manifold Γ.

In the next two sections we shall briefly discuss first order equations in more than two independent variables.

§5.2 Semilinear and Quasilinear Equations in More than Two Independent Variables

We first consider here the semilinear equation of the type

$$a_\alpha \frac{\partial u}{\partial x_\alpha} = c \tag{5.28}$$

where a_α are functions of x_α only and c may depend on u. The left hand side of the equation represents a directional derivative in the direction (a_α) in (x_α)-space. As in § 2.1, the characteristic ordinary differential equations are given by

$$\frac{dx_\alpha}{d\sigma} = a_\alpha. \tag{5.29}$$

Following the discussion of equation (5.1), the solution of (5.29) gives a $(m-1)$-parameter family of characteristic curves and along these curves the variation of u is given by

$$\frac{du}{d\sigma} = c. \tag{5.30}$$

Equation (5.30) is the *compatibility condition* along the characteristic curves. Equations (5.29) and (5.30) together give the *Monge curves* in (x_α, u)-space.

Unlike the case of two independent variables, where every noncharacteristic Cauchy problem has a unique solution, we face here a more complicated case: the characteristic curves defined above are one-dimensional manifolds and the Cauchy data has been prescribed on an $(m-1)$-dimenisional manifold (5.26). To make a statement analogous to that in the case of two independent variables we define the characteristic surface of (5.28) in the following way. The *characteristic surface*, C, is an $m-1$ dimensional manifold in (x_α)-space such that a Cauchy problem in which the data is prescribed on C, does not have a unique solution. Suppose the data is prescribed on the manifold γ whose equation is

$$\varphi(x_\alpha) = 0. \tag{5.31}$$

From the Cauchy data on γ we can determine all inner derivatives. If the solution exists uniquely, we should be able to determine one exterior derivative uniquely with the help of the partial differential equation.

Introducing a new set of independent variables (η_r, φ) instead of (x_α) and substituting in (5.28) we have

$$(a_\alpha \varphi_{x_\alpha}) u_\varphi + a_\alpha \frac{\partial \eta_r}{\partial x_\alpha} u_{\eta_r} = c. \tag{5.32}$$

The derivatives u_{η_r} when evaluated at $\varphi = 0$ are inner derivatives for the datum manifold γ and hence they are known from the prescribed values of u on γ. The exterior derivative u_φ cannot be determined uniquely from (5.32) if on $\varphi = 0$

$$a_\alpha \varphi_{x_\alpha} = 0 \tag{5.33}$$

which is, therefore, the condition that the manifold $\varphi=0$ is a characteristic surface. The equation (5.33) is called the *characteristic equation.* We note following points:

(i) In the case of two independent variables, the characteristic surface coincides with a characteristic curve.

(ii) The characteristic equation (5.33) is a single first order homogenous partial differential equation in m independent variables. The principal part of the operator in the original equation (5.28) is the same as that operating on the function φ in (5.33).

(iii) From (5.29) and (5.33) it follows that the characteristic curves are tangential to the characteristic surface $C:\varphi=0$. The left hand side of (5.28) is now an inner derivative for the surface C and this inner derivative is in the direction of the characteristic curves along which the compatibility condition

$$a_\alpha \frac{\partial \eta_r}{\partial x_\alpha} u_{\eta_r} = c \tag{5.34}$$

must be satisfied by the value of u on C.

We can prove that the charateristic surface C can be generated by an $(m-2)$-parameter family of the characteristic curves and vice-versa (see Problem 5, Exercise 5.1). Following the procedure of § 2.2, we can show that every integral surface S of (5.28) is generated by an m-parameter family of Monge curves and vice-versa.

Now we shall briefly mention here a method of solving a noncharacteristic Cauchy problem. Suppose the datum manifold γ is represented parametrically in the form (5.26) and the values of u are prescribed as in (5.27) where $x_{\alpha 0}$, u_0 are continuously differentiable functions. Solving the ordinary differential equations (5.29) and (5.30) with initial conditions

$$x_\alpha = x_{\alpha 0}(\eta_r), \qquad u=u_0(\eta_r) \text{ at } \sigma=0 \tag{5.35}$$

we get

$$x_\beta = x_\beta(\sigma, \eta_r), \qquad u=u(\sigma, \eta_r). \tag{5.36}$$

If the matrix

$$\begin{bmatrix} \frac{\partial x_1}{\partial \sigma} & \cdots\cdots & \frac{\partial x_m}{\partial \sigma} \\ \frac{\partial x_1}{\partial \eta_1} & \cdots\cdots & \frac{\partial x_m}{\partial \eta_1} \\ \cdot & \cdots\cdots & \cdot \\ \frac{\partial x_1}{\partial \eta_{m-1}} & \cdots\cdots & \frac{\partial x_m}{\partial \eta_{m-1}} \end{bmatrix}_{\sigma=0} = \begin{bmatrix} a_1 & \cdots & \cdots & a_m \\ \frac{\partial x_1}{\partial \eta_1} & \cdots & \cdots & \frac{\partial x_m}{\partial \eta_1} \\ \cdot & \cdots & \cdots & \cdot \\ \frac{\partial x_1}{\partial \eta_{m-1}} & \cdots & \cdots & \frac{\partial x_m}{\partial \eta_{m-1}} \end{bmatrix}_{\sigma=0} \tag{5.37}$$

is nonsingular, we can use (5.36) to solve σ, η_r in terms of x_α and express u as a function of x_α in a neighbourhood of the initial manifold. We can easily show that this gives a unique solution of the Cauchy problem. The non-vanishing of the determinant of the above matrix is a condition that the datum manifold is not a characteristic manifold.

For the discussion of the equation (5.28) when the coefficients are functions of u also, the analysis presented in this section remains unchanged with slight modifications and hence we need not repeat the whole thing again. We only note that the characteristic curves and characteristic surfaces now depend on the solution itself.

Uptil now we have discussed only solutions of the Cauchy problem. For a quasilinear system it is easy to deduce the form of the general solution. A first integral of the system of $(m+1)$ ordinary differential equations (5.29), (5.30) represents an integral surface of the equations (5.28). We know that this system has m independent first integrals. Let us assume that

$$f_\beta(x_\alpha, u) = c_\beta, \quad \beta = 1, 2, \dots, m \tag{5.38}$$

where c_β are the constants, are these m independent relations representing integral surfaces of equation (5.28). Therefore we have

$$a_\alpha \frac{\partial f_\beta}{\partial x_\alpha} + c\frac{\partial f_\beta}{\partial u} = 0. \tag{5.39}$$

Let

$$\Phi(x_\alpha, u) = 0 \tag{5.40}$$

be the equation of any other integral surface, then

$$a_\alpha \frac{\partial \Phi}{\partial x_\alpha} + c\frac{\partial \Phi}{\partial u} = 0. \tag{5.41}$$

Since a_α, c are not all identically zero, from (5.39) and (5.40) we deduce that

$$\frac{\partial(\Phi, f_1, f_2, \cdots, f_m)}{\partial(x_1, x_2, \cdots, x_m, u)} = 0. \tag{5.42}$$

Therefore

$$\Phi = \Phi(f_1, \cdots, f_m) \tag{5.43}$$

where Φ is an arbitrary function of its arguments. This shows that if independent functions $f_1, \dots, f_m$ are known such that (5.38) are equations of integral surfaces, then the general solution of (5.28) is of the form

$$\Phi(f_1, f_2, \cdots, f_m) = 0. \tag{5.44}$$

As discussed in the end of the section 2 for the case of two independent variables, the general solution given by (5.44) can be easily used to solve a Cauchy problem.

*§5.3 Nonlinear First Order Equations

We consider here equations of the form (5.22) where the function F is nonlinear in p_α. The theory of *Monge cones* (an m-dimensional hypercone in (x_α, u)-space) and *Monge strips* can be introduced exactly in the same manner as in the case of two independent variables and this gives the system of Charpit's equations

$$\frac{dx_\alpha}{d\sigma}=F_{p_\alpha},\ (m \text{ equations}) \tag{5.45}$$

$$\frac{du}{d\sigma}=p_\alpha F_{p_\alpha},\ (\text{one equation}) \tag{5.46}$$

and

$$\frac{dp_\alpha}{d\sigma}=-F_{x_\alpha}-p_\alpha F_u,\ (m \text{ equations}). \tag{5.47}$$

Solution of the Charpit's equations satisfy the strip condition with respect to σ, namely

$$\frac{\partial u}{\partial\sigma}=p_\alpha\frac{\partial x_\alpha}{\partial\sigma} \tag{5.48}$$

and if they further satisfy the partial differential equation (5.22), we get a $(2m-1)$-parameter family of Monge strips in (x_α, u)-space. The projection of the curve associated with a Monge strip on (x_α)-space is a *characteristic curve* which can be determined only for a given solution. As in the case of two independent variables we can easily develop the corresponding theorems giving relationship between the Monge strips and an integral surface and also discuss the existence, uniqueness and method of solution of a Cauchy problem when the datum manifold is not a characteristic manifold. We omit these but ask the question 'what are characteristic manifolds or surfaces?'

We take a datum manifold γ: $\phi(x_\alpha)=0$, where u is prescribed as $u_0(\eta_r)$. Consider now a set of values $p_{\alpha 0}(\eta_r)$ of p_α satisfying the strip condition with respect to η_r:

$$\frac{\partial u_0}{\partial\eta_r}=p_{\alpha 0}\frac{\partial x_{\alpha_0}}{\partial\eta_r},\ r=1,2,\ldots,m-1 \tag{5.49}$$

and the differential equation (5.22) on γ. The interior derivatives of u on γ, i.e. $\frac{\partial u_0}{\partial\eta_r}$ are known from the distribution of u on γ. To get the exterior derivative $\frac{\partial u}{\partial\varphi}$ on γ from the above data, we have to introduce new independent variables ϕ, η_r as in § 5.1. Then the partial differential equation on γ becomes:

$$F\left(x_{\alpha 0}(\eta_r),\ u_0(\eta_r),\ (\phi_{x_\alpha})_\gamma\left(\frac{\partial u}{\partial\varphi}\right)_\gamma+\left(\frac{\partial\eta_r}{\partial x_\alpha}\right)_\gamma\frac{\partial u_0}{\partial\eta_r}\right)=0 \tag{5.50}$$

where the subscript γ denotes the value of the quantity on γ and where all quantities except $\left(\frac{\partial u}{\partial\varphi}\right)_\gamma$ are known. Equation (5.50) can be solved for $\left(\frac{\partial u}{\partial\varphi}\right)_\gamma$ uniquely if

$$\phi_{x_\alpha}F_{p_\alpha}\neq 0 \quad \text{on } \gamma. \tag{5.51}$$

If

$$\phi_{x_\alpha}F_{p_\alpha}=0 \quad \text{on } \gamma \tag{5.52}$$

we do not get a unique solution for $\left(\frac{\partial u}{\partial \varphi}\right)_\gamma$ from (5.50) and the manifold γ is called a *characteristic surface* of the partial differential eqution.

We can show that (5.51) is also the condition that all higher order (if they exist) partial derivatives of u can be determined from the equation (5.22) and the strip condition (5.49).

The discussion of the present chapter shows that for first order partial differential equations (it is meaningful to consider only those first order equations which are satisfied for real values of p_α in a domain D_3 of (x_α, u, p_α)-space the characteristic curves are always real and the solution of the Cauchy problem (with smooth Cauchy data) exists and is unique provided the datum manifold is nowhere tangential to the characteristic curves. From the method of construction of the solution of a noncharacteristic Cauchy problem (i.e. solution with the help of Charpit equtions), it is clear that in any finite domain the solution depends continuously on the Cauchy data, i.e. the solution is stable. Equations, for which the solution of a Cauchy problem have these three properties: existence, uniqueness, and stability, are classified as hyperbolic equations. A detailed theory of such equations will be discussed in the section 4 of the Chapter 2 and in the Chapter 3. Here we only note that the first order partial differential equations are the simplest examples of hyperbolic equations. Moreover, we have seen that the problem of finding a solution of a first order equation can be reduced to that of finding the solution of a set of ordinary differential equations, which consists of characteristic equations and the corresponding compatability conditions. ▲

EXERCISE 5.1

1. If $u = u(x_1, x_2, \dots, x_m)$ be any solution of the differential equation

$$x_1 u_{x_1} + x_2 u_{x_2} + \dots + x_m u_{x_m} = \alpha u$$

where α is constant, show that for any constant $k \neq 0$

$$u(kx_1, kx_2, \dots, kx_m) = k^\alpha\, u(x_1, x_2, \dots, x_m).$$

Note: The above equation is called Euler's differential equation for homogeneous functions.

2. Show that the general solution of the partial differential equation

$$(y-z)u_x + (z-x)u_y + (x-y)u_z = x^2y + y^2z + z^2x - xy^2 - yz^2 - zx^2.$$

is
$$u = xyz + f(x+y+z,\ x^2+y^2+z^2)$$

where f is an arbitrary function of its arguments.

3. Show that the genuine solution of the following Cauchy problem:

$$uu_x + uu_y + u_z = 0, \qquad -\infty < x, y < \infty; \qquad 0 < z < \infty$$

$$u(x, y, 0) = \frac{1}{x^2+y^2+1}, \ -\infty < x,\ y < \infty$$

ceases to exist for values of $z > z_0$, where

$$z_0 = \min_{(x,\,y)} \frac{(x^2+y^2+1)^2}{2\,|\,x+y\,|}.$$

Determine z_0 explicitly.

4. Let $\varphi(x_1, x_2, \ldots, x_m) = t$ be a one parameter family of surfaces in (x_α)-space depending on t as the parameter and assume that these surfaces represent the successive positions of a wave front in the propagation of light waves, so that φ satisfies

$$\varphi_{x_1}^2 + \varphi_{x_2}^2 + \ldots + \varphi_{x_m}^2 = \frac{1}{c^2}$$

where c is the velocity of light in the medium. Assuming that c is constant, show that the surface S for a given time t can be obtained from its initial position S_0 by passing through S_0 straight lines which are normal to it and then measuring along these lines a constant distance ct (see also section 7.7 of Chapter 3).

5. For the semilinear partial differential equation

$$a_\alpha u_{x_\alpha} = c$$

show that a characteristic surface $\phi(x_\alpha) = 0$ can be generated by an $(m-2)$-parameter subfamily of characteristic cuvres.

*§6 APPLICATIONS OF THE THEORY OF A SINGLE FIRST ORDER EQUATION

A first order equation directly appears in a large number of practical problems in science and technology, especially in physical problems involving wave propagation [Whitham (1974)]. In this section we shall briefly indicate a few applications. We shall start with the simplest first order equation and interpret the properties of the solution in the language of wave propagation. We shall single out one of the independent variables and denote it by t representing time coordinate and all other independent variables by $x_1, x_2, \ldots, x_m$ spatial coordinates.

*§6.1 An Example: A Wave Equation

Consider the partial differential equation

$$u_t + c_\alpha u_{x_\alpha} = Ku, \quad -\infty < x_\alpha < \infty,\ t > 0,\ \alpha = 1, 2, \ldots, m \tag{6.1}$$

where c_α and K are constants.

Solving the characteristic equations and the compatibility condition, i.e. the ordinary differential equations

$$\frac{dx_\alpha}{dt} = c_\alpha, \quad \frac{du}{dt} = Ku$$

we get

$$x_\alpha - c_\alpha t = \text{constant} = \eta_\alpha, \quad ue^{-Kt} = \text{constant} = u_0.$$

Therefore, the general solution of (6.1) is

$$u = e^{Kt} f(x_1 - c_1 t, x_2 - c_2 t, \cdots, x_m - c_m t) \tag{6.2}$$

where f is an arbitrary function of its arguments. We note that the value of the function f remains unchanged when x_α and t are replaced by $x_\alpha + c_\alpha \tau$, $t + \tau$ showing that the solution (6.2) represents a wave which moves with a constant velocity given by the vector $(c_1, c_2, \cdots, c_m)$ in $(x_1, x_2, \cdots, x_m)$-space. Due to the presence of the factor e^{Kt}, the amplitude of the wave increases or decreases exponentially with time depending on whether $K > 0$ or $K < 0$.

We can define a wave front to be a moving surface in (x_α)-space such that on the wave front the "phase function" f maintains the same constant value. At any time t, the unit normal to the wave front is given by (n_α) where

$$n_\alpha = \frac{f_{x_\alpha}}{\sqrt{f_{x_\beta} f_{x_\beta}}}. \tag{6.3}$$

The 'wave front velocity', i.e. the velocity of displacement of a wave front in its normal direction is

$$n_\alpha c_\alpha = \frac{c_\alpha f_{x_\alpha}}{(f_{x_\alpha} f_{x_\alpha})^{1/2}}. \tag{6.4}$$

The initial manifold $t = 0$ is a noncharacteristic manifold and we can prescribe the initial state of the system described by the function $u(x_\alpha, t)$ by the Cauchy data

$$u(x_\alpha, 0) = u_0(x_\alpha), \quad -\infty < x_\alpha < \infty. \tag{6.5}$$

In this case we can determine the arbitrary function f in equation (6.2) as

$$f = u_0. \tag{6.6}$$

When $K = 0$, equation (6.1) becomes

$$u_t + c_\alpha u_{x_\alpha} = 0 \tag{6.7}$$

which represents a wave motion free from "diffusion", 'dispersion" and deformation. It is to be noted that (6.7) is the simplest "wave equation" in m space-dimensions. This comment is important due to the fact that in almost all discussions of a wave motion in a textbook on partial differential equations, what is treated is a very special equation, the so-called wave equation

$$u_{tt} = c^2 u_{x_\alpha x_\alpha} \tag{6.8}$$

*§6.2 The Hamilton-Jacobi Theory

Our discussion of equation (5.22) or (5.28) has shown that the solution of a partial differential equation of first order can be obtained with the help of the solution of a system of ordinary differential equations, namely the characteristic equations and compatibility conditions. Hamilton and

Jacobi recognised that this process can be reversed for a special class of the system of ordinary differential equations, i.e. their solutions can be obtained with the help of a certain partial differential equation of the first order. This special class consists of $2m$ equations of the form

$$\frac{dx_\alpha}{dt} = H_{p_\alpha}, \quad \frac{dp_\alpha}{dt} = -H_{x_\alpha} \tag{6.9}$$

where

$$H = H(t;\, x_1, \cdots, x_m;\, p_1, \cdots, p_m) \tag{6.10}$$

is a known function of its arguments.

The system (6.9) is referred to as *Hamilton's canonical equations* of a conservative mechanical system with m degrees of freedom (Goldstein (1950)). The motion of the mechanical system is described by *m generalised coordinates* $x_1, x_2, \ldots, x_m$ and the corresponding *m generalised momenta* $p_1, p_2, \cdots, p_m$. The independent variable t is time and the function H is the *Hamiltonian* of the system.

If we look at the Charpit's equations (5.45)-(5.47) little more carefully, we can easily construct a first order partial differential equation for a function $u = u(x_\alpha, t)$ of $m+1$ independent variables $x_1, x_2, \cdots, x_m, t$ such that equations (6.9) are nothing but its Charpit's equations. Defining q and p_α by

$$q = u_t, \qquad p_\alpha = u_{x_\alpha} \tag{6.11}$$

we find that the partial differential equation in question is

$$q + H(t;\, x_1, x_2, \cdots, x_m;\, p_1, p_2, \cdots, p_m) = 0 \tag{6.12}$$

where the dependent variable does not appear explicitly. Equation (6.12) is called the *Hamilton-Jacobi equation*.

The Charpit's equations of (6.12) are

$$\frac{dx_\alpha}{d\sigma} = H_{p_\alpha} \tag{6.13}$$

$$\frac{dt}{d\sigma} = 1, \tag{6.14}$$

$$\frac{du}{d\sigma} = q + p_\alpha H_{p_\alpha} \tag{6.15}$$

$$\frac{dp_\alpha}{d\sigma} = -H_{x_\alpha} \tag{6.16}$$

and

$$\frac{dq}{d\sigma} = -H_t. \tag{6.17}$$

Equation (6.14) shows that in the Charpit equations we can replace the variable σ by t. Then equations (6.13) and (6.16) are nothing but the Hamilton's canonical equations (6.9). Since u and q do not appear in the arguments of the function H, equations (6.13) and (6.16) form a completely determined system of equations and it is not necessary to retain the equations (6.15) and (6.17). In fact once x_α and p_α have been determined from Hamilton's canonical equations, q and u can be determined afterwards from (6.12) and (6.15).

The totality of all Monge strips of (6.12) form a $(2m+1)$-parameter family of strips (see §5.3). Since u does not appear explicitly in the equations the strips depend only on $2m$ parameters. Therefore, the set of all Monge strips give all possible $2m$ parameter solutions of the Hamilton's canonical equations (6.9). In order to show a complete equivalence between the solutions of (6.9) and the Monge strips of (6.12) we only have to show now that every solution of (6.9) leads to a Monge strip. This is simple. We know that the function $q+H$ remains constant along every solution of the Charpit's equations. Therefore, once solutions $x_\alpha = x_\alpha(t)$, $p_\alpha = p_\alpha(t)$ of (6.9) are given we select the initial value (at $t=0$) of q such that $q(0)+H(0, x_\alpha(0), p_\alpha(0))=0$ and solve (6.17) and then solve (6.15) with an arbitrary initial value of u. This gives a $(2m+1)$-parameter Monge strips of the Hamilton-Jacobi equations.

Now we proceed to prove the statement which we made in the beginning of this section, namely "all solutions of the ordinary differential equations (6.9) can be obtained with the help of a partial differential equation". We first note that if u be any solution of the partial differential equation (6.12), then for any constant b, $u+b$ is also a solution of (6.12). Let us take a solution $u=\varphi(x_1, x_2, \dots, x_m; t; a_1, a_2, \dots, a_m)$ of (6.12) depending on m arbitrary parameters $a_1, a_2, \dots, a_m$ such that the determinant

$$\Delta = |\varphi_{x_\alpha a_\beta}| \neq 0. \tag{6.18}$$

Then

$$u=\varphi(x_1, x_2, \dots, x_m; t; a_1, a_2, \dots, a_m)+b \tag{6.19}$$

depending on $m+1$ parameters a_α, b, is a *complete integral* of (6.12). The general solution of (6.12) is obtained by assuming $b=b(a_1, a_2, \dots, a_m)$ as an arbitrary function of a_α and forming the envelope of the m-parameter family of solutions (6.19) thus obtained. This is given by eliminating $a_1, a_2, \dots, a_m$ from (6.19) and the m relations obtained by differentiating (6.19) with respect to a_α

$$\varphi_{a_\alpha} + b_\alpha = 0 \tag{6.20}$$

where $b_\alpha = \dfrac{\partial b}{\partial a_\alpha}(a_1, a_2, \dots, a_m)$.

As we discussed in the end of the section 4.2, if we treat a_α, b and b_α as arbitrary constants, then we can show that the $m+1$ relations (6.19) and (6.20) represent the $(2m+1)$-parameter family of Monge curves of (6.12) in (x_α, t, u)-space. The variation of p_α and q along these curves is given by

$$p_\alpha = \varphi_{x_\alpha}(x_1, x_2, \dots, x_m; t; a_1, a_2, \dots, a_m) \tag{6.21}$$

and (6.12), i.e. for constant values of a_α, b and b_α, the equations (6.12), (6.19), (6.20) and (6.21) give the $(2m+1)$-parameter family of Monge strips of (6.12). We note that u, q and b do not appear in (6.20) and (6.21). Therefore these equations together represent the $2m$-parameter family of solutions of the Hamilton's canonical equations (6.9) (for a more explicit and direct proof of this statement, reference may be made to Courant and Hilbert—*Methods of Mathematical Physics*, Vol. II, § 8.3 Chapter II, pages 108 and 109). Thus we have Hamilton and Jacobi's result that all solutions of (6.9) can be obtained with the help of a complete integral of the partial differential equation (6.12).

*§6.3 Traffic Flow

Let us discuss here an interesting kinematical theory of the movement of heavy traffic on a highway. This example shows how a simple mathematical model can be constructed for a very complex problem. More than this, it also shows how new information of great practical value can be obtained by the use of the simple theory of first order quasilinear equations.

Consider the heavy traffic flow on a very long stretch of a highway without entries or exits. Let x represent the distance along the highway from some fixed point. In such a case, the number of cars (or other vehicles) per unit length (in kilometres) is quite high and we may develop a mathematical model based on continuum hypothesis for the flow of traffic in terms of the density $\rho(x, t)$ of the cars per unit length and the associated flux $q(x, t)$ of cars crossing a position x at a given time t. The total number of cars between two points x_1 and x_2 at a given time t is $\int_{x_2}^{x_1} \rho(x, t)\, dx$. In the stretch of the highway without entries and exists the total number of cars remains constant. The excess of the cars entering into a segment $x_1 < x < x_2$ of the highway from the end at x_1 over the cars leaving the same segment from the end at x_2 in time δt is $\delta t\{q(x_1, t) - q(x_2, t)\}$. From the law of conservation of the number of cars on the highway, difference in the flux multiplied by δt must be equal to the increase in the number of cars in the segment (x_1, x_2) in the time δt. This gives

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x, t)\, dx = q(x_1, t) - q(x_2, t). \tag{6.22}$$

If $\rho(x, t)$ and $q(x, t)$ have continuous partial derivatives, we may let $x_2 \to x_1$ and deduce from (6.22) the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{6.23}$$

This is a single partial differential equation in two quantities ρ and q. In order that the traffic flow is fully determined, we need further either a relation between ρ and q or another differential equation connecting their partial derivatives. To achieve this we put forward the following simplified argument assuming that the highway contains maximum number of cars it can afford without accidents at any given speed. It is a common experience that in such a traffic movement, smaller the free space between two cars, lower the speed with which they move since there must be sufficient time for the driver of the following car to respond to the changes in the speed of the car just ahead. When the space between the two cars is zero, i.e. the cars are bumper to bumper, i.e. when $\rho=\rho_{max}$, the flow velocity must be zero. Therefore, to the first approximation we can assume the flow velocity, i.e. the velocity v of the cars to be a function of the local density, i.e. $v=v(\rho)$ where $v(\rho)$ is a decreasing function of ρ and $v(\rho_{max})=0$. The flux q is also a function of ρ and related to v by

$$v(\rho)=\frac{q(\rho)}{\rho}. \tag{6.24}$$

Further, when the road is free, there is maximum velocity with which the cars can move, i.e. as ρ tends to zero, v attains a maximum value

$$v(0)=v_{max}. \tag{6.25}$$

Figure 6.1 shows the graph of the relation between q and ρ. The function $q(\rho)$ has a maximum value for some value of ρ, say $\rho=\rho_m$.

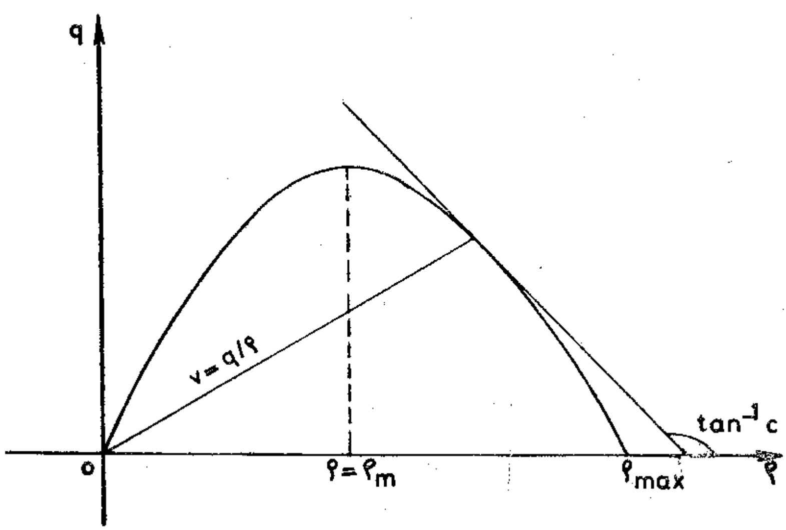

Fig. 6.1 Graph of the function $q(\rho)$.

Substituting the relation (6.24) in (6.23) we get

$$\rho_t+(v+\rho v')\rho_x=0 \tag{6.26}$$

showing that the changes in the density ρ propagates with a velocity c

$$c(\rho)\equiv q'(\rho)=v(\rho)+\rho v'(\rho). \tag{6.27}$$

Any acceleration or deceleration of a car is felt by other drivers and its effect travels with the 'sound speed' c given by (6.27). Since v is a decreasing function of ρ, $v'(\rho)<0$ and hence the velocity of propagation relative to the velocity of the traffic is negative, i.e. the effect is felt by the drivers of the following cars.

From the graph of the $q(\rho)$ it is evident that the maximum traffic flow occurs not at the maximum traffic velocity (where $\rho=0$) but at $\rho=\rho_m$ satisfying $0 < \rho_m < \rho_{\max}$. For $\rho < \rho_m$, the figure shows that $c=q'(\rho) > 0$ and for $\rho > \rho_m$ it is less than zero. Therefore the signals moving with the sound velocity c in the traffic flow, move forward or backward relative to the road according as $\rho < \rho_m$ or $\rho > \rho_m$.

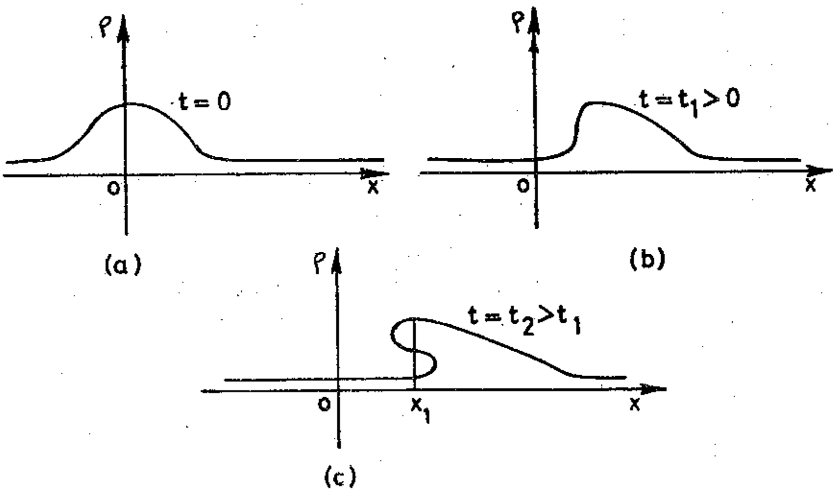

Fig. 6.2. Nonlinear deformation.

Assume that the initial car density in the traffic in a certain portion of the road be as shown in the Fig. 6.2a. If the velocity $c(\rho)$ was independent of ρ, all parts of the traffic would have been moving with a constant velocity and the given distribution of the cars would have propagated without any change in the shape (in this case the governing equation would have been linear). However, we notice that when q depends on ρ as shown in the Fig. 6.1, the signal velocity c is a decreasing function of ρ as ρ increases from 0 to $\rho_{\max}$. In this case, a portion of the pulse (Fig. 6.2a) where ρ is larger propagates with smaller velocity than that where ρ is smaller. The result is that the portion of the pulse where the slope $\partial\rho/\partial x$ is negative becomes less steep and that where the slope $\partial\rho/\partial x$ is positive becomes more steep (as in Fig. 6.2b). This nonlinear deformation of the pulse continues till it folds itself at the back in the profile as shown in Fig. 6.2c. In this case there is a portion of the x-axis where there are three values of ρ, for the same value of x. This leads to a catastrophic situation not acceptable in a normal traffic flow. However, much before this happens the "diffusive effects" such as drivers awareness of the conditions ahead and the time lag in the response of the driver and his car to changes in the flow conditions become important. Since these effects are not taken account of in our model, our equation (6.26) ceases to be valid even before the above unrealistic situation occurs [Whitham (1974), (§3.1)].

CHAPTER 2

Linear Second Order Partial Differential Equations

§1 CLASSIFICATION OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Solutions of partial differential equations of order higher than the first exhibit distinctive types of behaviour, that it is worthwhile to classify these equations according to their different properties and method of construction of solutions. We will start with a brief comment on the simplest case, namely the semilinear second order equation in two independent variables. The discussion of this case will give us the insight to the method of classification of equations in a space of higher dimensions.

§1.1 Linear Equation in Two Independent Variables

We consider an equation of the form

$$a(x, y)\, u_{xx} + 2b(x, y)\, u_{xy} + c(x, y) u_{yy} + d(x, y, u, u_x, u_y) = 0 \tag{1.1}$$

where $a(x, y)$, $b(x, y)$ and $c(x, y)$ are real-valued differentiable functions of x and y, in a domain D. This form of (1.1) is typical of a class of equations referred to as semilinear equations. If

$$d(x, y, u, u_x, u_y) \equiv e(x, y)u_x + f(x, y)u_y + g(x, y)u + h(x, y) \tag{1.2}$$

then (1.1) is said to be a linear equation.

Our object is to transform the differential equation (1.1) into a simple standard or normal form by introducing new independent variables ξ, η through a real transformation:

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y). \tag{1.3}$$

The transformed equation is of the form

$$A(\xi, \eta)u_{\xi\xi} + 2B(\xi, \eta)\, u_{\xi\eta} + C(\xi, \eta)u_{\eta\eta} + D(\xi, \eta, u, u_\xi, u_\eta) = 0 \tag{1.4}$$

where

$$\left.\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\ B &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \\ C &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2. \end{aligned}\right] \tag{1.5}$$

One of the relations satisfied by A, B, C and a, b, c is

$$AC-B^2=(ac-b^2)(\xi_x\eta_y-\xi_y\eta_x)^2. \tag{1.6}$$

If we confine ourselves to transformations that are locally one-to-one, then the Jacobian $J\begin{pmatrix}\xi & \eta\\ x & y\end{pmatrix}\equiv(\xi_x\eta_y-\xi_y\eta_x)$ is different from zero. Equation (1.6) therefore shows that the sign of the expression b^2-ac remains invariant under the transformation. This fact will help us to classify second order semilinear equations based on the sign of b^2-ac.

With equation (1.1), we associate a polynomial—'a characteristic quadratic form'—defined by

$$Q(l, m)=al^2+2blm+cm^2. \tag{1.7}$$

If we substitute

$$l=\lambda\xi_x+\mu\eta_x, \quad m=\lambda\xi_y+\mu\eta_y \tag{1.8}$$

in (1.7), we obtain

$$Q(l, m)=A\lambda^2+2B\lambda\mu+C\mu^2 \tag{1.9}$$

which is precisely the polynomial associated with the transformed equation (1.4). Since two real functions $\xi(x, y)$ and $\eta(x, y)$ are at our disposal, we choose two conditions on the coefficients of the transformed equation aiming at simple normal forms of the transformed equation (1.4)*. This leads to exactly three different cases as will be shown under the heading "method of reduction to normal form":

Case I :

$$\left.\begin{aligned} &Q(l, m)=A(\lambda^2+\mu^2)\\ \text{i.e.}\quad &A=C,\ B=0\end{aligned}\right\} \tag{1.10}$$

Case II :

$$\left.\begin{aligned} &Q(l, m)=A(\lambda^2-\mu^2)\\ \text{i.e.}\quad &A=-C,\ B=0\end{aligned}\right\} \tag{1.11}$$

or equivalently

$$\left.\begin{aligned} &Q(l, m)=B\lambda_1\mu_1\\ &\text{i.e. } \lambda_1=\lambda-\mu,\ \mu_1=\lambda+\mu,\ A=C=0\end{aligned}\right\} \tag{1.12}$$

Case III :

$$\left.\begin{aligned} &Q(l, m)=A\lambda^2\\ \text{i.e.}\quad &B=C=0.\end{aligned}\right\} \tag{1.13}$$

Case I corresponds to the case where the discriminant of the quadratic form (1.7), i.e. $ac-b^2>0$. This arises when the associated real symmetric matrix $M=\begin{pmatrix}a & b\\ b & c\end{pmatrix}$ has non-zero characteristic roots of the same sign. All equations (1.1) with $ac-b^2>0$, can be reduced to (1.4) with $A=C$,

*This can be regarded as a transformation of $Q(l, m)$ to principal axes. By the Principal Axes Theorem (Finkbeiner, 1970), any real quadratic form can be reduced to the algebraic sum of squares by an appropriate linear transformation.

$B=0$. Geometrically, in the (l, m)-plane for fixed (x, y) the quadratic curve $Q(l, m)=1$ is an ellipse. We call equation (1.1) in Case I an *elliptic* equation. Its normal form is

$$u_{\xi\xi}+u_{\eta\eta}+D_1(\xi, \eta, u, u_\xi, u_\eta)=0 \tag{1.14}$$

Case II arises when $Q(l, m)$ has the algebraic property that its discriminant $ac-b^2 < 0$. This corresponds to the case when the matrix M has non-zero characteristic roots of different signs.

Here $Q(l, m)=1$ represents a hyperbola in the l, m plane. We call equation (1.1) in Case II as a *hyperbolic* equation. Its normal form is

$$u_{\xi\xi}-u_{\eta\eta}+D_1(\xi, \eta, u, u_\xi, u_\eta)=0 \tag{1.15}$$

or, equivalently,

$$u_{\bar{\xi}\bar{\eta}}+\bar{D}_2(\bar{\xi}, \bar{\eta}, u, u_{\bar{\xi}}, u_{\bar{\eta}})=0 \tag{1.16}$$

where

$$\bar{\xi}=(\xi+\eta)/2, \qquad \bar{\eta}=(\xi-\eta)/2.$$

Case III arises when the discriminant $ac-b^2$ of $Q(l, m)$ is zero. In this case the matrix M has one zero characteristic root. $Q(l, m)=1$ represents two parallel lines in the l, m plane. We call equation (1.1) in Case III as a *parabolic* equation. Its normal form is

$$u_{\xi\xi}+D_3(\xi, \eta, u, u_\xi, u_\eta)=0. \tag{1.17}$$

Method of reduction to normal form

We assign to the quadratic form (1.7), the partial differential equation of the first order given by

$$Q(\varphi_x, \varphi_y) \equiv a\varphi_x\varphi_x+2b\varphi_x\varphi_y+c\varphi_y\varphi_y=0.* \tag{1.18}$$

In Case I this has no real solution for $(-\varphi_x/\varphi_y)$ as $ac-b^2 > 0$. Solving for $(-\varphi_x/\varphi_y)$, we get two complex conjugate values

$$-\frac{\varphi_x}{\varphi_y}=\rho(x, y)\pm i\delta\,(x, y). \tag{1.19}$$

If we were to choose $\xi(x, y)$ and $\eta(x, y)$ such that

$$-\frac{(\xi\pm i\eta)_x}{(\xi\pm i\eta)_y}=\rho\pm i\delta \tag{1.20}$$

then we would have

$$a\xi_x^2+2b\xi_x\xi_y+c\xi_y^2=a\eta_x^2+2b\eta_x\eta_y+c\eta_y^2 \tag{1.21}$$

and

$$a\xi_x\eta_x+b(\eta_x\xi_y+\xi_x\eta_y)+c\xi_y\eta_y=0 \tag{1.22}$$

which are precisely, from (1.5),

$$A=C, \qquad B=0. \tag{1.23}$$

*This equation is referred to as the characteristic partial differential equation of equation (1.1).

The transformation (1.3) which will reduce (1.1) to normal form (1.14) is obtained from the relation (1.20).

To find the expressions for ξ and η, set

$$\xi + i\eta = \sigma,\ \xi - i\eta = \tau \tag{1.24}$$

then σ, τ satisfy the equations

$$-\frac{\sigma_x}{\sigma_y} = \left.\frac{dy}{dx}\right|_{\sigma=\text{const.}} = \rho + i\delta,\ -\frac{\tau_x}{\tau_y} = \left.\frac{dy}{dx}\right|_{\tau=\text{const.}} = \rho - i\delta. \tag{1.25}$$

The notation $\left.\frac{dy}{dx}\right|_{\sigma=\text{const.}}$ or $\left.\frac{dy}{dx}\right|_{\sigma}$ represents the slope of the curve $\sigma(x, y)=$ constant at a point on it. σ and τ can be determined by solving the first order partial differential equations (1.25). We get $\sigma = r(x, y)$, $\tau = s(x, y)$, where $r(x, y) =$ constant and $s(x, y) =$ constant are obtained by integrating the ordinary differential equations $dy/dx = \rho \pm i\delta$, respectively. ξ and η are obtained from the relation (1.24). In Case II, equation (1.18) has distinct real solutions, ρ_1 and ρ_2 for $-\varphi_x/\varphi_y$. If we were to choose ξ and η such that

$$-\frac{\xi_x}{\xi_y} = \left.\frac{dy}{dx}\right|_{\xi=\text{const.}} = \rho_1, -\frac{\eta_x}{\eta_y} = \left.\frac{dy}{dx}\right|_{\eta=\text{const.}} = \rho_2 \tag{1.26}$$

then we would get from (1.5)

$$A = C = 0$$

which would reduce (1.1) to the normal form (1.16). The equivalent normal form (1.15) can then be easily obtained. Differential equations (1.26) help to determine the transformation (1.3).

In case III, equation (1.8) has coincident real roots for $(-\varphi_x/\varphi_y)$. If we were to choose η such that

$$-\frac{\eta_x}{\eta_y} = \left.\frac{dy}{dx}\right|_{\eta} = \rho_1 \tag{1.27}$$

then we would get

$$B = C = 0$$

which would reduce (1.1) to the normal form (1.17). ξ can be chosen arbitrarily, such that ξ and η (given by equation (1.27)) are functionally independent, i.e. $\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$.

Since the coefficients a, b and c are functions of (x, y) it may so happen that in some regions $ac - b^2 > 0$ while in others it is less than or equal to zero. Such equations are called "mixed type" of equations and occur, for example, in transonic flows. In the case of quasilinear equations where a, b and c are functions of (x, y, u, u_x, u_y) one must substitute a given solution $u = u(x, y)$ for u before one can classify the equation. Therefore the same equation may be hyperbolic for one solution in a certain domain, but elliptic or parabolic for another solution in the same domain.

Example 1.1 The Tricomi Equation. Reduce to normal form the equation

$$u_{xx}+xu_{yy}=0 \text{ for all } x,\ y. \tag{1.28}$$

In this equation

$$ac-b^2=x$$

and therefore the equation is of mixed type, hyperbolic for $x<0$ and elliptic for $x>0$. In the half plane $x<0$, we can determine ξ and η by the relation

$$\left.\frac{dy}{dx}\right|_{\xi}=\sqrt{-x}\,,\quad \left.\frac{dy}{dx}\right|_{\eta}=-\sqrt{-x}$$

Therefore $\xi(x,y)\equiv\dfrac{3}{2}\,y+(\sqrt{-x})^3,\ \eta(x,y)\equiv\dfrac{3}{2}y-(\sqrt{-x})^3.$ (1.29)

The normal form of the equation for $x<0$ is

$$u_{\xi\eta}-\frac{1}{6(\xi-\eta)}(u_{\xi}-u_{\eta})=0 \tag{1.30}$$

In the half plane $x>0$ (see equation (1.25)),

$$\left.\frac{dy}{dx}\right|_{\sigma}=i\sqrt{x}\,,\quad \left.\frac{dy}{dx}\right|_{\tau}=-i\sqrt{x}$$

$$\sigma(x,y)=\frac{3}{2}\,y-i(\sqrt{x}\,)^3,\quad \tau(x,y)=\frac{3}{2}\,y+i(\sqrt{x}\,)^3.$$

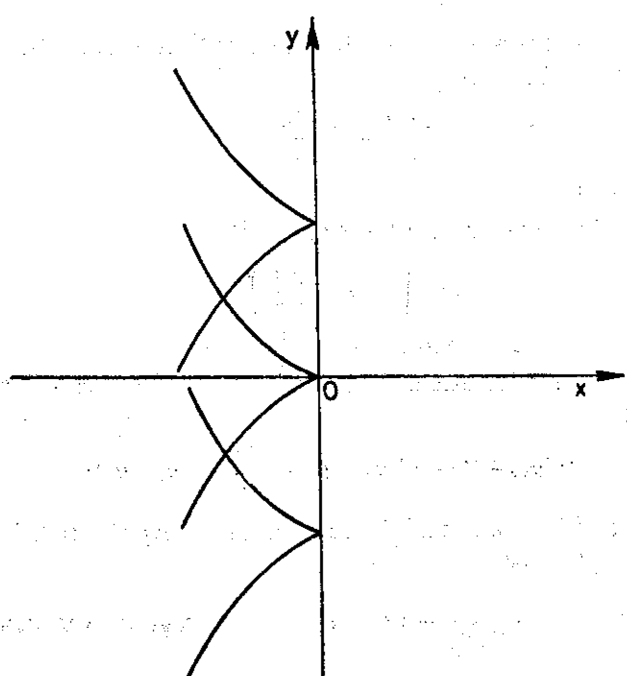

Fig. 1.1 The characteristic curves for $x<0$ are cubic parabolas.

From equation (1.24), we then have

$$\xi(x, y) = \frac{3}{2} y, \qquad \eta(x, y) = -(\sqrt{x})^3 \tag{1.31}$$

The normal form of the equation for $x > 0$ is

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta} u_\eta = 0 \tag{1.32}$$

EXERCISE 1.1

1. Find the type of the following differential equations, and reduce them to normal form:

   (i) $u_{xx} - y^2 u_{yy} + u_x - u + x^2 = 0$
   (ii) $(1 - x^2) u_{xx} - u_{yy} = 0$
   (iii) $y^2 u_{xx} + 2y u_{xy} + u_{yy} - u_x = 0$
   (iv) $y^2 u_{xx} + u_{yy} = 0$

2. Transform the following equation to normal form

$$u_{xy} + y u_{yy} + \sin (x + y) = 0$$

   and obtain its general solution in the form

$$u = e^x \int_0^x e^{-\alpha} \cos (\alpha + y e^{\alpha - x})\, d\alpha + e^x F(y e^{-x}) + G(x)$$

   where F and G are arbitrary functions of their arguments.

3. Transform the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} = y^2 \frac{\partial^2 u}{\partial y^2}$$

   to normal form.
   Hence show that the general solution is

$$u = f(xy) + x g\left(\frac{y}{x}\right)$$

   where f and g are arbitrary functions.

4. Find the transformations $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ which will reduce the equation

$$4y^2 u_{xx} + 2(1 - y^2) u_{xy} - u_{yy} - \frac{2y}{1 + y^2}(2u_x - u_y) = 0$$

   to normal form and find the solution satisfying the initial condition $u(x, 0) = f(x)$, $u_y(x, 0) = F(x)$.

§1.2 Linear Equation in More than Two Independent Variables

In the case of more than two independent variables, it is usually not possible to reduce a linear partial differential equation of the second order to a simple canonical form throughout a region. Only in the case of equations with constant coefficients or in the neighbourhood of a given point, can a suitable canonical form be achieved. Consider the equation (a repeated index denoting summation over the range 1 to m)

$$a_{\alpha\beta}u_{x_\alpha x_\beta} + b_\alpha u_{x_\alpha} + cu = d;\ \alpha, \beta = 1, 2, \ldots, m \tag{1.33}$$

where $a_{\alpha\beta}$, b_α, c and d are functions of the independent variables $x_1, x_2, \ldots, x_m$ and $a_{\alpha\beta} = a_{\beta\alpha}$.

The associated real symmetric matrix in this case is $M = (a_{\alpha\beta})$. As in section 1.1, consider a one-to-one transformation

$$\xi_\alpha = \xi_\alpha(x_1, x_2, \ldots, x_m),\ \alpha = 1, 2, \ldots, m. \tag{1.34}$$

Equation (1.33) then transforms to

$$A_{\gamma\delta}u_{\xi_\gamma\xi_\delta} + D(\xi_1, \xi_2, \ldots, \xi_m; u; u_{\xi_1}, \ldots, u_{\xi_m}) = 0 \tag{1.35}$$

$$\gamma, \delta = 1, 2, \ldots m,$$

where $A_{\gamma\delta} = a_{\alpha\beta}(\xi_\gamma)_{x_\alpha}(\xi_\delta)_{x_\beta}$.

As we have only m functions ξ_α at our disposal, we cannot, in general, get rid of all $m(m-1)/2$ mixed derivatives by setting $A_{\gamma\delta} = 0$, $\gamma \neq \delta$, as it would lead to an over-determined system of equations for ξ_α's.

At a given point, we are able to reduce equation (1.33) to a canonical form by treating the coefficients $a_{\alpha\beta}$ to be constants. Without loss of generality, we can choose this point as the origin, and let

$$\xi_\alpha = f_{\alpha\beta}x_\beta;\ \alpha, \beta = 1, 2, \ldots, m \tag{1.36}$$

where $f_{\alpha\beta}$'s are constants, then the transformed equation (1.35) has coefficients

$$A_{\gamma\delta} = a_{\alpha\beta}f_{\gamma\alpha}f_{\delta\beta};\ \gamma, \delta = 1, 2, \ldots, m \tag{1.37}$$

as constants. The same would hold if equations (1.33) had constant coefficients and we restrict ourselves to linear transformations (1.34). The characteristic quadratic form $Q(\lambda)$ associated with equation (1.33) is

$$Q(\lambda) = a_{\alpha\beta}\lambda_\alpha\lambda_\beta = A_{\alpha\beta}\mu_\alpha\mu_\beta \tag{1.38}$$

where λ's and μ's are related as follows:

$$\lambda_\alpha = \mu_\beta(\xi_\beta)_{x_\alpha} = \mu_\beta f_{\beta\alpha};\ \alpha, \beta = 1, 2, \ldots, m. \tag{1.39}$$

Our aim is to transform $Q(\lambda)$ to principal axes (so that $A_{\alpha\beta} = 0$ for $\alpha \neq \beta$) so as to obtain the normal form of the equation (1.33). According to the Principal Axes Theorem of linear algebra, any real symmetric matrix M is simultaneously similar to and congruent to a diagonal matrix D [Finkbeiner, 1970]. That is, there exists an orthogonal matrix P, such that

$$D = PMP^{-1}$$

is diagonal with special diagonal values. This means, by suitably choosing the constants $f_{\alpha\beta}$ and scaling ξ_α's, we have

$$A_{\alpha\beta}=0 \text{ for } \alpha \neq \beta$$

$$A_{\alpha\alpha}=\left.\begin{array}{r} +1 \\ -1 \\ 0 \end{array}\right\} \begin{array}{l} \alpha=1, 2, \ldots, p \\ \alpha=p+1, \ldots, r \quad \text{(no summation on } \alpha) \\ \alpha=r+1, \ldots, m \end{array} \tag{1.40}$$

depending on the fact that A has p positive eigen values, $(r-p)$ negative eigenvalues and $(m-r)$ zero eigenvalues.

Case I. If all the eigenvalues of A are non-zero and of the same sign, the equation is of elliptic type with the normal form

$$u_{\xi_1\xi_1}+u_{\xi_2\xi_2}+\cdots+u_{\xi_m\xi_m}+D(\xi_1, \xi_2, \ldots, \xi_m; u; u_{\xi_1}, \ldots, u_{\xi_m})=0. \tag{1.41}$$

The quadratic form Q is then either positive definite or negative definite.

Case II. If all the eigen values are non-zero and have the same sign, except precisely one of them, the equation is of normal hyperbolic type with the normal form

$$u_{\xi_1\xi_1}-u_{\xi_2\xi_2}-u_{\xi_3\xi_3}\cdots-u_{\xi_m\xi_m}+D(\xi_1, \ldots, \xi_m, u, u_{\xi_1}, \ldots, u_{\xi_m})=0. \tag{1.42}$$

Case III. If all the eigenvalues are non-zero and at least two eigenvalues are present with positive sign and at least two with negative sign, then the equation is of ultra hyperbolic type. This situation can only occur when $n \geqslant 4$, the simplest case being the equation

$$u_{\xi_1\xi_1}+u_{\xi_2\xi_2}=u_{\xi_3\xi_3}+u_{\xi_4\xi_4} \tag{1.43}$$

in four independent variables.

Case IV. If any of the eigenvalues is zero, the equation is of parabolic type. The heat or diffusion equation

$$u_t=u_{\xi_1\xi_1}+u_{\xi_2\xi_2}+\cdots+u_{\xi_{m-1}\xi_{m-1}} \tag{1.44}$$

in $m-1$ space variables and one time variable is the best known parabolic equation

Method of reduction to normal form

As in section 1.1, we assign to the quadratic form $Q(\lambda)$, the first order partial differential equation

$$Q_1(\varphi) \equiv a_{\alpha\beta}\varphi_{x_\alpha}\varphi_{x_\beta}=0. \tag{1.45}$$

This is the characteristic equation of equation (1.33). The importance of this equation will be seen in the solution of initial value problems in section 1.3.

In general, given equation (1.33) it is not necessary to find the orthogonal matrix P or the eigenvalues of M before classifying the equation. It is sufficient to express the quadratic form (1.38) as a sum of squares, which is merely the process of 'completing the square'. This is equivalent to a non-singular (not necessarily orthogonal) linear transformation of the variables. By Sylvester's law of inertia (Birkhoff and MacLane, 1965), the number of positive and negative squares does not depend on the particular nonsingular linear transformation used and is an invariant of the associated matrix. Thus the classification can be done depending on the number of positive and negative squares in the transformed quadratic form, as it will be the same as the number of positive and negative eigenvalues of the associated matrix.

First we consider an example and then explain the procedure.

Example 1.2

Classify the equation

$$u_{xx}+3u_{yy}+84u_{zz}+28u_{yz}+16u_{zx}+2u_{xy}=0. \tag{1.46}$$

The associated matrix is

$$M=\begin{bmatrix} 1 & 1 & 8 \\ 1 & 3 & 14 \\ 8 & 14 & 84 \end{bmatrix}$$

It is extremely complicated to find the eigenvalues of M. On the other hand complete the squares for $Q(\lambda)$ in λ_1, λ_2 and λ_3 in turn:

$$\begin{aligned} Q(\lambda) &= \lambda_1^2+3\lambda_2^2+84\lambda_3^2+2\lambda_1\lambda_2+16\lambda_1\lambda_3+28\lambda_2\lambda_3 \\ &= (\lambda_1+\lambda_2+8\lambda_3)^2+2(\lambda_2+3\lambda_3)^2+2\lambda_3^2 \\ &= (\lambda_1+\lambda_2+8\lambda_3)^2+[\sqrt{2}(\lambda_2+3\lambda_3)]^2+(\sqrt{2}\,\lambda_3)^2. \end{aligned} \tag{1.47}$$

The equation is elliptic as $Q(\lambda)$ can be expressed as the sum of three squares. By means of the transformation (see (1.54))

$$\xi_1=x,\ \xi_2=\frac{1}{\sqrt{2}}(y-x),\ \xi_3=\frac{1}{\sqrt{2}}(z-5x-3y) \tag{1.48}$$

the given equation transforms to

$$u_{\xi_1\xi_1}+u_{\xi_2\xi_2}+u_{\xi_3\xi_3}=0. \tag{1.49}$$

We can derive a working rule for using the "completing the square of the quadratic form" to determine the normal form of the linear second order equation (1.33) and the transformation (1.34) leading to it.

Consider the quadratic form associated with the matrix M, namely X^TMX, where superscript T stands for the transpose. Completing the square requires finding a variable Z such that

$$X^TMX=Z^TDZ \tag{1.50}$$

where D is a diagonal matrix with elements $+1$, -1, 0. When equation (1.33) is not parabolic, the m elements of the vector Z are determinable as linear combinations of $x_1, x_2, \cdots, x_m$ and we can find a nonsingular matrix Q such that

$$Z=QX \tag{1.51}$$

This is so because the linear combinations in each square constituting the elements of Z form a linearly independent set. In the parabolic case not all the m elements of Z are determinable, but still Q can be completed suitably so as to be nonsingular.

We have from (1.50)

$$M = Q^T D Q$$

or

$$D = (Q^{-1})^T M Q^{-1}. \qquad (1.52)$$

Consider the transformation $Y = PX$ of the independent variable. By this transformation, equation (1.33) is transformed to equation (1.35) with associated matrix N such that

$$N = PMP^T. \qquad (1.53)$$

Choose $P = (Q^{-1})^T$; then $N = D$ and we have our required normal form. The transformation which reduces (1.33) to the normal form is

$$Y = (Q^{-1})^T X. \qquad (1.54)$$

In the previous example

$$Z = \begin{bmatrix} x_1 + x_2 + 8x_3 \\ \sqrt{2}(x_2 + 3x_3) \\ \sqrt{2}\, x_3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 & 8 \\ 0 & \sqrt{2} & 3\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \quad (Q^{-1})^T = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{5}{\sqrt{2}} & -\frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

EXERCISE 1.2

1. Reduce the following to normal form:
   (i) $4u_{xy} + 2u_{yz} + 2u_{zx} = 0$
   (ii) $u_{xx} + 2u_{yy} + 3u_{zz} + 4u_{tt} + 2u_{xy} + 2u_{zx}$
   $+ 2u_{xt} + 4u_{yz} + 4u_{yt} + 6u_{zt} = 0$
2. What are the possible normal forms for a second order partial differential equation in three independent variables?
3. Classify the following equation by finding the eigenvalues of the associated matrix:

   $$3u_{xx} + 3u_{zz} + 4u_{xy} + 8u_{zx} + 4u_{yz} = 0.$$

   Verify by completing the squares of the associated form.

§1.3 The Cauchy Problem

We start with the general quasilinear second order equation for a function $u(x, y)$ of two independent variables:

$$au_{xx}+2bu_{xy}+cu_{yy}=d \tag{1.55}$$

where a, b, c, d depend on x, y, u, u_x, u_y. The Cauchy problem consists in finding a solution of (1.55) with given values of u and its normal derivative on a curve C in the (x, y) plane.

Let the parametric representation of C be: $x=x_0(s), y=y_0(s), s \in I$, where I is an interval on the real line. We are given two functions $u_0(s)$ and $u_1(s)$, $s \in I$. The Cauchy problem consists in finding a solution $u(x, y)$ of (1.55) which satisfies the following conditions:

$$\left.\begin{aligned} &u(x_0(s), y_0(s))=u_0(s),\ s \in I \\ \text{and}\quad &\frac{\partial u}{\partial \nu}(x_0(s), y_0(s))=u_1(s),\ s \in I \end{aligned}\right\} \tag{1.56}$$

where $\frac{\partial}{\partial \nu}$ denotes a normal derivative to C.

For discussion of the Cauchy problem here, we assume that a, b, c and d are analytic functions, regular in some domain D. Our aim is to examine whether there exists a unique *analytic* solution of (1.55), which takes given values on C. To do so, we formally construct a solution using a Taylor's series expansion about any point of C. The first step in such a solution is to show that the partial derivatives of u of all orders are uniquely determined at every point of C. Let suffix 0 denote the values of partial derivatives of u at point of C. i.e. $u_x(x_0(s), y_0(s))=u_{x0}(s)$ and so on. Then $u_{x0}(s)$ and $u_{y0}(s)$ satisfy the following linear equations:

$$\begin{aligned} &x_0' u_{x0}(s)+y_0' u_{y0}(s)=u_0'(s) \\ \text{and}\quad &-y_0' u_{x0}(s)+x_0' u_{y0}(s)=\sqrt{x_0'^2+y_0'^2}\, u_1(s) \end{aligned} \tag{1.57}$$

where a prime ($'$) denotes differentiation with respect to s. Except at points where x_0' and y_0' vanish simultaneously u_{x0} and u_{y0} can be determined uniquely.

Regarding second order derivatives, namely, $u_{xx0}(s)$, $u_{xy0}(s)$ and $u_{yy0}(s)$, they can be determined as solutions of the linear equations:

$$\left.\begin{aligned} &au_{xx0}(s)+2bu_{xy0}(s)+cu_{yy0}(s)=d \\ &x_0'(s)u_{xx0}(s)+y_0'(s)u_{xy0}(s) = \{u_{x0}(s)\}' \\ &x_0'(s)u_{xy0}(s)+y_0'(s)u_{yy0}(s) = \{u_{y0}(s)\}'. \end{aligned}\right\} \tag{1.58}$$

These equations determine $u_{xx0}(s)$, $u_{xy0}(s)$ and $u_{yy0}(s)$ uniquely provided the determinant of the coefficient matrix is nonzero. This requires that

$$\begin{aligned} &ay_0'^2-2bx_0'y_0'+cx_0'^2 \neq 0, \\ \text{or}\quad &Q(-y_0', x_0') \neq 0, \end{aligned} \tag{1.59}$$

where Q is the characteristic quadratic form described in (1.7). Further we can show that the derivatives of u of all orders can be uniquely determined at points of C, provided

$$Q(-y_0', x_0') \neq 0.$$

In this way we can formally develop a unique Taylor's series expansion solution in the neighbourhood of any point of C, satisfying the given conditions on C. The difficulty is to show that such an expansion is convergent in some region around C. The Cauchy-Kowalewski method (see Garabedian, 1964) provides a majorant series ensuring convergence.

On the other hand if $Q(-y_0', x_0')=0$, then the partial derivatives of u on the curve C cannot be determined uniquely. The exceptional curves C, on which if u and its normal derivative are prescribed, no unique solution of (1.55) can be found satisfying these conditions, are called characteristic curves. These curves satisfy the homogeneous equation

$$Q(-y_0', x_0')=0.$$

If the curve C: $x=x_0(s)$, $y=y_0(s)$ in the (x, y) plane, is given by the equation

$$\varphi(x, y)=\text{constant}$$

(by eliminating s), then φ satisfies the partial differential equation

$$Q(\varphi_x, \varphi_y)=0 \text{ on } \varphi(x, y)=\text{constant}; \tag{1.60}$$

since

$$-\frac{y_0'}{x_0'}=\frac{\varphi_x}{\varphi_y}\bigg| = -\frac{dy}{dx}\bigg|_C.$$

From the results of section (1.1), it follows that there are two distinct families of characteristic curves satisfying (1.60), if the equation is hyperbolic. They are precisely $\xi(x, y)=\text{constant}$ and $\eta(x, y)=\text{constant}$. ξ and η are referred to as characteristic variables or coordinates.

No real characteristic curves are found, when the equation is elliptic, and one family of characteristic curves exist when the equation is parabolic.

For a hyperbolic equation in its normal form, namely

$$u_{\xi\eta}+D(\xi, \eta, u, u_\xi, u_\eta)=0 \tag{1.61}$$

$\xi=\text{constant}$ and $\eta=\text{constant}$ are the characteristic curves. If, for example in the Cauchy problem u and u_ξ are prescribed on a characteristic curve C: $\xi=\text{constant}$, then we cannot determine $u_{\xi\xi}$ uniquely on $\xi=\text{constant}$ from the given equation (since the coefficient of $u_{\xi\xi}$ is zero in the linear second order equation (1.61)). Since u and u_ξ are prescribed on $\xi=\text{constant}$ as $u_0(\eta)$ and $u_1(\eta)$, say, respectively, $u_{\xi\eta}$ and u_η can be computed on $\xi=\text{constant}$ and the equation (1.61) will reduce to the compatibility condition

$$u_1'(\eta)+D(\xi, \eta, u_0, u_1, u_0')=0$$

on $\xi=\text{constant}$. Compatibility conditions to be satisfied on characteristic curves are typical, as the equation gives no additional information in this case (like the value of $u_{\xi\xi}$ in (1.61)), but merely insists on a relation between already known quantities. If the compatibility condition is satisfied there will be an infinity of solutions of the Cauchy problem (choosing $u_{\xi\xi}$ arbitrarily in (1.61)), or else there will be no solution. The above discussion holds for data prescribed on $\eta=$ constant as well. For a hyperbolic equation, we have two compatibility conditions, one each on the characteristic curves $\xi=$ constant and $\eta=$ constant. For a parabolic equation, we have one compatibility condition on the single family of characteristic curves.

In the canonical elliptic form, $u_{\xi\xi}$ and $u_{\eta\eta}$ can always be determined whenever u and its normal derivative are prescribed on any curve in the (x, y)-plane, since $Q(\varphi_x, \varphi_y) \neq 0$ on any real curve $\varphi(x, y)=$ constant. We can always find a unique solution for the Cauchy problem in this case.

In the case of m independent variables, those surfaces $\varphi(x_1, x_2, \cdots, x_m)=0$ on which, when the function and its normal derivative are prescribed, no unique solution exists satisfying the prescribed conditions, are called characteristic surfaces. Following a similar process, as in the case of two independent variables, it follows that φ satisfies the equation (1.45), namely

$$Q_1(\varphi) \equiv a_{\alpha\beta}\, \varphi_{x_\alpha} \varphi_{x_\beta} = 0 \text{ on } \varphi=0. \tag{1.62}$$

The characteristic condition $Q_1(\varphi)=0$ is required to be satisfied on $\varphi=0$ but this does not require that φ satisfies the equation $Q_1(\varphi)=0$ identically. What then is the relation between a solution of the partial differential equation (1.62) and a characteristic surface of (1.33)? This has been discussed in Chapter 3, §6.

EXERCISE 1.3

1. Let $u(x, y)$ satisfy the equation

$$u_{xx}-2u_{xy}+u_{yy}+3u_x-u+1=0$$

in a region of the (x, y) plane. Classify the equation and find its characteristics. Construct a solution, if it exists, for each of the following Cauchy data:

(i) $u=2$, $u_y=0$ on the line $y=0$.

(ii) $u=2$, $u_x=0$ on the line $x+y=0$.

§1.4 Propagation of Discontinuities

The characteristic curves are closely associated with the propagation of singularities of certain types. Let u be a function of class C^1 in a certain domain D of the (x, y)-plane. Let C, given in parametric form by $x=x_0(s)$, $y=y_0(s)$, be a curve which divides this domain D into two domains I and II, such that u^{I} and u^{II}, the values of u in domains I and II, respectively, are of class C^2 in these domains. In each of these domains, let u (denoted by u^{I} and u^{II}, respectively) satisfy the equation:

$$a(x, y)u_{xx}+2b(x, y)u_{xy}+c(x, y)u_{yy}+d(x, y, u, u_x, u_y)=0 \tag{1.63}$$

The second derivatives of u have a jump discontinuity across C. Such a solution can be considered to be a generalised solution of equation (1.63). Let

$$[g]=g^{\text{II}}(s)-g^{\text{I}}(s) \text{ on } C \tag{1.64}$$

denote the jump discontinuity in the quantity $g(x, y)$ across the curve C. Using (1.58) for the function $u(x, y)$ along neighbouring curves to C on the left and right, we obtain in the limit the following jump relations*:

$$\left.\begin{aligned} a[u_{xx}]+2b[u_{xy}]+c[u_{yy}] &= 0 \\ x_0'[u_{xx}]+y_0'[u_{xy}] &= 0 \\ x_0'[u_{xy}]+y_0'[u_{yy}] &= 0. \end{aligned}\right\} \tag{1.65}$$

For a non-zero solution of $[u_{xx}]$, $[u_{xy}]$, $[u_{yy}]$ from the above system, we require:

$$ay_0'^2-2bx_0'y_0'+cx_0'^2=0 \tag{1.66}$$

i.e. $$Q(-y_0', x_0')=0.$$

C, the curve of discontinuity of second order derivatives of u, must be a characteristic curve of equation (1.63). For a C^1 function, the jumps in the second derivatives are not independent as seen from equations (1.65). Knowing the jump of u_{xx}, we uniquely get those of u_{xy}, u_{yy}. Characteristic curves are the carriers of possible discontinuities in the solution.

Let C be a characteristic curve given by

$$\xi(x, y)=\text{constant}.$$

Then across C, $[u_{\xi\eta}]=\dfrac{\partial}{\partial\eta}[u_\xi]=0$, $[u_{\eta\eta}]=\dfrac{\partial}{\partial\eta}[u_\eta]=0$ but $[u_{\xi\xi}]\neq 0$, where η is a function independent of ξ.

Changing from x, y to ξ, η coordinates, as in §1.1, the first of equation (1.65) will take the form

$$Q(\xi_x, \xi_y)[u_{\xi\xi}]=0.$$

Since $[u_{\xi\xi}]$ is assumed to be non-zero, from (1.5),

$$Q(\xi_x, \xi_y)\equiv A(\xi, \eta)=0 \text{ along } C.$$

If the transformed equation (1.4) is differentiated with respect to ξ and jumps across C are considered, we have

$$2B[u_{\xi\xi\eta}]+(A_\xi+Du_\xi)[u_{\xi\xi}]=0. \tag{1.67}$$

Since $[u_{\xi\xi}]$ is purely a function of η, (1.67) is equivalent to the ordinary differential equation

$$\frac{d}{d\eta}[u_{\xi\xi}]+f(\eta)[u_{\xi\xi}]=0 \tag{1.68}$$

which has a solution of the form

$$[u_{\xi\xi}]=[u_{\xi\xi}]_{\eta_0}\exp\left(-\int_{\eta_0}^{\eta}f(\zeta)\,d\zeta\right). \tag{1.69}$$

*Any sufficiently smooth curve can be embedded in a one-parameter family of "parallel" curves.

We conclude that if at any point $\eta=\eta_0$ of C a discontinuity in the second order derivatives of u is present, it will persist (be non-zero) for all points on C. The variation of the jump in $[u_{\xi\xi}]$ across $\xi=$ constant is governed by the equation (1.68), also called the transport equation.

EXERCISE 1.4

1. Consider any straight line $ax+by+c=0$ in the (x, y) plane where a, b, c are constants. For any point (x, y) in the plane define $u(x, y) = |ax+by+c|$. Show that u satisfies the equation $u_{xx}+u_{yy}=0$ on each side of the line. Show that u is continuous across the line, but its normal derivative is not. Comment on the statment that any straight line can serve as a carrier of a discontinuity in the first derivative for the above equation, which has no real characteristics.
2. Find the possible curves along which discontinuity in the second order derivatives of u could occur when u is a weak solution of the equation:

$$xy^2u_{xx}-xu_{yy}+y^2u_x+\frac{x}{y}u_y=0.$$

Find the transport equation and solve it to find out how the discontinuity grows or decays.

§2 POTENTIAL THEORY AND ELLIPTIC DIFFERENTIAL EQUATIONS

Boundary data rather than initial data serve to fix properly the solution of an elliptic differential equation. It is usually necessary to find an answer "in the large," namely in the domain bounded by a closed boundary, and this need for "global" constructions, rather than "local" treatment makes it especially difficult to study nonlinear elliptic equations. We shall restrict ourselves mainly to the linear potential equation or Laplace's equation in m-space variables. The boundary value problems of potential theory are suggested by physical phenomena from such varied field as electrostatics, steady heat conduction and incompressible fluid flow [see Sneddon, 1957].

§2.1 Boundary Value Problems and Cauchy Problem

The general linear homogeneous second order partial differential equation in m-space variables $x_1, x_2, \ldots, x_m$ is

$$Lu \equiv a_{\alpha\beta}u_{x_\alpha x_\beta}+b_\alpha u_{x_\alpha}+cu=0, \quad \alpha, \beta=1, 2, \ldots, m \tag{2.1}$$

where the coefficients $a_{\alpha\beta}$, b_α and c are continuous functions of the independent variables $x_1, x_2, \ldots, x_m$ and $a_{\alpha\beta}=a_{\beta\alpha}$. Equation (2.1) is said to be elliptic in a domain D of m-dimensional space, when the quadratic form

$$Q(\lambda)=a_{\alpha\beta}\lambda_\alpha\lambda_\beta \tag{2.2}$$

can be expressed as the sum of squares with coefficients of the same sign, or equivalently, $Q(\lambda)$ is either positive or negative definite in D. The simplest case is that of the Laplace equation or potential equation:

$$\Delta_m u = u_{x_\alpha x_\alpha} = 0. \tag{2.3}$$

We shall first state three boundary value problems associated with Laplace equation and then consider the Cauchy problem. Let D be a domain in $(x_1, x_2, \ldots, x_m)$-space bounded by a piecewise smooth boundary ∂D. Let continuous boundary values be prescribed on ∂D, by means of a function f.

The first boundary value problem, also called the *Dirichlet problem*, requires a solution u of the Laplace equation (2.3) in the domain D, which is continuous in $D+\partial D$ and coincides with f on ∂D, i.e.

$$u=f \text{ on } \partial D. \tag{2.4}$$

The second boundary value problem, also called the Neumann problem, requires the determination of the solution u in the domain D, which is continuous with continuous first order partial derivatives in $D+\partial D$, such that the normal derivative $\partial u/\partial \nu$ of u on ∂D takes prescribed values f, i.e.

$$\frac{\partial u}{\partial \nu}=f \text{ on } \partial D. \tag{2.5}$$

Here ∂D must have a continuously varying normal.

The third boundary value problem is a modification of the first two boundary value problems, where the solution u is such that a linear combination of u and $\partial u/\partial \nu$, rather than either of them separately, takes prescribed values on ∂D i.e.

$$\frac{\partial u}{\partial \nu}+\alpha u=f \text{ on } \partial D \tag{2.6}$$

where α is a constant.

Before we discuss the Cauchy problem, we shall examine, in general, the requirements to be satisfied by a reasonable mathematical problem. There are two requirements:

1. Existence requirement—There is at least one u satisfying the equation and the given boundary/Cauchy data.
2. Uniqueness requirement—There is utmost one such u.

   If the mathematical problem is to be also physically realistic an extra requirement has to be satisfied:
3. Stability requirement—Small changes in the boundary or Cauchy data result in small changes in the solution u.

The first two requirements ensure the existence and uniqueness of the solution of a mathematical problem, while all three requirements ensure, further, stability or continuous dependence on given data for a physical problem. If the three requirements are satisfied by a problem, it is said to be *well posed*.

The Cauchy-Kowalewski theorem shows that the solution of an analytic Cauchy problem for an elliptic equation exists and is unique. However, a Cauchy problem for Laplace's equation is not always well posed.

Hadamard gave an example of a Cauchy problem, which violates the stability requirement. Consider the Laplace equation in two independent variables x, y with the following initial conditions:

$$\text{(a) } u(x, 0)=0, \quad u_y(x, 0)=0$$

$$\text{(b) } u(x, 0)=0, \quad u_y(x, 0)=\frac{\sin kx}{k}. \tag{2.7}$$

A solution satisfying condition (a) is

$$u(x, y)=0. \tag{2.8}$$

A solution satisfying condition (b) is

$$u(x, y)=\frac{1}{k^2} \sin kx \sinh ky. \tag{2.9}$$

For sufficiently large k, the Cauchy or initial values (a) and (b) are arbitrarily close, but the solutions are not, since $\sinh ky$ behaves like e^{ky} for large k.

Having noted that a Cauchy problem could be illposed for an elliptic equation, we shall concentrate our attention hereafter only on the three boundary value problems mentioned earlier and show that they are really wellposed.

Definition: A function $u(x)$ is called *harmonic* in D, if $u(x) \in C^0$ in $D+\partial D$, $\in C^2$ in D and $\Delta_m u=0$ in D.

In the case of two or three variables, the "general solution" of the potential equation can be easily obtained. For $m=2$ $(x_1=x,\ x_2=y)$, this is the real or imaginary part of any analytic function of the complex variable $x+iy$. For $m=3$ $(x_1=x,\ x_2=y,\ x_3=z)$, consider an arbitrary function $p(w, t)$ analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function:

$$u=p(z+ix \cos t+iy \sin t,\ t) \tag{2.10}$$

of the real variables x, y, z are solutions of the equation $\Delta u=0$. Further solutions may now be obtained by superposition.

$$u=\int_a^b p(z+ix \cos t+iy \sin t,\ t)\, dt. \tag{2.11}$$

If $u(x, y)$ is a solution of the Laplace's equation in a domain D of the (x, y) plane, the function

$$v(x, y)=u\left(\frac{x}{r^2}, \frac{y}{r^2}\right), \quad r^2=x^2+y^2 \tag{2.12}$$

also satisfies the potential equation and is regular in the domain D' obtained from D by inversion with respect to the unit circle.

In general in m-dimensions, if $u(x_1, x_2, \cdots, x_m)$ satisfies the potential equation in a bounded domain D, then

$$v=\frac{1}{r^{m-2}}u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \cdots, \frac{x_m}{r^2}\right), \quad r^2=x_\alpha x_\alpha \tag{2.13}$$

also satisfies the potential equation and is regular in the region D' obtained from D by inversion with respect to the m-dimensional unit sphere. Therefore, except for the factor r^{2-m}, the harmonic character of a function is invariant under inversions with respect to spheres. Besides, the harmonic property is retained completely under rotations, magnifications, translations and simple reflections across planes.

Example 2.1 Dirichlet problem for a circle in the x, y plane.

Let the circle C be given by $|\zeta|=R$, where $\zeta=x+iy$. The problem is to find $u(x, y)$ such that

$$\Delta_2 u=u_{xx}+u_{yy}=0$$

where (2.14)

$$u=f(\theta) \text{ on } C$$

where θ is the angular coordinate on C, i.e. $\zeta=Re^{i\theta}$ on C.

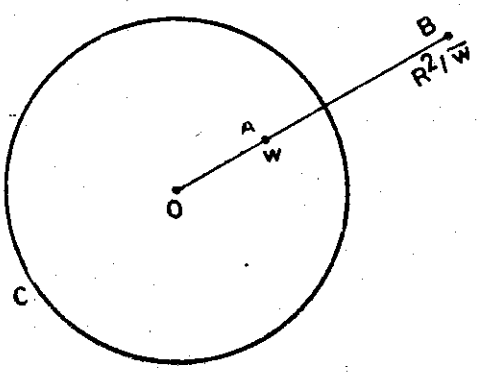

Fig. 2.1 w and $R^2/\bar{w}$ are inverse points with respect to $C: |z|=R$.

We shall present here a constructional proof for the existence of the solution, i.e. we shall derive an expression for the solution. Let $F(\zeta)$ be an analytic function in the region enclosed by C, such that the real part of $F(\zeta)$ on $|\zeta)|=R$ is $f(\theta)$. Let w be a complex number in the region. The inverse point of w with respect to C is $R^2/\bar{w}$, which lies outside C. Here $\bar{w}$ denotes the complex conjugate of w. According to the Cauchy integral formula

$$F(w)=\frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta-w}\,d\zeta$$

$$0=\frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta-R^2/\bar{w}}\,d\zeta.$$

Subtracting

$$F(w) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)(R^2 - w\bar{w})}{\zeta(R^2 + w\bar{w}) - \bar{w}\zeta^2 - wR^2}\, d\zeta.$$

As ζ lies on C and w inside C, set

$$\zeta = Re^{i\theta},\ w = re^{i\varphi},\ r < R.$$

Then,

$$F(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\theta})\, \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta - \varphi)}\, d\theta.$$

Taking the real part on both sides, we get

$$u(x, y) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(\theta)\, d\theta}{R^2 + r^2 - 2rR\cos(\theta - \varphi)} \tag{2.15}$$

where $r^2 = x^2 + y^2$, $\tan\varphi = y/x$.

Equation (2.15) is called Poisson's integral formula in two dimensions. This completes the proof of the existence of the solution.

EXERCISE 2.1

1. If $u(x, y)$ satisfies Laplace's equation in the unit circle and is equal to $\cos i\theta$ on the boundary, show that $u(x, y)$ is a rational function of x and y.
2. Given that $f(w, t) = w^n e^{iht}$, where n and h are integers, use formula (2.11) to find a solution of the Laplace equation in three dimensions.
3. Prove that $\Delta_m u(\mathbf{x}) = 0$ implies that $\Delta_m\left(|\mathbf{x}|^{2-m} u\left(\frac{\mathbf{x}}{|\mathbf{x}|^2}\right)\right) = 0$ for $\frac{\mathbf{x}}{|\mathbf{x}|^2}$ in the domain of definition of u.

§2.2 Singularity Functions and the Fundamental Solution: Green's Function

In R^m the solutions $v(r)$ of the potential equation $\Delta_m u = 0$, which depend only on the distance $r(\neq 0)$ of a point $\mathbf{x}$ from a fixed point $\mathbf{a}$, say, are given by the equation

$$\frac{d^2 v}{dr^2} + \frac{m-1}{r}\frac{dv}{dr} = 0,$$
$$r = |\mathbf{x} - \mathbf{a}| = \sqrt{(x_\alpha - a_\alpha)(x_\alpha - a_\alpha)}. \tag{2.16}$$

This equation has solutions

$$v(r) = c_1 + c_2 r^{2-m}, \qquad m > 2$$
$$= c_1 + c_2 \log r, \qquad m = 2 \tag{2.17}$$

where c_1 and c_2 are arbitrary constants. These solutions exhibit the so-called characteristic singularity at $r=0$. We call

$$\left.\begin{aligned} s(\mathbf{a}, \mathbf{x}) &= \frac{1}{(m-2)\omega_m} \mid \mathbf{a}-\mathbf{x} \mid^{2-m}, \qquad & m>2 \\ &= -\frac{1}{2\pi} \log \mid \mathbf{a}-\mathbf{x} \mid, \qquad & m=2 \end{aligned}\right\} \tag{2.18}$$

the *singularity function* for $\Delta_m u=0$, where ω_m is the surface area of the unit sphere in m-dimensions given by

$$\omega_m = 2(\sqrt{\pi})^m / \Gamma\left(\frac{m}{2}\right). \tag{2.19}$$

$s(\mathbf{a}, \mathbf{x})$ has the property that $s \in C^\infty$ and $\Delta_m s=0$ for $\mathbf{x} \neq \mathbf{a}$, with a singularity at $\mathbf{x}=\mathbf{a}$. For $m=3$, $s(\mathbf{a}, \mathbf{x})$ corresponds physically to the gravitational potential at the point $\mathbf{x}$ of a unit mass concentrated at the point $\mathbf{a}$. Every solution of the potential equation $\Delta_m u=0$ in D of the form

$$\gamma(\mathbf{a}, \mathbf{x}) = s(\mathbf{a}, \mathbf{x}) + \varphi(\mathbf{x}),\ \mathbf{a} \in D \tag{2.20}$$

where $\varphi(\mathbf{x}) \in C^2$ in D and $\varphi(\mathbf{x}) \in C^1$ in the closed region $D+\partial D$ and $\Delta_m \varphi=0$ in D, is called a *fundamental solution* relative to D with a singularity at $\mathbf{a}$.

Two important relations required for the study of the potential equation are the well known Green's formulae. These formulae help in obtaining the properties of the fundamental solution. Let u and v be two functions defined in a domain D bounded by a piecewise smooth surface ∂D. Then

1. $$\int_D \nabla_m u \nabla_m v \, d\mathbf{x} + \int_D v \Delta_m u \, d\mathbf{x} = \int_{\partial D} v \frac{\partial u}{\partial \nu} \, dS \tag{2.21}$$

and

$$\int_D (u\Delta_m v - v\Delta_m u) \, d\mathbf{x} = \int_{\partial D} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS \tag{2.22}$$

where $\partial/\partial\nu$ represents differentiation in the direction of the outward drawn normal to S. In (2.11) we assume continuity of u and v in $D+\partial D$, continuity of the first derivative of v in D, continuity of the first derivatives of u in $D+\partial D$ and continuity of the second derivatives of u in D. In (2.22), we assume continuity of the first derivatives of u and v in $D+\partial D$ and continuity of the second derivatives in D [for proof, see Courant and Hilbert (1962)].

If $\Delta_m u=0$ and $v=1$, we obtain from (2.21)

$$\int_{\partial D} u_\nu \, dS = 0 \tag{2.23}$$

where the suffix ν denotes normal derivative to the surface ∂D. That means, if a function satisfies $\Delta_m u=0$ in a bounded region D and is continuously differentiable in $D+\partial D$, then the surface integral of its normal derivative is zero. This lays a restriction on the Neumann boundary value problem (see (2.5)), where f cannot be prescribed arbitrarily. We require the condition that

$$\int_{\partial D} f\,dS=0.$$

Green's formulae undergo an important modification if for v we substitute a function having the characteristic singularity of potential function at an interior point. Let us choose v to be a fundamental solution:

$$v=\gamma(\mathbf{a},\,\mathbf{x})=s(\mathbf{a},\,\mathbf{x})+\varphi(\mathbf{x}). \tag{2.24}$$

This leads to the following theorem:

Theorem 2.1 If $u(\mathbf{x}) \in C^1$ in $D+\partial D$ and $\in C^2$ in D and $u(\mathbf{x})$ is a solution of the potential equation then for an arbitrary point $\mathbf{a} \in D$

$$u(\mathbf{a})=\int_{\partial D}\left\{\gamma(\mathbf{a},\,\mathbf{x})u_\nu(\mathbf{x})-u(\mathbf{x})\gamma_\nu(\mathbf{a},\,\mathbf{x})\right\}dS. \tag{2.25}$$

Proof Since D is open, the ball $V:|\,\mathbf{a}-\mathbf{x}\,|\leqslant\rho$ about any point $\mathbf{a}$ in D is contained in D for sufficiently small ρ. Since $\gamma(\mathbf{a},\,\mathbf{x})$ becomes singular at $\mathbf{x}=\mathbf{a}$, we remove V from D and apply Green's formula (2.22) for γ and u on $D-V$. We obtain

$$\int_{D-V}(\gamma\Delta_m u-u\Delta_m\gamma)\,d\mathbf{x}=\int_{\partial D}(\gamma u_\nu-u\gamma_\nu)\,dS$$
$$+\int_{|\,\mathbf{x}-\mathbf{a}\,|\,=\rho}(\gamma u_\nu-u\gamma_\nu)\,dS.$$

Since $\Delta_m\gamma=0$, $\Delta_m u=0$ in $D-V$, the left hand side is zero.

For the last integral on the right hand side, using the fact that $\gamma(\mathbf{a},\,\mathbf{x})=s(\mathbf{a},\,\mathbf{x})+\varphi(\mathbf{x})$, we get the sum of two integrals I_1 and I_2:

$$I_1=\int_{|\,\mathbf{x}-\mathbf{a}\,|\,=\rho}(\varphi u_\nu-u\varphi_\nu)\,dS.$$

The integrand is continuous in $|\,\mathbf{x}-\mathbf{a}\,|\leqslant\rho$, so that the integral tends to zero as $\rho\to 0$.

$$I_2=\int_{|\,\mathbf{x}-\mathbf{a}\,|\,=\rho}(su_\nu-us_\nu)\,dS.$$

At any point on the surface of V, $\mathbf{x}=\mathbf{a}+\rho\boldsymbol{\nu}$ and $dS=\rho^{m-1}\,d\omega$. Therefore

$$\int_{|\,\mathbf{x}-\mathbf{a}\,|\,=\rho}su_\nu\,dS=\rho^{m-1}\int_{|\,\boldsymbol{\nu}\,|\,=1}s(\mathbf{a},\,\mathbf{a}+\rho\boldsymbol{\nu})u_\nu(\mathbf{a}+\rho\boldsymbol{\nu})\,d\omega$$
$$=\frac{\rho}{(m-2)\omega_m}\int_{|\,\boldsymbol{\nu}\,|\,=1}u_\nu(\mathbf{a}+\rho\boldsymbol{\nu})\,d\omega.$$

This tends to zero as $\rho\to 0$.

Further,

$$-\int_{|\,\mathbf{x}-\mathbf{a}\,|\,=\rho}s_\nu u\,dS=-\rho^{m-1}\int_{|\,\boldsymbol{\nu}\,|\,=1}u(\mathbf{a}+\boldsymbol{\nu}\rho)s_\nu(\mathbf{a},\,\mathbf{a}+\rho\boldsymbol{\nu})\,d\omega$$
$$=-\frac{\rho^{m-1}}{(m-2)\omega_m}\int_{|\,\boldsymbol{\nu}\,|\,=1}u(\mathbf{a}+\boldsymbol{\nu}\rho)\frac{\partial}{\partial\boldsymbol{\nu}}\,\rho^{2-m}\,d\omega.$$

Here $$\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial \rho}.$$ Therefore

$$-\int_{|\mathbf{x}-\mathbf{a}|=\rho} s_\nu u \, dS = -\frac{1}{\omega_m}\int_{|\nu|=1} u(\mathbf{a}+\boldsymbol{\nu}\rho)\, d\omega.$$

As $\rho \to 0$, this tends to $-u(\mathbf{a})$. This proves the result (2.25), namely

$$u(\mathbf{a}) = \int_{\partial D}\Big\{\gamma(\mathbf{a}, \mathbf{x})u_\nu(\mathbf{x}) - u(\mathbf{x})\gamma_\nu(\mathbf{a}, \mathbf{x})\Big\} dS.$$

This formula holds for any $\mathbf{a} \in D$.

If instead **a** lies on ∂D, then it is easy to show using the same procedure that

$$u(\mathbf{a}) = 2\int_{\partial D}\Big\{\gamma(\mathbf{a}, \mathbf{x})u_\nu(\mathbf{x}) - u(\mathbf{x})\gamma_\nu(\mathbf{a}, \mathbf{x})\Big\} dS \tag{2.26}$$

whereas if **a** lies outside D, then

$$\int_{\partial D}\Big\{\gamma(\mathbf{a}, \mathbf{x})u_\nu(\mathbf{x}) - u(\mathbf{x})\gamma_\nu(\mathbf{a}, \mathbf{x})\Big\} dS = 0.$$

In particular in 3-dimensions $m=3$, if $\varphi(\mathbf{x})=0$, then we have for $\mathbf{a} \in D$:

$$u(\mathbf{a}) = \frac{1}{4\pi}\int_{\partial D}\left\{\frac{1}{r}\,\frac{\partial u}{\partial \nu} - u\,\frac{\partial}{\partial \nu}\left(\frac{1}{r}\right)\right\} dS. \tag{2.27}$$

u can be considered as the potential of a distribution consisting of a single layer surface distribution of density $(1/4\pi)\partial u/\partial \nu$ and a double layer dipole distribution of density $-u/4\pi$ on the boundary surface ∂D.

We have defined a fundamental solution as an ordinary function having a characteristic singularity. We shall briefly indicate how a fundamental solution can be treated as a distribution [see Smith (1967)] satisfying a partial differential equation. If instead of taking u as a solution of Laplace's equation, we had taken it to be an arbitrary function, then for $\mathbf{a} \in D$ the formula (2.25) is modified as:

$$u(\mathbf{a}) = -\int_D \gamma \Delta_m u \, d\mathbf{x} + \int_{\partial D}\left(\gamma\,\frac{\partial u}{\partial \nu} - u\,\frac{\partial \gamma}{\partial \nu}\right) dS.$$

Formally, using Green's formula (2.22) for the right-hand side, we have

$$u(\mathbf{a}) = -\int_D u \Delta_m \gamma \, d\mathbf{x} \equiv -\Delta_m \gamma(u(\mathbf{x})).$$

In a distribution sense, the integral can be interpreted as the action of $\Delta_m \gamma$ on u.

Therefore, we formally identify γ as satisfying the equation

$$\Delta_m \gamma = -\delta_a$$

where δ_a is the "Dirac Delta function". δ_a is a linear functional defined on the space of all continuously differentiable functions on D, such that

$$\delta_a(u(\mathbf{x})) = u(\mathbf{a}).$$

Therefore, we can define a fundamental solution $\gamma(\mathbf{x}, \mathbf{a})$ of the equation $\Delta_m u = 0$ as a solution of the nonhomogeneous equation

$$\Delta_m u = -\delta(\mathbf{x}, \mathbf{a}) \tag{2.28}$$

where the equality is understood in the sense of distributions.

Now we return to the use of formula (2.25). From this it appears that to determine the solution at an interior point, both u and u_ν must be prescribed on ∂D. This is not true as will be seen from the following procedure. Using the formula (2.25), we will show how to represent the solution of the Dirichlet boundary value problem by means of a Green's function, which is independent of the particular boundary values specified, say $f(\mathbf{x})$ on ∂D. We define a Green's function $G(\mathbf{a}, \mathbf{x})$ of the differential expression $\Delta_m u$ for the region D as a specific fundamental solution of $\Delta_m u = 0$, depending on the parameter $\mathbf{a}$, of the form

$$\begin{aligned} G(\mathbf{a}, \mathbf{x}) &= G(a_1, a_2, \cdots, a_n; x_1, x_2, \cdots, x_n) \\ &= s(\mathbf{a}, \mathbf{x}) + \varphi \end{aligned} \tag{2.29}$$

which vanishes at all points $\mathbf{x}$ on ∂D and for which the component φ is continuous in $D+\partial D$ and harmonic in D. Assuming the existence of a Green's function G, we replace γ in (2.25) by G and get the solution of a Dirichlet boundary value problem at a point $\mathbf{a} \in D$ as

$$u(\mathbf{a}) = -\int_{\partial D} f(x) \frac{\partial G}{\partial \nu}(\mathbf{a}, \mathbf{x})\, dS. \tag{2.30}$$

A Green's function is associated with the boundary ∂D, and depends only on its shape and not on the data prescribed on it. For plane boundaries, $\varphi(\mathbf{x})$ is obtained by taking reflections $\mathbf{a}_1$ of $\mathbf{a}$ in the boundary and replacing $\mathbf{a}$ by $\mathbf{a}_1$ in $s(\mathbf{a}, \mathbf{x})$. For circular or spherical boundaries, we take the inverse $\mathbf{a}_1$ of $\mathbf{a}$ in the boundary and again suitably substitute for $\mathbf{a}$ in $s(\mathbf{a}, \mathbf{x})$ to get $\varphi(\mathbf{x})$. This ensures that $G(\mathbf{a}, \mathbf{x}) = 0$ on ∂D.

Example 2.2 Green's function for a sphere of radius R, with centre at origin in m-dimensions.

Here

$$G(\mathbf{a}, \mathbf{x}) = \psi(r) - \psi\left(\frac{|\mathbf{a}|}{R} r_1\right) \tag{2.31}$$

where

$$\begin{aligned} \psi(r) &= \frac{1}{(m-2)\omega_m} |\mathbf{x}-\mathbf{a}|^{2-m}, && m>2 \\ &= \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{a}|}, && m=2 \end{aligned}$$

$$|\mathbf{a}|^2 = a_\alpha a_\alpha,\quad r_1^2 = \left(x_\alpha - \frac{R^2}{|\mathbf{a}|^2} a_\alpha\right)\left(x_\alpha - \frac{R^2}{|\mathbf{a}|^2} a_\alpha\right)$$

$$= \left| \mathbf{x} - \frac{R^2}{|\mathbf{a}|^2} \mathbf{a} \right|^2$$

r_1 denotes the distance of the point **x** from the reflected image of the point **a** in the sphere. This function satisfies all the requirements of the Green's function since (i) it is of the form $s(\mathbf{a}, \mathbf{x}), +\varphi(\mathbf{x})$, where $\varphi(\mathbf{x})$ is regular in D and continuous in $D+\partial D$ and (ii) it vanishes on ∂D, since

$$r = \frac{|\mathbf{a}|}{R} r_1 \text{ on } \partial D.$$

Example 2.3 Green's function for a positive half plane bounded by $x_1=0$, $m > 2$.

Here for 'a' in the positive half plane

$$G(\mathbf{a}, \mathbf{x}) = \frac{1}{(m-2)\omega_m} |\mathbf{x}-\mathbf{a}|^{2-m} - \varphi(\mathbf{x}), \quad m > 2 \tag{2.32}$$

where

$$\varphi(\mathbf{x}) = \frac{1}{(m-2)\omega_m} \left[(x_1+a_1)^2 + \sum_{\alpha=2}^{m} (x_\alpha - a_\alpha)^2\right]^{(2-m)/2}. \tag{2.33}$$

$\varphi(\mathbf{x})$ is obtained by taking the image of 'a' in the boundary $x_1=0$. We can find the solution at a by

$$u(\mathbf{a}) = \int_{x_1=0} f(\mathbf{x}) \left\{\frac{\partial}{\partial x_1} G(\mathbf{a}, \mathbf{x})\right\}\Big|_{x_1=0} dS. \tag{2.34}$$

Example 2.4 Green's function for a circle and Poisson's formula, $m=2$.

This is a special case of example 2.2

$$G(\mathbf{a}, \mathbf{x}) = \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{a}|} - \frac{1}{2\pi} \log \left| \mathbf{x} - \frac{R^2}{|\mathbf{a}|^2}\mathbf{a} \right|. \tag{2.35}$$

Of special interest is the case of the Green's function in two dimensions, for a domain D which can be mapped conformally onto the unit circle. We present here a method of obtaining this.

**Theorem 2.2* Let $F(x+iy) = u+iv$ represent a mapping of $D+\partial D$ onto the unit circle in the u, v plane, where $F(x+iy)$ is a simple analytic function of the complex variable $x+iy$. Then the Green's function for D is given by

$$G(a_1, a_2; x, y) = -\frac{1}{2\pi} Re \log \left[\frac{F(a_1+ia_2) - F(x+iy)}{\overline{F(a_1+ia_2)}F(x+iy) - 1}\right] \tag{2.36}$$

where Re denotes the real part of a complex quantity.

Proof To show that $G=0$ on the boundary ∂D, we note that ∂D is mapped by F onto the boundary of the unit circle in the z plane. Therefore

$$F(x+iy) = e^{i\theta}$$

when (x, y) lies on ∂D. Set $z = x+iy$, $\alpha = a_1 + ia_1$.

$$\text{On } \partial D,\ G(a_1, a_2; x, y) = -\frac{1}{2\pi} Re \log \left[\frac{F(\alpha) - e^{i\theta}}{\overline{F(\alpha)}e^{i\theta} - 1}\right]$$

$$= -\frac{1}{2\pi} Re \log\left[\frac{F(\alpha)-e^{i\theta}}{(F(\alpha)-e^{i\theta})e^{i\theta}}\right]$$

$$= -\frac{1}{2\pi} \log 1 = 0.$$

To show that G is a fundamental solution of the Laplace equation, we have to show that it satisfies $\Delta_2 u = 0$ except at $(x, y) = (a_1, a_2)$ and in the neighbourhood of this point

$$2\pi G = -\log \mid (x, y) - (a_1, a_2) \mid + \text{a harmonic function.}$$

Since F is a one-to-one mapping

$$F(z) - F(\alpha) = 0 \text{ only for } z = \alpha.$$

For any two points (x, y), (a_1, a_2) belonging to D

$$\mid F(\alpha) \mid < 1, \mid F(z) \mid < 1. \text{ Therefore, } \overline{F(\alpha)}F(z) - 1 \neq 0.$$

Except for $(x, y) = (a_1, a_2)$, the function

$$\log\left[\frac{F(\alpha)-F(z)}{\overline{F(\alpha)}F(z)-1}\right]$$

is analytic in z and its real part satisfies Laplace's equation. In the neighbourhood of α

$$F(z) - F(\alpha) = (z-\alpha)F'(\alpha) - \frac{(z-\alpha)^2}{2!}F''(\alpha) + \ldots$$

$$= (z-\alpha)\left[F'(\alpha) - \frac{(z-\alpha)}{2!}F''(\alpha) + \ldots\right]$$

$$= (z-\alpha)H(z)$$

where $H(z)$ is analytic. Due to the conformal nature of the mapping, $H(z)$ is non-zero in a suitable neighbourhood of α. Therefore

$$2\pi G(\alpha, z) = -Re \log (z-\alpha) - Re \log\left(\frac{-H(z)}{\overline{F(\alpha)}F(z)-1}\right)$$

$$= -\log \mid z-\alpha \mid + \text{harmonic function.}$$

Therefore, G is the desired Green's function.

EXERCISE 2.2

1. Prove theorem 2.1 in the case $m = 2$.
2. Find the Green's function for the first quadrant of the (x, y)-plane.
3. Introduce the Green's function in any region D as a fundamental solution $G_3 = G_3(\mathbf{a}, \mathbf{x})$ satisfying the mixed boundary condition

$$\frac{\partial G_3}{\partial \nu} + \alpha G_3 = 0$$

for x on the boundary ∂D. Show that a solution of the Churchill problem has the representation

$$u(\mathbf{a}) = -\int_{\partial D} G_3(\mathbf{a}, \mathbf{x}) \left[\frac{\partial u(\mathbf{x})}{\partial \nu} + \alpha u(\mathbf{x})\right] dS$$

in terms of this Green's function.

4. Show that for **a** on ∂D in (2.26)

$$u(\mathbf{a}) = 2\int_{\partial D} \{\gamma(\mathbf{a}, \mathbf{x}) u_\nu(\mathbf{x}) - u(\mathbf{x}) \gamma_\nu(\mathbf{a}, \mathbf{x})\}\, dS.$$

5. Justify the integral formula (2.34) for the Dirichlet problem of Laplace's equation in a half space by performing the intermediate integration over a large hemisphere and passing to the limit as its radius becomes infinite.
6. Use the method of images to find the Green's function for Laplace's equation of the infinite strip $a < x < b$ in the (x, y)-plane.
7. Establish the formula

$$u(a_1, a_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x) \log\,[(x-a_1)^2 + a_2^2]\, dx + \text{constant}.$$

for the solution of the Neumann problem for Laplace's equation in the upper half plane, where $g(x) = u_y(x, 0)$. Generalise this result to the case of $m > 2$ independent variables.

8. The function $f(\xi)$ is bounded and continuous. Prove that

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\, \frac{y\, d\xi}{(x-\xi)^2 + y^2}$$

is harmonic in $y > 0$. Show that as $(x, y) \to (a, 0)$ on any path in $y > 0$, $u(x, y) \to f(a)$. Show also that $u(x, y)$ is harmonic in $y < 0$ but is discontinuous across the x-axis. Deduce that

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi) \log\,[(x-\xi)^2 + y^2]\, d\xi$$

is the solution of the Neumann problem for the half plane.

9. $w = e^z$, $z = x + iy$, maps the infinite strip $0 < y < \pi$ conformally on the half plane $\text{Im}\, w > 0$. Hence show that the Green's function for the strip is

$$G(a_1, a_2;\, x, y) = \frac{1}{2} \log \frac{e^{2x} - 2e^{x+a_1} \cos\,(y + a_2) + e^{2a_1}}{e^{2x} - 2e^{x+a_1} \cos\,(y - a_2) + e^{2a_1}}$$

Deduce that if u is harmonic in the strip, then

$$u(a_1, a_2) = \frac{\sin a_2}{2\pi}\int_{-\infty}^{\infty} \left\{\frac{u(x, 0)}{\cosh\,(x - a_1) - \cos a_2} + \frac{u(x, \pi)}{\cosh\,(x - a_1) + \cos a_2}\right\} dx.$$

10. Prove the symmetric property of the Green's function:

$$G(\mathbf{a}, \mathbf{x}) = G(\mathbf{x}, \mathbf{a}).$$

§2.3 Poisson's Theorem

This theorem gives the solution of Dirichlet's problem in a sphere of radius R about the origin.

If $f(\mathbf{x})\in C^0$ on $|\mathbf{x}|=R$, then for $m\geqslant 2$

$$
{}^{*}\quad u(\mathbf{x})=\begin{cases}\dfrac{R^2-|\mathbf{x}|^2}{R\omega_m}\displaystyle\int_{|\mathbf{y}|=R}\frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^m}\,dS & \text{for } |\mathbf{x}|<R \qquad (2.37\text{a})\\[2ex] f(\mathbf{x}) & \text{for } |\mathbf{x}|=R \qquad (2.37\text{b})\end{cases}
$$

belongs to C^0 in $|\mathbf{x}|\leqslant R$, to C^2 in $|\mathbf{x}|<R$ and $u(\mathbf{x})$ is a solution of the problem

$$\Delta_m u=0 \text{ for } |\mathbf{x}|<R,\ u=f(\mathbf{x}) \text{ for } |\mathbf{x}|=R. \tag{2.38}$$

We can further show that in $|x|<R$, $u\in C^\infty$.

Proof

Step 1 We shall first show that u given by (2.37) satisfies $\Delta_m u=0$. If $|\mathbf{x}|<R$, then $|\mathbf{x}-\mathbf{y}|\neq 0$ in the integrand and (2.37a) can be differentiated under the integral sign arbitrarily often.

$$\Delta_m u(\mathbf{x})=\frac{1}{R\omega_m}\int_{|\mathbf{y}|=R} f(\mathbf{y})\Delta_m\left\{\frac{R^2-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{y}|^m}\right\}dS=0 \text{ for } |\mathbf{x}|<R,$$

since $\Delta_m\left\{\dfrac{R^2-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{y}|^m}\right\}=0$ for $|\mathbf{y}|=R$.

In particular $u(\mathbf{x})\equiv 1$ is a solution of $\Delta_m u=0$ in D and $u\in C^2$ in $D+\partial D$. Applying (2.30) in this case, we get

$$1=\frac{R^2-|\mathbf{x}|^2}{R\omega_m}\int_{|\mathbf{y}|=R}\frac{dS}{|\mathbf{x}-\mathbf{y}|^m}. \tag{2.39}$$

Step 2 To show that on approaching the boundary ∂D, $u(\mathbf{x})$ as given by (2.37a) tends to the prescribed boundary value $f(\mathbf{x})$.

Let $0<\rho<a$, $\mathbf{x}^0$ is an arbitrary point on ∂D and $\mathbf{x}$ is such that $|\mathbf{x}-\mathbf{x}^0|<\rho/2$. Then from (2.37a) and (2.39), we get

$$u(\mathbf{x})-f(\mathbf{x}^0)=\frac{R^2-|\mathbf{x}|^2}{R\omega_m}\int_{|\mathbf{y}|=R}\frac{f(\mathbf{y})-f(\mathbf{x}^0)}{|\mathbf{x}-\mathbf{y}|^m}\,dS$$

About $\mathbf{x}^0$ construct a sphere with radius ρ and the part of ∂D which lies in this sphere we denote by S_1, i.e. S_1: $|\mathbf{y}|=R$, $|\mathbf{y}-\mathbf{x}^0|\leqslant\rho$, S_2: $|\mathbf{y}|=R$, $|\mathbf{y}-\mathbf{x}^0|>\rho$, $\partial D=S_1+S_2$.

$$\int_{|\mathbf{y}|=R}\frac{f(\mathbf{y})-f(\mathbf{x}^0)}{|\mathbf{x}-\mathbf{y}|^m}\,dS=\left(\int_{S_1}+\int_{S_2}\right)\frac{f(\mathbf{y})-f(\mathbf{x}^0)}{|\mathbf{x}-\mathbf{y}|^m}\,dS.$$

*This form of the solution is suggested by the Green's function in Example 2.2.

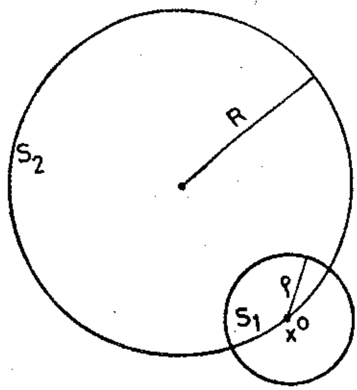

Fig. 2.2 Dividing the boundary into two parts S_1 and S_2 for an arbitrary point x^0 on the boundary

Consider each of these integrals in turn.

$$\left|\frac{R^2-|\mathbf{x}|^2}{R\omega_m}\int_{S_1}\frac{f(\mathbf{y})-f(\mathbf{x}^0)}{|\mathbf{x}-\mathbf{y}|^m}dS\right|$$

$$< \frac{R^2-|\mathbf{x}|^2}{R\omega_m}\max_{\mathbf{y}\in S_1}|f(\mathbf{y})-f(\mathbf{x}^0)|\int_{S_1}\frac{dS}{|\mathbf{x}-\mathbf{y}|^m}$$

$$\leqslant \max_{\mathbf{y}\in S_1}|f(\mathbf{y})-f(\mathbf{x}^0)|\frac{R^2-|\mathbf{x}|^2}{R\omega_m}\int_{|\mathbf{y}|=R}\frac{dS}{|\mathbf{x}-\mathbf{y}|^m}$$

$$= \max_{\mathbf{y}\in S_1}|f(\mathbf{y})-f(\mathbf{x}^0)| \text{ from (2.39).}$$

Next $$\max_{\mathbf{y}\in S_2}\frac{R^2-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{y}|^m}=\max_{\mathbf{y}\in S_2}\frac{(R-|\mathbf{x}|)(R+|\mathbf{x}|)}{|\mathbf{x}-\mathbf{y}|^m}\leqslant\frac{2R(R-|\mathbf{x}|)}{(\rho/2)^m}$$

since

$$|\mathbf{x}-\mathbf{y}|=|\mathbf{y}-\mathbf{x}^0+\mathbf{x}^0-\mathbf{x}|\geqslant|\mathbf{y}-\mathbf{x}^0|-|\mathbf{x}^0-\mathbf{x}|$$
$$\geqslant\rho-\rho/2=\rho/2.$$

If we set $\max_{\mathbf{y}\in\partial D}|f(\mathbf{y})|=M$, we obtain the estimate for the second term

$$\left|\frac{R^2-|\mathbf{x}|^2}{R\omega_m}\int_{S_2}\frac{f(\mathbf{y})-f(\mathbf{x}_0)}{|\mathbf{x}-\mathbf{y}|^m}dS\right|<\frac{2(R-|\mathbf{x}|)}{\omega_m(\rho/2)^m}2M\int_{|\mathbf{y}|=R}dS$$
$$=\frac{4MR^{m-1}(R-|\mathbf{x}|)}{(\rho/2)^m}.$$

Adding we have

$$|u(\mathbf{x})-f(\mathbf{x}^0)|<\max_{\mathbf{y}\in S_1}|f(\mathbf{y})-f(\mathbf{x}^0)|+\frac{4MR^{m-1}(R-|\mathbf{x}|)}{(\rho/2)^m}$$

Since f is continuous, given $\epsilon>0$ we can choose ρ such that

$$\max_{\mathbf{y}\in S_1}|f(\mathbf{y})-f(\mathbf{x}^0)|<\epsilon/2.$$

For such a ρ, we can find $\delta > 0$ so that

$$\frac{4MR^{m-1}(R-|\mathbf{x}|)}{\rho^{m/2}} < \epsilon/2 \text{ for all } \mathbf{x} \text{ with } |\mathbf{x}-\mathbf{x}^0| \leqslant \delta, \text{ since}$$

$$R-|\mathbf{x}| = |\mathbf{x}^0| - |\mathbf{x}| \leqslant |\mathbf{x}-\mathbf{x}^0|.$$

Therefore $|u(\mathbf{x})-f(\mathbf{x}^0)| < \epsilon$ for $|\mathbf{x}-\mathbf{x}^0| \leqslant \delta$ which shows $u \in C^0$ in $|\mathbf{x}| \leqslant R$ and has the correct boundary values.

That the solution is unique will be shown in §2.4. To show the stability of the solution, assuming uniqueness, consider a solution $\bar{u}(\mathbf{x})$ which can also be represented by (2.37a) with f replaced by $\bar{f}$. Then

$$|\bar{u}(\mathbf{x})-u(\mathbf{x})| \leqslant \max_{|\mathbf{y}|=R} |\bar{f}(\mathbf{y})-f(\mathbf{y})| \frac{R^2-|\mathbf{x}|^2}{R\omega_m} \int_{|\mathbf{y}|=R} \frac{dS}{|\mathbf{x}-\mathbf{y}|^m}$$

$$= \max_{|\mathbf{y}|=R} |\bar{f}(\mathbf{y})-f(\mathbf{y})|$$

which implies stability.

EXERCISE 2.3

1. Derive the three-dimensional analogue of Poisson's integral formula.
2. The gravitational force of attraction exerted on a unit mass located at the point $x=(x_1, x_2, x_3)$ by a solid D with density $\mu(\mathbf{x})$, according to Newton's law, is given by the vector

$$\mathbf{F}(\mathbf{x}) = \gamma \int_D \frac{\mu(\mathbf{y})(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} dy$$

where γ is universal gravitational constant. Prove that if the three components F_1, F_2, F_3 of the force $\mathbf{F}(\mathbf{x})$ have the form

$$F_i(\mathbf{x}) = \frac{\partial u}{\partial x_i}, \quad i=1, 2, 3$$

where $u(\mathbf{x})$ is the gravitational potential then $u(\mathbf{x})$ is a solution of

$$\Delta u = \mu(\mathbf{x}).$$

Calculate the potential and the force of attraction of a solid sphere D of radius a with centre at the origin and of constant density μ.

§2.4 The Mean Value and the Maximum-Minimum Properties

Definition $u(\mathbf{x})$ has the first mean value property in D if $u(\mathbf{x}) \in C^0$ in $D+\partial D$ and if for every sphere $|\mathbf{y}-\mathbf{x}| < r$ with centre at $\mathbf{x}$ in $D+\partial D$

$$u(\mathbf{x}) = \frac{1}{\omega_m r^{m-1}} \int_{|\mathbf{y}-\mathbf{x}|=r} u(\mathbf{y})\, dS. \tag{2.40}$$

Geometrically, we can interpret this as "the value of u at the centre of a sphere is equal to the mean value of u over the surface of the sphere".

If instead $u(\mathbf{x})$ satisfies

$$u(\mathbf{x}) = \frac{m}{\omega_m r^m} \int_{|\mathbf{y}-\mathbf{x}| \leqslant r} u(\mathbf{y})\, d\mathbf{y} \tag{2.41}$$

we say that $u(\mathbf{x})$ has the second mean value property. Here the value of u at the centre of the sphere is equal to the mean value of u averaged over the volume of the sphere.

The two mean value properties are equivalent.

From (2.40)

$$u(\mathbf{x})\rho^{m-1} = \frac{1}{\omega_m} \int_{|\mathbf{x}-\mathbf{y}| = \rho} u(\mathbf{y})\, dS.$$

Therefore

$$u(\mathbf{x}) \int_0^r \rho^{m-1}\, d\rho = \frac{1}{\omega_m} \int_0^r d\rho \int_{|\mathbf{y}-\mathbf{x}| = \rho} u(\mathbf{y})\, dS$$

$$= \frac{1}{\omega_m} \int_{|\mathbf{y}-\mathbf{x}| \leqslant r} u(\mathbf{y})\, d\mathbf{y}$$

$$\frac{r^m u(\mathbf{x})}{m} = \frac{1}{\omega_m} \int_{|\mathbf{y}-\mathbf{x}| \leqslant r} u(\mathbf{y})\, d\mathbf{y}$$

$$= \frac{1}{\omega_m} \int_0^r d\rho \int_{|\mathbf{y}-\mathbf{x}| = \rho} u(\mathbf{y})\, dS$$

which is the second mean value property. If we differentiate this with respect to r, we get the first mean value property.

We will now derive properties associated with solutions of Laplace's equation.

Theorem 2.5 Maximum-Minimum Principle.
If $u(\mathbf{x})$ has the mean value property in D, then the maximum and minimum of $u(\mathbf{x})$ occur on ∂D.

Proof Since $u(\mathbf{x}) \in C^0$ in the closed region $D + \partial D$ there is at least one point $\mathbf{x}^0 \in D + \partial D$ at which $u(\mathbf{x})$ attains its maximum. Let $u(\mathbf{x}^0) = M$.

1. If $u(\mathbf{x}) \equiv M$ in $D + \partial D$, every point is a maximum and a minimum point and the theorem is proved.

2. If $u(\mathbf{x}) \not\equiv M$ consider the set $\mathscr{M}$ of all these points belonging to D at which u attains a maximum. Therefore

$$\mathscr{M} : \mathbf{x} \in D \text{ with } u(\mathbf{x}) = M$$

Now $\mathscr{M} \subset D$. We shall show that $\mathscr{M}$ is both open and closed in D.

(a) $\mathscr{M}$ is open in D. If $\tilde{\mathbf{x}} \in \mathscr{M}$ and $\bar{S} : |\mathbf{x} - \tilde{\mathbf{x}}| < r$ is any sphere with centre $\tilde{\mathbf{x}}$ contained in D, then $u \equiv M$ in $\bar{S}$. If not there is an $\mathbf{x} \in \bar{S}$ at which $u(\mathbf{x}) < M$.

From the continuity of u, we have

$$M = u(\tilde{\mathbf{x}}) = \frac{m}{\omega_m r^m} \int_{|\mathbf{x}-\tilde{\mathbf{x}}| \leqslant r} u(\mathbf{x})\, d\mathbf{x} < \frac{Mm}{\omega_m r^m} \int_{|\mathbf{x}-\tilde{\mathbf{x}}| \leqslant r} d\mathbf{x} = M$$

which is a contradiction. Therefore $\tilde{\mathbf{x}} \in \bar{S} \subseteq \mathcal{M}$, which implies that $\mathcal{M}$ is open in D.

(b) $\mathcal{M}$ is closed in D. If $\{\mathbf{x}^i\}$ is a convergent sequence of maximum points $u(\mathbf{x}^i) = M$ such that $\lim_{i\to\infty} \mathbf{x}^i = \mathbf{x}$ in D, then from continuity of u, we obtain $u(\mathbf{x}) = \lim_{i\to\infty} u(\mathbf{x}^i) = M$ which implies $\mathbf{x} \subset \mathcal{M}$.

$\mathcal{M}$ is therefore both open and closed relative to D. But since D is connected the only subsets which are both open and closed in D is the null set or D itself. If $\mathcal{M}$ is non-empty, then $\mathcal{M} \equiv D$ or $u(\mathbf{x}) \equiv M$ in D.

By continuity $u(\mathbf{x}) \equiv M$ in $D + \partial D$ which contradicts $u(\mathbf{x}) \not\equiv M$. Hence $\mathcal{M}$ is empty. Therefore $u(\mathbf{x})$ will attain its maximum only on the boundary ∂D.

For the minimum, similar arguments hold.

Theorem 2.4 $u(\mathbf{x})$ is harmonic in D if and only if it possesses the mean value property.

Proof We first prove the 'necessary' part.

(a) Let us assume that $u(\mathbf{x})$ is harmonic in D.

Let S be a sphere of radius ρ and centre $\mathbf{x}$ in D, such that $S + \partial S \subset D$. In Green's identity (2.22) for the sphere S, choosing $v \equiv 1$, we have

$$\int_{|\mathbf{x}-\mathbf{y}| = \rho} \frac{\partial u}{\partial \nu}\, dS = 0$$

$$\int_{|\boldsymbol{\nu}| = 1} \frac{\partial u}{\partial \rho} (\mathbf{x} + \rho\boldsymbol{\nu}) \rho^{m-1}\, dS = 0$$

i.e.

$$\rho^{m-1} \frac{\partial}{\partial \rho} \int_{|\boldsymbol{\nu}| = 1} u(\mathbf{x} + \rho\boldsymbol{\nu})\, d\omega_m = 0.$$

ω (omega)

This implies that

$$\int_{|\boldsymbol{\nu}| = 1} u(\mathbf{x} + \rho\boldsymbol{\nu})\, d\omega_m \text{ is independent of } \rho.$$

Therefore

$$\int_{|\boldsymbol{\nu}| = 1} u(\mathbf{x} + \rho\boldsymbol{\nu})\, d\omega_m = \lim_{\rho\to 0} \int_{|\boldsymbol{\nu}| = 1} u(\mathbf{x} + \rho\boldsymbol{\nu})\, d\omega_m = u(\mathbf{x})\omega_m.$$

It follows that

$$u(\mathbf{x}) = \frac{1}{\omega_m} \int_{|\boldsymbol{\nu}| = 1} u(\mathbf{x} + \rho\boldsymbol{\nu})\, d\omega_m = \frac{1}{\rho^{m-1}\omega_m} \int_{|\mathbf{y}-\mathbf{x}| = \rho} u(\mathbf{y})\, dS$$

i.e. $u(\mathbf{x})$ has the mean value property in D.

(b) We now prove the "sufficient" part. Let us assume that $u(\mathbf{x})$ is continuous in $D+\partial D$ and satisfies the mean value property in D.

By differentiating under the integral sign, we can show that all order derivatives of u exist and have the mean value property in D. Therefore $u \in C^2$ in D.

It remains to show that u is harmonic in D. Let H contained in D be an arbitrary sphere with surface ∂H. Suppose $v(\mathbf{x})$ is a solution of the boundary value problem $\Delta_m v=0$ in H with $v=u$ on ∂H. By Poisson's Theorem, v can be determined. $v(\mathbf{x})$ is harmonic in H and thus by the first part of the proof, has the mean value property. Let $w(\mathbf{x})=v(\mathbf{x})-u(\mathbf{x})$. $w(\mathbf{x})$ has the mean value property in H and $w=0$ on ∂H. Since the maxima and minima of w lie on ∂H, $w \equiv 0$ in $H+\partial H$ or $u=v$ in $H+\partial H$. $u(\mathbf{x})$ is harmonic on any arbitrary sphere in D and hence harmonic in all of D. This completes the proof.

So far we have assumed that the mean value property is satisfied for every sphere entirely contained in D, where D can be an arbitrary finite or infinite domain. However, it turns out that if D is a finite domain, it is not necessary to assume the mean value property for every sphere, but only for a sphere about every point $\mathbf{x}$.

Theorem 2.5 Let D be a bounded domain for which the Dirichlet problem for the equation $\Delta_m u=0$ is solvable for arbitrary continuous boundary values. If a function w is continuous in $D+\partial D$ and has the mean value property at every point P of D for at least *one single* sphere with centre at P and radius greater than zero, such that the sphere is contained in $D+\partial D$, then w is a harmonic function in D. For proof, see Courant and Hilbert (1962), Chap. IV, §3.

As an immediate consequence of the above theorems, we obtain a uniqueness theorem for the Dirichlet problem. We prove the theorem for Poisson's equation $\Delta_m u=d(\mathbf{x})$, of which the Laplace equation is a particular case for $d \equiv 0$.

Theorem 2.6 The boundary value problem

$$\left.\begin{aligned} \Delta_m u &= d(\mathbf{x}) \text{ in } D \\ u &= f(\mathbf{x}) \text{ on } \partial D \end{aligned}\right\} \tag{2.42}$$

$d(\mathbf{x}) \in C^0$ in $D+\partial D, f(\mathbf{x}) \in C^0$ on ∂D, has at most one solution $u(\mathbf{x}) \in C^0$ in $D+\partial D$, and $\in C^2$ in D.

Proof Suppose $u^1(\mathbf{x}), u^2(\mathbf{x})$ are two solutions. Then for $v=u^1-u^2$, $\Delta_m v=0$ in D, $v=0$ on S. Hence v is harmonic. Therefore it has the mean value property and so its maxima and minima lie on ∂D. It follows that $v=0$ in $D+\partial D$. This completes the proof.

Theorem 2.7 If $u^i(\mathbf{x})$ are solutions of $\Delta_m u^i=d(\mathbf{x})$ in D, $u^i=f^i(\mathbf{x})$ on ∂D, $i=1, 2$, where $\max\limits_{\mathbf{x}\in\partial D} | f^1(\mathbf{x})-f^2(\mathbf{x}) | =\epsilon$, then $| u^1(\mathbf{x})-u^2(\mathbf{x}) | \leqslant \epsilon$ in $D+\partial D$.

Proof $v=u^1-u^2$ satisfies the problem $\Delta_m v=0$, $v=f^1(\mathbf{x})-f^2(\mathbf{x})$ on ∂D. As before v must have its maxima and minima on ∂D; hence the result. This theorem implies the stability of the solution of a Dirichlet problem. It is also of practical importance in that, if we change the boundary value slightly so that the problem is easily solvable, then we shall get an appropriate solution, which is sufficiently close to the original solution.

The following properties of harmonic functions can be derived naturally using Poisson's theorem and the mean value property.

Weierstrass convergence theorem. Consider a sequence $\{u_n(\mathbf{x})\}$ of harmonic functions in a bounded domain D, which are continuous in $D+\partial D$ and which have boundary values $\{f_n\}$ converging uniformly to f on ∂D. Then this sequence $\{u_n(\mathbf{x})\}$ converges uniformly in D to a harmonic function $u(\mathbf{x})$ with boundary values $f=\lim_{n\to\infty} f_n$.

Proof (a) To show that $u(\mathbf{x})$ exists and $u_n(\mathbf{x})$ tends uniformly to $u(\mathbf{x})$ in D.

Consider $w_{nm}=u_n-u_m$ for arbitrary n and m. w_{nm} is harmonic in D, continuous in $D+\partial D$ and on the boundary

$$| w_{nm} | = | u_n-u_m | = | f_n-f_m | < \epsilon$$

for $n, m > N(\epsilon)$ due to the uniform convergence of the sequence $\{f_n\}$. By the maximum principle $| w_{nm} | < \epsilon$ in D, which implies that $\{u_n\}$ is a Cauchy sequence, which converges uniformly to some limit function u.

(b) To show that u is harmonic in D.

That u is continuous in $D+\partial D$ follows from the fact that a uniformly convergent sequence of continuous functions converges to a continuous function. u_n has the mean value property in D, so

$$\begin{aligned} u(\mathbf{x}) &= \lim_{n\to\infty} u_n(\mathbf{x}) \\ &= \lim_{n\to\infty} \frac{1}{\omega_m r^{m-1}} \int_{|\mathbf{y}-\mathbf{x}|=r} u_n(\mathbf{y})\, dS \\ &= \frac{1}{\omega_m r^{m-1}} \int_{|\mathbf{y}-\mathbf{x}|=r} \lim_{n\to\infty} u_n(\mathbf{y})\, dS. \end{aligned}$$

This is possible since the sequence $\{u_n\}$ is uniformly convergent and we can pass to the limit under the integral sign

$$u(\mathbf{x}) = \frac{1}{\omega_m r^{m-1}} \int_{|\mathbf{y}-\mathbf{x}|=r} u(\mathbf{y})\, dS.$$

u is continuous in $D+\partial D$ and has the mean value property in D, and therefore it is harmonic in D.

(c) To show that u takes on proper boundary values.

On the boundary

$$\begin{aligned} | f-u | &\leqslant | f-f_n | + | f_n-u_n | + | u_n-u | \\ &\leqslant \epsilon_1+0+\epsilon_2=\epsilon \end{aligned}$$

Since ϵ can be chosen arbitrarily small by choosing n large, it follow that

$$u = f \text{ on } \partial D.$$

Next we discuss a property of positive harmonic functions.

**Theorem 2.8* (Harnack's inequality): Let $u(\mathbf{x})$ be harmonic in $|\mathbf{x}| < R$. Suppose that $u(\mathbf{x}) > 0$, then for $|\mathbf{x}| < R$, the following inequality holds

$$\frac{1 - \frac{|\mathbf{x}|}{R}}{\left(1 + \frac{|\mathbf{x}|}{R}\right)^{m-1}} u(0) \leqslant u(\mathbf{x}) \leqslant \frac{1 + \frac{|\mathbf{x}|}{R}}{\left(1 - \frac{|\mathbf{x}|}{R}\right)^{m-1}} u(0). \tag{2.43}$$

Proof By Poisson's theorem for $|\mathbf{x}| < R$

$$u(\mathbf{x}) = \frac{1}{R\omega_m} \int_{|\mathbf{y}|=R} \frac{|\mathbf{y}|^2 - |\mathbf{x}|^2}{|\mathbf{x} - \mathbf{y}|^m} u(\mathbf{y})\, dS.$$

For $|\mathbf{y}| = R$, $|\mathbf{x}| < R$

$$\frac{R^2 - |\mathbf{x}|^2}{(R + |\mathbf{x}|)^m} < \frac{|\mathbf{y}|^2 - |\mathbf{x}|^2}{|\mathbf{x} - \mathbf{y}|^m} < \frac{R^2 - |\mathbf{x}|^2}{(R - |\mathbf{x}|)^m}.$$

Since $u(\mathbf{x}) \geqslant 0$, we can multiply this inequality by $u(\mathbf{y})/R\omega_m$. Integrating, we get

$$\frac{R^2 - |\mathbf{x}|^2}{(R + |\mathbf{x}|)^m} \frac{1}{R\omega_m} \int_{|\mathbf{y}|=R} u(\mathbf{y})\, dS \leqslant u(\mathbf{x})$$
$$\leqslant \frac{R^2 - |\mathbf{x}|^2}{(R - |\mathbf{x}|)^m} \frac{1}{R\omega_m} \int_{|\mathbf{y}|=R} u(\mathbf{y})\, dS.$$

Since from (2.40) $\frac{1}{R^{m-1}\omega_m} \int_{|\mathbf{y}|=R} u(\mathbf{y})\, dS = u(0)$, we have

$$\frac{1 - |\mathbf{x}|/R}{(1 + |\mathbf{x}|/R)^{m-1}} u(0) \leqslant u(\mathbf{x}) \leqslant \frac{1 + |\mathbf{x}|/R}{(1 - |\mathbf{x}|/R)^{m-1}} u(0).$$

This inequality lays bounds on the values of positive harmonic function in a sphere of radius R.

**Theorem 2.9* Let $u(\mathbf{x})$ be harmonic in the whole of R_m. Suppose $u(\mathbf{x})$ is bounded from above, i.e. $u(\mathbf{x}) \leqslant C$, then $u(\mathbf{x})$ is a constant.

Proof The function $U(\mathbf{x}) = C - u(\mathbf{x})$ is harmonic in every sphere $|\mathbf{x}| < R$ and $U(\mathbf{x}) \geqslant 0$. By Harnack's inequality

$$\frac{1 - |\mathbf{x}|/R}{(1 + |\mathbf{x}|/R)^{m-1}} U(0) \leqslant U(\mathbf{x}) \leqslant \frac{1 + |\mathbf{x}|/R}{(1 - |\mathbf{x}|/R)^{m-1}} U(0).$$

If we keep $\mathbf{x}$ fixed and let $R \to \infty$, we get $U(\mathbf{x}) = U(0)$. This implies $u(\mathbf{x}) = u(0)$, a constant. This can be compared to Liouville's Theorem in complex variables.

EXERCISE 2.4

1. Find the fundamental solutions of the reduced wave equation, $\Delta u+\lambda^2 u=0(\lambda > 0)$ in three dimensions. Use these to prove that not all solutions of the above equation obey the maximum-minimum principle.
2. Prove the Harnack convergence theorem:
   Any monotonically increasing sequence of harmonic functions in a bounded domain must either approach infinity everywhere or else converge to a limit function that is again harmonic.

*§2.5 Dirichlet's Principle

Dirichlet's problem for a general domain can be studied using Dirichlet's principle.

The Principle concerns a certain minimum property of an integral associated with the function u. Suppose we are given a bounded domain D whose boundary ∂D is piecewise continuous. Let v be a member of a class of functions which belong to C^1 in $D+\partial D$ and which takes on the same boundary values as the harmonic function u, namely $v=f$ on ∂D. For each v in this class we define the Dirichlet integral for v as

$$DI(v)=\int_D\left[\left(\frac{\partial v}{\partial x_1}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2+\dots+\left(\frac{\partial v}{\partial x_m}\right)^2\right]dx \tag{2.44}$$

and consider those v for which the integral exists. Denote this class of functions by $\bar{D}$.

Dirichlet's Principle states that the solution u of Laplace's equation has the property that for all $v \in C^1(\bar{D})$

$$DI(u) \leqslant DI(v). \tag{2.45}$$

That is, u is the function that minimizes the Dirichlet integral DI.

Proof of the Dirichlet principle in the case when $u \in C^1$ in $D+\partial D$.

Let $g=v-u$, then

$$DI(v)=DI(g)+DI(u)+2\int_D\left[\frac{\partial g}{\partial x_1}\frac{\partial u}{\partial x_1}+\dots+\frac{\partial g}{\partial x_m}\frac{\partial u}{\partial x_m}\right]dx$$

$$=DI(g)+DI(u)-2\int_D g\Delta u\,dx+2\int_{\partial D} g\frac{\partial u}{\partial \nu}\,dS, \text{ from (2.21)}$$

$$=DI(g)+DI(u)$$

since $\Delta u=0$ in D and $g=0$ on ∂D.

Since $DI(g) \geqslant 0$, we have

$$DI(u) \leqslant DI(v).$$

This can be viewed as an example of the direct method of the calculus of variations for extremum problems involving multiple integrals. Physically the Dirichlet integral represents the energy of the system and the Dirichlet principle asserts that stable equilibrium corresponds to the configuration in which energy is a minimum.

It is difficult to establish that there exists a function in $\bar{D}$ for which $DI(v)$ attains its minimum; however, assuming $\bar{D}$ to be non empty and since $DI(v)$ is bounded from below by zero, we can always find a sequence of functions v_n, $n=1, 2, \ldots$ such that

$$\lim_{n\to\infty} DI(v_n) = \underset{v \in D}{glb}\, DI(v) \equiv L. \tag{2.46}$$

It is from this minimizing sequence of functions that we will construct the solution u.

Consider a sphere B in D with centre ξ and radius ρ and form the functions

$$k_n(\xi, \rho) = \frac{1}{\omega_m \rho^m} \int_B v_n(x)\, d\mathbf{x}, \quad n=1, 2, \ldots. \tag{2.47}$$

That is, we consider the mean value of the minimizing sequence $v_1, v_2, \ldots$. The following statements hold for k_n [For Proof see F. John (1975)]:

1. For fixed ρ, the k_n's are continuous functions of ξ in every closed subset of D and converge uniformly to a continuous function $k(\xi, \rho)$ as $n \to \infty$ uniformly in ξ.

2. The limit function $k(\xi, \rho)$ is independent of ρ.

The $k(\xi)$ so defined is the solution of the Dirichlet problem. It can be shown that k satisfies the mean value property in D and hence is harmonic in D, also k takes on the boundary value on ∂D.

EXERCISE 2.5

1. Expand the Poisson integral formula in two dimensions into a Fourier Series

$$u(r, \varphi) = a_0 + \sum_{k=1}^{\infty} (a_k r^k \cos k\varphi + b_k r^k \sin k\varphi)$$

for the solution u of the Laplace's equation

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0$$

inside the circle $r < R$. Derive the expression

$$\int_0^{2\pi} \int_0^R \left[u_r^2 + \frac{1}{r^2} u_\varphi^2 \right] r\, dr\, d\varphi = \pi \sum_{k=1}^{\infty} k R^{2k} (a_k^2 + b_k^2)$$

for the Dirichlet integral of u.

2. Show that the series $f(\theta) = \sum_1^{\infty} \dfrac{\sin k^4 \theta}{k^2}$ defines a continuous distribution of boundary values on the circle of radius R such that the corresponding formal Fourier series solution of Dirichlet's problem has an infinite Dirichlet integral.

3. Let D be plane region bounded by the simple closed curves C_1 and C_2 and let w be a smooth function possessing a finite Dirichlet integral over D. Show that among all functions reducing to w on C_1 the one that satisfies Laplace's equation in D and the boundary condition $\partial u/\partial \nu=0$ along C_2 minimises the Dirichlet integral.

*§ 2.6 General Second Order Linear Elliptic Equation

Consider the linear partial differential equation

$$Lu \equiv Au+cu=d, \text{ where } Au=a_{\alpha\beta}\, u_{x_\alpha x_\beta}+b_\alpha u_{x_\alpha} \tag{2.48}$$

in a domain D of R^m. Let $a_{\alpha\beta}(\mathbf{x}), b_\alpha(\mathbf{x}), c(\mathbf{x}), d(\mathbf{x}) \in C^0$ in $D+\partial D$. The characteristic quadratic form associated with equation (2.48) is

$$a_{\alpha\beta}(\mathbf{x})\lambda_\alpha\lambda_\beta, \quad \alpha, \beta=1, 2, \ldots m.$$

We assume that equation (2.48) is elliptic. This implies that for any point $\mathbf{x} \in D$, the characteristic form is a positive (or negative) definite in $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Theorem 2.10 If $c \leqslant 0, d \geqslant 0$ $(d \leqslant 0)$ in $D+\partial D$, then every nonconstant solution $u(\mathbf{x})$ for which a positive maximum (negative minimum) exists, attains this value on ∂D and not in D.

Before proving the theorem, we will prove Hopf's first lemma: If $Au \geqslant 0$ in D and if $\mathbf{x}^0$ is a point in D such that $u(\mathbf{x}) \leqslant u(\mathbf{x}^0)$ for all $\mathbf{x} \in D+\partial D$, then $u(\mathbf{x})=u(\mathbf{x}^0)$ in $D+\partial D$. Here we will consider those functions u for which $u \in C^0$ in $D+\partial D$ and $u \in C^2$ in D.

Proof of lemma Put $u(\mathbf{x}^0)=M$. Denote by $\mathcal{M}$ the set of all $\mathbf{x} \in D$ for which $u(\mathbf{x})=M$. We assume $u(\mathbf{x})\not\equiv M$ in $D+\partial D$ and since $u \in C^0$ in $D+\partial D$, $u(\mathbf{x})\not\equiv M$ in D. $\mathcal{M}$ is a proper subset of D. This implies that there is an $\mathbf{x}^*$ in D with $u(\mathbf{x}^*) < M$, and which has a smaller distance from points of $\mathcal{M}$ than those from ∂D. This is because of the connectedness of D. Because of the continuity of $u(\mathbf{x})$, there is a sphere $S_{\mathbf{x}^*}$ about $\mathbf{x}^*$ which lies entirely in D and which contains points of $\mathcal{M}$ on its boundary and only there. Let one of the points be $\mathbf{x}^0$.

Step 1 $u(\mathbf{x}) < M$ within $S_{\mathbf{x}^*}$. Let $S_1 \subseteq S_{\mathbf{x}^*}$ be a smaller ball with radius r_1 so that $\mathbf{x}^0$ also lies on the boundary of S_1. $u<M$ in ∂S_1 except at the one point $\mathbf{x}^0$ where $u=M$. Finally, let $S_2 \subseteq D$ be another closed ball with centre $\mathbf{x}^0$ and radius $r_2 < r_1$. The boundary of S_2 denoted by ∂S_2 can be decomposed as $\partial S_2=\partial T_1+\partial T_2$, with $\partial T_1=\partial S_2 \cap S_1$. ∂T_1 is closed. $u < M$ on ∂T_1 and ∂T_1 is closed, so we can find an ϵ such that $u < M-\epsilon$ on ∂T_1, $u \leqslant M$ on ∂T_2.

Step 2 Transfer the origin to $\mathbf{x}^*$. A translation does not change the form of Au. Consider the auxiliary function

$$h(x)=\exp\,(-k\mid \mathbf{x}\mid^2-\exp\,(-kr_1^2), k > 0. \tag{2.49}$$

Then

$$\exp\,(k\mid \mathbf{x}\mid^2)\, Ah=4k^2 a_{\alpha\beta}x_\alpha x_\beta-2k(a_{\alpha\alpha}+b_\alpha x_\alpha), \quad \alpha, \beta=1, 2, \ldots, m.$$

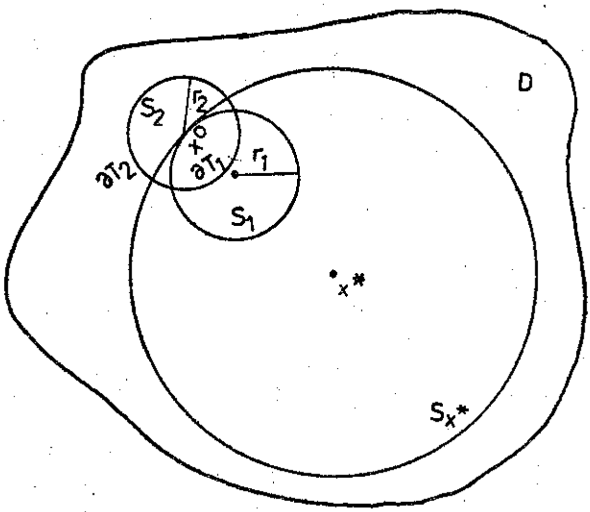

Fig. 2.3 x^0 is a point in D at which $u(\mathbf{x})$ attains a possible maximum

Since $r_2 < r_1$, the origin is not contained in S_2. S_2 is closed and (2.48) is elliptic; hence we have

$$a_{\alpha\beta}x_\alpha x_\beta \geqslant l > 0,\ \mathbf{x} \in S_2$$

If we choose k sufficiently large, $Ah > 0$ in S_2.

Step 3 Set $v(\mathbf{x}) = u(\mathbf{x}) + \delta h(\mathbf{x})$, $\delta > 0$. According to Step 1, δ can be chosen so that $v < M$ on ∂T_1. The form of $h(\mathbf{x})$ implies $h < 0$ on ∂T_2, so $v < M$ on ∂T_2. Therefore $v(\mathbf{x}) < M$ on ∂S_2. But $v(\mathbf{x}^0) = u(\mathbf{x}^0) = M$. So $v(\mathbf{x})$ has a maximum in S_2. Suppose it is attained at $\mathbf{x}^1 \in S_2$. Then for each vector $(y_1, y_2, \cdots, y_m)$, we have

$$v_{x_\alpha}(\mathbf{x}^1) = 0,\ \alpha = 1, 2, \cdots, m;\ v_{x_\alpha x_\beta}(\mathbf{x}^1) y_\alpha y_\beta \leqslant 0. \tag{2.50}$$

Because of elliptic nature of the equation, we have

$$a_{\alpha\beta}(\mathbf{x}^1)\ y_\alpha y_\beta \geqslant 0, \text{ for all vectors } (y_1, y_2, \cdots, y_m).$$

The elementary theorem on traces from the theory of matrices gives

$$a_{\alpha\beta}(\mathbf{x}^1) v_{x_\alpha x_\beta}(\mathbf{x}^1) \leqslant 0$$

i.e. at $\mathbf{x}^1$ we get

$$0 \geqslant Av = Au + \delta Ah.$$

But $Ah > 0$ from second step. So $Au < 0$ at $\mathbf{x}^1$. This is a contradiction to our assumption. Hopf's lemma is proved.

Proof of the Theorem Let $u(\mathbf{x})$ have a positive maximum at $\bar{\mathbf{x}}$ for which $u(\bar{\mathbf{x}}) = M$. Let $u(\mathbf{x}) \not\equiv M$ in D, i.e. u is not a constant M in D. Consider the set $\mathcal{M}$ of points $\mathbf{x} \in D$ for which $u(\mathbf{x}) = M$. $\mathcal{M}$ is non-empty. We will now prove that $\mathcal{M}$ is open relative to D. To prove it, choose an $\mathbf{x}^0 \in \mathcal{M}$; then $u(\mathbf{x}) \leqslant u(\mathbf{x}^0)$ in a suitable ball $\bar{S}$ with centre $\mathbf{x}^0$. Also $u(\mathbf{x}) > 0$ in $\bar{S}$ which is possible because $M > 0$ and u is continuous in D. In $\bar{S}$, $Au = -cu + d \geqslant 0$.

From Hopf's Lemma, $u(\mathrm{x}) \equiv M$ in $\bar{S}$ and so $\mathcal{M}$ is open. From the continuity of u for a set of points $\{\mathrm{x}^i\} \in D$, when $\mathrm{x}^i \to \mathrm{x}$, $u(\mathrm{x}^i) \to u(\mathrm{x})$. If $\{\mathrm{x}^i\}$ is a convergent sequence of points with $u(\mathrm{x}^i) = M$, then $u(\mathrm{x}) = M$. Therefore $\mathcal{M}$ is closed. Because of the connectedness of D, $\mathcal{M} \equiv D$. By reasons of continuity, $u(\mathrm{x}) \equiv M$ in $D + \partial D$. This is a contradiction. Therefore $u(\mathrm{x})$ has a positive maximum only on ∂D.

We now prove the uniqueness and stability of a solution of a boundary value problem.

Theorem 2.11 If in (2.48), $d \in C^0$ in $D + \partial D$, and $c \leqslant 0$ in $D + \partial D$, then the boundary value problem,

$$Lu = d \text{ in } D, \; u = f \text{ on } \partial D$$

with $f \in C^0$ on ∂D, has at most one solution $u(\mathrm{x}) \in C^0$ in $D + \partial D$, $\in C^2$ in D.

Proof If u^1, u^2 are two solutions, then $v = u^1 - u^2$ satisfies $Lv = 0$ in D, $v = 0$ on ∂D. By the previous theorem $v = 0$ in $D + \partial D$. This proves the uniqueness of the solution.

Theorem 2.12 If $u^i(\mathrm{x})$ are solutions of $Lu^i = d$ in D, $u^i = f^i(\mathrm{x})$ on ∂D, $i = 1, 2$ where $\max\limits_{\mathrm{x} \in S} | f^1(\mathrm{x}) - f^2(\mathrm{x}) | = \epsilon$ then $| u^1(\mathrm{x}) - u^2(\mathrm{x}) | < \epsilon$ in $D + \partial D$.

Proof $v = u^1 - u^2$ satisfies the equation $Lv = 0$, $v = f^1 - f^2$ on ∂D. If $v > \epsilon$ in D, then $v > \epsilon$ on ∂D, which is not true. Hence the proof. This shows the stability of the solution.

EXERCISE 2.6

1. If equation (2.48) in two independent variables is quasilinear (i.e. a_{11}, a_{22}, b_1, b_2 depend on x, y, u, u_x and u_y) and c, d are zero, show that the maximum principle is still valid for any solution fulfilling the ellipticity requirement

$$a_{12}^2 - a_{11}a_{22} < 0.$$

2. Let D be a plane region bounded by a smooth curve C and suppose that u is a non-negative harmonic function in D possessing continuous first partial derivatives on C. If u vanishes at a point of C, show that $\partial u / \partial \nu \geqslant 0$ there. Prove that the equality sign can hold only when u vanishes identically.

§3. THE DIFFUSION EQUATION AND PARABOLIC DIFFERENTIAL EQUATIONS

The diffusion equation is of the form

$$u_t = k \Delta_m u \tag{3.1}$$

in the $m+1$ variables $x_1, x_2, \cdots, x_m, t$, where k is a constant greater than zero. By a suitable change of scale of t or $\mathbf{x}$ we can always make $k=1$. This equation is also called the heat equation. For $m=3$ the equation is satisfied by the temperature distribution in a heat conducting medium, provided the density and specific heat of the material are constant. It also governs diffusion processes.

For the equation

$$u_t = \Delta_m u \tag{3.2}$$

we can determine the characteristic surfaces $\zeta(x_1, x_2, \cdots, x_m, t) = \text{constant}$ from the relation

$$Q_1(\zeta) \equiv \sum_{i=1}^{m} \zeta_{x_i}^2 = 0. \tag{3.3}$$

Therefore, ζ is independent of $x_1, x_2, \cdots, x_m$ and the family of characteristic hypersurfaces are given by

$$t = \text{constant}. \tag{3.4}$$

Equation (3.2) changes when we replace t by $-t$, which is not the case in a conservative system without dissipation of energy. This indicates that it describes irreversible processes and makes a distinction between the past and the future.

§3.1 Existence and Uniqueness Theorems for the Initial Value Problem in an Infinite Domain

We first look for a solution of the type

$$u = \exp\{i(\lambda t + \mathbf{x}\cdot\boldsymbol{\xi})\}, \quad \mathbf{x}\cdot\boldsymbol{\xi} = x_\alpha \xi_\alpha, \quad \alpha = 1, 2, \cdots, m \tag{3.5}$$

involving the exponential function.

This satisfies equation (3.2) provided

$$i\lambda = -\xi_\alpha \xi_\alpha = -|\boldsymbol{\xi}|^2. \tag{3.6}$$

(3.5) can be used to obtain a formal solution of the initial value problem

$$u_t = \Delta_m u, \; u(\mathbf{x}, 0) = f(\mathbf{x}), \; -\infty < x_\alpha < \infty, \; t > 0. \tag{3.7}$$

u defined by (3.5) has the initial value $\exp(i\mathbf{x}\boldsymbol{\xi})$. This suggests the use of a Fourier integral method. Representation of $f(\mathbf{x})$ as a Fourier integral

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{m/2}} \int \exp(i\mathbf{x}\cdot\boldsymbol{\xi}) \hat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \tag{3.8}$$

$$d\boldsymbol{\xi} = d\xi_1\, d\xi_2 \cdots d\xi_m$$

where the integration is over R_m. It leads formally to the solution

$$u(\mathbf{x}, t) = \frac{1}{(2\pi)^{m/2}} \int \exp(i\mathbf{x}\cdot\boldsymbol{\xi} - |\boldsymbol{\xi}|^2 t)\, \hat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{3.9}$$

Using the expression

$$\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{m/2}} \int \exp(-i\mathbf{y}\boldsymbol{\xi}) f(\mathbf{y})\, dy \quad \text{for } t > 0 \tag{3.10}$$

for $\hat{f}(\boldsymbol{\xi})$

and interchanging integrations, we have

$$u(\mathbf{x}, t)=\int K(\mathbf{x}-\mathbf{y}, t) f(\mathbf{y})\, d\mathbf{y} \tag{3.11}$$

where the Kernel K is given by

$$K(\mathbf{x}-\mathbf{y}, t)=\frac{1}{(2\pi)^m}\int \exp\{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}-|\boldsymbol{\xi}|^2 t\}\, d\boldsymbol{\xi}. \tag{3.12}$$

Equation (3.11) is a formal solution of the initial value problem (3.7). The integral (3.12) can be evaluated by completing the squares in the exponent

$$K(\mathbf{x}-\mathbf{y}, t)=\frac{1}{(2\pi)^m}\int \exp\left(-t\left|\boldsymbol{\xi}-i\frac{(\mathbf{x}-\mathbf{y})}{2t}\right|^2-\frac{|(\mathbf{x}-\mathbf{y})|^2}{4t}\right)d\boldsymbol{\xi}$$

$$=\frac{\exp\left(-\dfrac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)}{(2\pi)^m t^{m/2}}\int\exp(-|\boldsymbol{\eta}|^2)\, d\boldsymbol{\eta}=\frac{\exp\left(-\dfrac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)}{(4\pi t)^{m/2}}$$

since $\displaystyle\int\exp(-|\boldsymbol{\eta}|^2)\, d\boldsymbol{\eta}=\prod_{k=1}^{m}\int\exp(-\eta_k^2)\, d\eta_k=\pi^{m/2}$.

The function

$$K(\mathbf{x}, t)=\frac{\exp(-|\mathbf{x}|^2/4t)}{(4\pi t)^{m/2}} \tag{3.13}$$

is called a singularity function. It has the following properties:

(i) It satisfies the heat equation for $t>0$.

$$\Delta_m K=K_{x_i x_i}=\frac{1}{(4\pi t)^{m/2}}\left(\frac{|\mathbf{x}|^2}{4t^2}-\frac{m}{2t}\right)\exp(-|\mathbf{x}|^2/(4t))=K_t.$$

By defining $K(\mathbf{x}, 0)$ as $\lim_{t\to 0+} K(\mathbf{x}, t)$ for $\mathbf{x}\neq 0$, we see that it has a singularity only at $t=0, \mathbf{x}=0$.

(ii) $K(\mathbf{x}, t)\in C^\infty$ for $t>0$

(ii) $K>0$ for $t>0$

(iv) $\displaystyle\int K(\mathbf{x}-\mathbf{y}, t)\, dy=1$ for $t>0$ (3.14)

(v) for any $\delta>0$ $\displaystyle\lim_{t\to 0}\int_{|\mathbf{y}-\mathbf{x}|>\delta} K(\mathbf{x}-\mathbf{y}, t)\, d\mathbf{y}=0.$ (3.15)

To prove (iv) and (v), consider

$$\int_{|\mathbf{y}-\mathbf{x}|>\delta} K(\mathbf{x}-\mathbf{y}, t)\, dy=\frac{1}{(4\pi t)^{m/2}}\int_{|\mathbf{y}-\mathbf{x}|>\delta}\exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)dy$$

$$=\frac{1}{\pi^{m/2}}\int_{|\boldsymbol{\xi}|>\frac{\delta}{2\sqrt{t}}}\exp(-|\boldsymbol{\xi}|^2)\, d\boldsymbol{\xi}.$$

The above integral is convergent, when the integration is performed over the entire space. As $t \to 0$, the integral tends to zero, giving (v). For $\delta = 0$, we get (iv).

As in the case of the potential equation and in contrast to the wave equation, there is no decisive dependence of the properties of the solution on the number m of the spatial variables for the diffusion equation. There is, therefore, no loss of generality in restricting ourselves to one space variable. In the rest of this section, we shall consider the diffusion equation in the two independent variables t and x only.

We will now justify that (3.11) is a solution of the initial value problem (3.7).

Theorem 3.1 Let $u_0(x) \in C$ in R^1 and $|u_0(x)| \leqslant Me^{Ax^2}$ with constants M, $A \geqslant 0$. Then

$$u(x, t) = \int_{-\infty}^{\infty} K(x-y, t) u_0(y)\, dy \text{ for } 0 < t \leqslant T \tag{3.16}$$

where $T < \dfrac{1}{4A}$, is the solution of the problem

$$\Delta u = u_t \text{ and } u(x, 0) = u_0(x)$$

with $u \in C^\infty$ in $-\infty < x < \infty$, $0 < t \leqslant T$. Further, $u(x, t)$ satisfies an estimate of the form

$$|u(x, t)| \leqslant M_1 \exp(A_1 x^2) \text{ with } M_1, A_1 > 0.$$

Note: The smaller we choose A, the better, for then T becomes larger.

Consider a function $f(x)$ continuous in $-\infty < x < \infty$ which satisfies

$$|f(x)| \leqslant M' e^{A'x^2}, -\infty < x < \infty.$$

Consider the set $\mathcal{A}$ of all constants A' (not necessarily positive), for which there exist corresponding constants M' such that the above inequality holds. In particular if $\tilde{A} \in \mathcal{A}$, then $\bar{A} \in \mathcal{A}$ whenever $\bar{A} \geqslant \tilde{A}$. The greatest lower bound of $\mathcal{A}$ is denoted by $\gamma(f(x))$. We will first prove a lemma, for those functions $f(x)$ for which $\gamma(f(x))$ exists.

Lemma 3.1 Let $f(x)$, $g(x)$ be continuous functions on $-\infty < x < \infty$, where $f(x)$, $g(x)$ satisfy the estimates $|f(x)| \leqslant Me^{Ax^2}$, $|g(x)| \leqslant \bar{M}e^{\bar{A}x^2}$. If $\gamma(f(x)) > 0$ and

$$\gamma(f(x)) + \gamma(g(x)) < 0 \tag{3.17}$$

then

$$k(x) = \int_{-\infty}^{\infty} g(x-y) f(y)\, dy \tag{3.18}$$

is continuous in $-\infty < x < \infty$ and satisfies the estimate

$$|k(x)| \leqslant \tilde{M} e^{\tilde{A}x^2} \text{ with}$$

$$\gamma(k(x)) \leqslant \frac{\gamma(f(x))\gamma(g(x))}{\gamma(f(x)) + \gamma(g(x))} \tag{3.19}$$

Proof

$$| k(x) | \leqslant M\bar{M}\int_{-\infty}^{\infty} \exp[Ay^2+\bar{A}(x-y)^2]\,dy$$

$$= M\bar{M}\exp\left(\frac{A\bar{A}}{A+\bar{A}}x^2\right)\int_{-\infty}^{\infty}\exp(A+\bar{A})\left(y-\frac{x\bar{A}}{A+\bar{A}}\right)^2 dy.$$

Because of (3.17), we can assume $A+\bar{A} < 0$; so we have

$$| k(x) | \leqslant M\bar{M}\sqrt{-\frac{\pi}{A+\bar{A}}}\exp\left(\frac{A\bar{A}}{A+\bar{A}}x^2\right)$$

$$\leqslant \tilde{M}e^{\tilde{A}x^2}.$$

Simple calculations show that

$$\gamma(k(x)) \leqslant \frac{\gamma(f(x))\gamma(g(x))}{\gamma(f(x))+\gamma(g(x))}.$$

Since the integral (3.18) for $k(x)$ converges uniformly on any finite interval, $k(x)$ is continuous.

Proof of the theorem 3.1. According to the Lemma, integral (3.16) is convergent whenever

$$\gamma(K(x, t))+\gamma(u_0(x)) < 0.$$

Here $$\gamma(K(x, t)) = -\frac{1}{4t}. \qquad (3.20)$$

So we require

$$-\frac{1}{4t}+\gamma(u_0(x)) < 0.$$

This implies the restriction

$$0 < t \leqslant T, \text{ where } T < \begin{cases} \dfrac{1}{4\gamma(u_0(x))} \text{ if } \gamma(u_0(x)) > 0 \\ \infty \text{ if } \gamma(u_0(x)) \leqslant 0. \end{cases} \qquad (3.21)$$

The integrals

$$\int_{-\infty}^{\infty} | K_t(x-y, t)u_0(y) |\, dy$$

and $$\int_{-\infty}^{\infty} | K_{xx}(x-y, t)u_0(y) |\, dy$$

are uniformly convergent in (x, t) in any finite region, $0 < T_1 \leqslant t \leqslant T$, $X_0 \leqslant x \leqslant X_1$. Therefore, differentiation with respect to x and t can be carried out under the integral sign and since $K_{xx}-K_t = 0$, we have $u(x, t)$ as the desired solution. It remains to prove that $u(x, t)$ is continuous at $t = 0$, i.e.

$$\lim_{(x,t)\to(x^0,0)} u(x, t) = u_0(x^0) \text{ for } -\infty < x^0 < \infty. \qquad (3.22)$$

Given ϵ, we choose δ so small that

$$|u_0(y)-u_0(x^0)| < \epsilon \text{ for } |y-x^0| < 2\delta.$$

Let $M=\sup |u_0(x)|$; then for $|x-x^0| < \delta$, $t > 0$,

$$|u(x,t)-u_0(x^0)| = \left| \int (K(x-y)t)(u_0(y)-u_0(x^0))\,dy \right| \quad \text{from (3.14)}$$

$$\leqslant \int_{|y-x|<\delta} K(x-y,t)\,|u_0(y)-u_0(x^0)|\,dy$$

$$+\int_{|y-x|>\delta} K(x-y,t)\,|u_0(y)-u_0(x^0)|\,dy.$$

For $|x-x^0| < \delta$ and $|y-x| < \delta$, $|y-x^0| < 2\delta$, therefore

$$|u(x,t)-u_0(x^0)| \leqslant \epsilon \int_{|y-x|<\delta} K(x-y,t)\,dy$$

$$+2M\int_{|y-x|>\delta} K(x-y,t)\,dy$$

$$\leqslant \epsilon\cdot 1+2M\epsilon$$

from (3.14) and (3.15) for sufficiently small $t > 0$. This leads to (3.22).

Also

$$\gamma(u(x,t)) \leqslant \frac{\gamma(K(x,t))\gamma(u_0(x))}{\gamma(K(x,t))+\gamma(u_0(x))}$$

$$= \frac{-\frac{1}{4t}\,\gamma(u_0(x))}{-\frac{1}{4t}+\gamma(u_0(x))} = \frac{\gamma(u_0(x))}{1-4t\gamma(u_0(x))}. \tag{3.23}$$

If $0 < t \leqslant T$, then $|u(x,t)| < M_1 e^{A_1 x^2}$ for suitable M_1, A_1. (3.24)
Since $K \in C^\infty$ and the integrals obtained by differentiating (3.16) with respect to x and t are uniformly convergent in (x,t) in any finite domain $u \in C^\infty$.

This completes the proof of the theorem.

By the same type of argument, we can prove more generally that if u_0 is measurable and satisfies an inequality

$$|u_0(x)| \leqslant Me^{A|x|^2}$$

for all x with constant A, then u given by (3.16) is a solution of $u_t=\Delta u$ of class C^∞ for $0 < t < 1/4A$ and $\lim_{(x,t)\to(x^0,0)} u(x,t)=u_0(x^0)$ at every point x^0 of continuity of u_0. In particular u_0 can have "jump discontinuities". Since for $t > 0$ the value of $u(x,t)$ for any x depends on the value of $u_0(y)$ for all y through an integral, these initial discontinuities smooth out instantaneously. Any finite "local point disturbance" u_0 at the time $t=0$ is not noticeable anywhere a moment later. Confined initial disturbances are felt everywhere a moment later. Here effects travel with infinite speed, indicating that the applicability of the diffusion or heat equation to a physical situation may be limited. Also for the solution (3.16) with initial values $u_0(x)$

$$\begin{aligned} |u(x,t)| &\leqslant \left(\int_{-\infty}^{\infty} K(x-y,t)\,dy\right) \sup_y |u_0(y)| \\ &\leqslant \sup_y |u_0(y)|. \end{aligned} \tag{3.25}$$

At no time can the value of u exceed the maximnm value of the initial distribution. This represents a maximum principle for the solutions of the heat equation.

A.N. Tikhonov proved the uniqueness theorem for the initial value problem of the equation of heat, which we shall discuss here.

Theorem 3.2 If $u_1(x, t)$ and $u_2(x, t)$ satisfy the heat equation in $0 < t \leqslant c$ (c, a constant) and have continuous first and second derivatives in a given region and if both tend to $u_0(x^0)$ as $(x, t) \to (x^0, 0)$ for all values of x^0 and if

$$|u_1(x,t)| < Me^{Ax^2} \text{ and } |u_2(x,t)| < Me^{Ax^2}$$

for some positive A, M, then $u_1(x, t)$ and $u_2(x, t)$ are identical in the strip.

The proof of this is based on two auxiliary theorems. Let D be the set of points in the (x, t)-plane satisfying the relations $-R < x < R$, $0 < t \leqslant c$. We define the boundary ∂D to consist of $x = \pm R$, $0 \leqslant t \leqslant c$ and $t=0$, $-R \leqslant x \leqslant R$.

Theorem 3.3 If $u(x, t)$ satisfies the diffusion equation in D and if

$$\liminf_{(x,\,t)\to(x^0,\,t^0)} u(x,t) \geqslant 0$$

for every point (x^0, t^0) of ∂D then $u(x, t) \geqslant 0$ in D.

Since ∂D is a bounded closed set, given $\epsilon > 0$, we can find a δ such that $u(x, t) > -\epsilon$ whenever (x, t) of D lies in an open disc with centre (x_0, t_0) and radius δ, where δ is independent of (x^0, t^0). Suppose the theorem is not true; then there exists a point (x_1, t_1) of D at which u is negative, i.e.

$$u(x_1, t_1) = -l < 0 \text{ for some } l > 0.$$

Let

$$v(x,t) = u(x,t) + \kappa(t - t_1)$$

where κ is a constant. Then v satisfies the equation

$$v_{xx} = v_t - \kappa.$$

Choose κ so that $0 < \kappa < l/t_1$ and let

$$0 < \epsilon < l - \kappa t_1.$$

For all (x, t) at a distance less than δ from ∂D

$$v(x,t) > -\epsilon + \kappa(t - t_1) > -l + \kappa t > -l.$$

Since $v(x_1, t_1) = -l$, the minimum of $v(x, t)$ cannot exceed $-l$ and so is attained at a point (x_2, t_2) of D. This is impossible, for at a minimum point $\partial v/\partial t = 0$ (or the one-sided derivative $\partial v/\partial t < 0$ if $t_2 = c$) and $\partial^2 v/\partial x^2 \geqslant 0$, yet

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} - \kappa$$

where $\kappa > 0$. Hence the assumption that $u(x, t)$ is negative at a point of D is false. Therefore

$$u(x, t) \geqslant 0 \text{ in } D. \tag{3.26}$$

Theorem 3.4 If $u(x, t)$ satisfies the diffusion equation in the strip $0 < t \leqslant c$ (c, a constant) and if $\lim_{(x, t)\to(x^0, 0)} u(x, t) = 0$ for all x^0 and if $| u(x, t) | \leqslant Me^{Ax^2}$ for some positive values of A, M, then $u(x, t)$ is identically zero in the strip.

Proof

Let
$$F(x) = \sup_{0<t\leqslant c} | u(x, t) | .$$

Then
$$F(x) \leqslant Me^{Ax^2}$$

for all x. Consider the function

$$U(x, t) = F(-R)K(x+R, t) + F(R)K(x-R, t). \tag{3.27}$$

$U(x, t)$ satisfies the heat equation in the region $-R < x < R$, $0 < t \leqslant c$ and

$$U(R, t) \geqslant F(R)K(0, t)) = \frac{F(R)}{2\sqrt{\pi t}}$$

If $0 < t \leqslant c$,

$$| u(R, t) | \leqslant F(R) \leqslant \frac{F(R)\sqrt{c}}{\sqrt{t}} < 2U(R, t)\sqrt{\pi c}$$

which implies

$$2\sqrt{\pi c}U(R, t) \pm u(R, t) \geqslant 0 \tag{3.28}$$

A similar result holds for R replaced by $-R$.

Let
$$w_1(x, t) = 2\sqrt{\pi c}U(x, t) + u(x, t) \tag{3.29}$$

$w_1(x, t)$ satisfies the heat equation in $-R < x < R$, $0 < t \leqslant c$ and is non-negative when $x = \pm R$, $0 < t \leqslant c$. If $(x, t) \to (x_0, 0)$ where $-R < x_0 < R$, then $w_1(x, t) \to 0$. Hence

$$\liminf_{(x,t)\to(x_0,t_0)} w_1(x, t) \geqslant 0$$

for every point of the boundary ∂D and by Theorem 3.3 $w_1(x, t) \geqslant 0$ in D. The same holds for

$$w_2(x, t) = 2\sqrt{\pi c}U(x, t) - u(x, t) \tag{3.30}$$

and we can show that

$$w_2(x, t) \geqslant 0 \text{ in } D.$$

Therefore

$$| u(x, t) | \leqslant 2\sqrt{\pi} cU(x, t)$$

in D.

Also,

$$F(R)K(x-R, t) \leqslant \frac{M}{2\sqrt{\pi t}} \exp\left\{AR^2 - \frac{(R-x)^2}{4t}\right\}$$

and a similar result holds for $F(-R)K(x+R, t)$. Hence $U(x, t) \to 0$ as $R \to \infty$ in $0 < t < 1/4A$. Since $u(x, t)$ is independent of R, and since $U(x, t) \to 0$, $R \to \infty$, $0 < t < 1/(4A)$ if $0 < c \leqslant 1/(4A)$, $u(x, t)$ is equal to zero for $0 < t \leqslant c$.

The argument can be repeated for the function $U(x, c+t)$ where $c < 1/(4A)$. The conclusion will be $u(x, t) \equiv 0$ for $0 < t \leqslant 1/(2A)$. Repeating the argument sufficient number of times we prove that $u(x, t) \equiv 0$ in any strip $0 < t \leqslant c$, where u satisfies the conditions of the theorem.

Proof of Theorem 3.2 We now prove the uniqueness of the initial value problem of the heat equation in the strip $0 < t \leqslant T$. Applying theorem 3.4 to $u_1 - u_2$, we get

$$u_1 = u_2 \text{ for all } x, \ 0 < t \leqslant T. \tag{3.31}$$

Hence the theorem.

The condition $| u(x, t) | < Me^{Ax^2}$ limiting the rate of growth of $u(x, t)$ as $| x | \to \infty$ is of great importance in proving uniqueness of the solution of the initial value problem in the case where $-\infty < x < \infty$. Tikhonov gave the following example of a non-zero function satisfying the diffusion equation with zero initial conditions and continuous for $t \geqslant 0$:

Consider

$$u(x, t) = \sum_{k=0}^{\infty} f^{(k)}(t) \frac{x^{2k}}{(2k)!} \quad \forall x, t \tag{3.32}$$

where the convergence is sufficiently good, so that term by term differentiation is valid.

$$u_{xx} = \sum_{k=1}^{\infty} f^{(k)}(t) \frac{x^{2k-2}}{(2k-2)!} = \sum_{j=0}^{\infty} f^{(j+1)}(t) \frac{x^{2j}}{(2j)!}$$

$$u_t = \sum_{k=0}^{\infty} f^{(k+1)}(t) \frac{x^{2k}}{(2k)!}$$

Therefore $\quad u_{xx} - u_t = 0 \ \forall \ x, t.$

This series certainly converges if $f(t)$ is analytic for all t. Tikhonov, however, set

$$f(t) = \begin{cases} e^{-1/t^2}, & t \neq 0 \\ 0 \ , & t = 0. \end{cases} \tag{3.33}$$

As a function of the real variable t, $f(t)$ is infinitely differentiable but not analytic at $t=0$. Consider the contour $|z-t|=\frac{1}{2}t$ in the complex z-plane. On this contour

$$\frac{1}{z}=\frac{4}{t}\,\frac{1+\dfrac{1}{2}e^{-i\varphi}}{5+4\cos\varphi},\quad 0<\varphi<2\pi$$

$$\mathrm{Re}\frac{1}{z^2}=\frac{16}{t^2}\,\frac{1+\cos\varphi+(\cos 2\varphi)/4}{(5+4\cos\varphi)^2}\geqslant\frac{4}{9t^2}. \tag{3.34}$$

By Cauchy's formula and (3.34)

$$|f^{(k)}(t)|\leqslant\frac{k!}{2\pi}\int_{|z-t|=(1/2)t}\frac{e^{-4/(9t^2)}}{|z-t|^{k+1}}\,dz=\frac{2^k k!}{t^k}\,e^{-4/(9t^2)}$$

It follows that

$$|u(x,t)|\leqslant\sum_{k=0}^{\infty}\frac{2^k k!}{(2k)!}\,\frac{x^{2k}}{t^k}\,e^{-4/(9t^2)}<\sum_{k=0}^{\infty}\frac{1}{k!}\,\frac{x^{2k}}{t^k}e^{-4/(9t^2)}$$

$$=\exp\{x^2/t-4/(9t^2)\}. \tag{3.35}$$

Therefore, $\lim_{t\to 0}u(x,t)=0$ uniformly in x. $u(x,t)$ satisfies the heat equation with zero initial conditions and is continuous for all t, including $t=0$. However this solution is not unique as $u(x,t)\equiv 0$ is a solution with the same initial values. The hypothesis $|u(x,t)|<Me^{Ax^2}$ is obviously violated, where t is such that $0<t\leqslant c$.

The initial value problem for the diffusion equation in an infinitely extending domain for x is well posed, as shown earlier. However, the Cauchy and Dirichlet problems for parabolic equations are not necessarily well posed. We consider an example of each. Consider a solution of the diffusion equation

$$u=y_1^{-N-1}\exp\{i(x\eta_0+ty_1)\} \tag{3.36}$$

where, $y_1>0$; N is a positive integer and η_0 is given by the equation:

$$\eta_0^2+iy_1=0.$$

The solution u corresponding to the root

$$\eta_0=(1-i)\sqrt{y_1/2} \tag{3.37}$$

tends to infinity as $y_1\to\infty$ for $x>0$. At $x=0$, u and its derivatives of order $\leqslant N$ tends to zero as $y_1\to\infty$. Consider now the corresponding Cauchy problem, where the data is prescribed on the curve $x=0$ as

$$u=y_1^{-N-1}\exp(ity_1),\quad u_x=\frac{(1+i)}{\sqrt{2}}y_1^{-N-1/2}\exp(ity_1),\ \forall\ t$$

The solution (3.36) with η_0 given by (3.37) is the required solution of the Cauchy problem. As $y_1\to\infty$, the Cauchy data approaches zero, but the solution for $x > 0$ tends to infinity and diverges from the zero solution. The stability requirement is not satisfied and the problem is not well posed.

Consider the Dirichlet problem for the following parabolic equation:

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} - xu_x - yu_y - u = 0 \tag{3.38}$$

with boundary conditions

$$u(x, \pm 1) = x^2, \; u(\pm 1, y) = y^2.$$

on the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

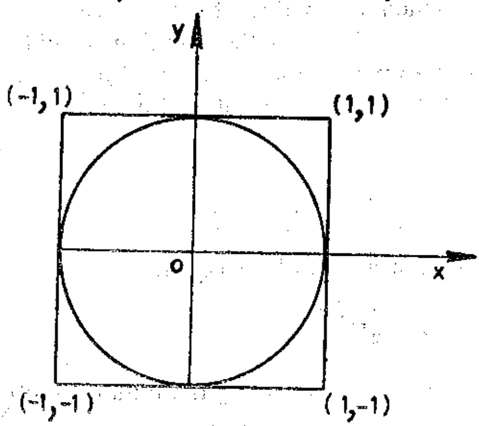

Fig. 3.1 Dirichlet problem for a parabolic equation in a region bounded by $x = \pm 1$, $y = \pm 1$.

This equation is parabolic and its characteristics consist of a single parameter family of concentric circles with centre at the origin. Transforming this equation to Polar coordinates ρ, θ we get the normal form

$$u_{\theta\theta} - u = 0 \tag{3.39}$$

an equation free of derivatives with respect to ρ. On any complete circular path in the square, for single valuedness, the solutions must return to the same value as θ changes by 2π. Both the solutions of equation (3.39) are real exponential functions and these will be periodic only if $u \equiv 0$ on this circular path. Thus $u \equiv 0$ inside the largest circle interior to the square. For the circular paths in the corners we must solve simple linear two-point boundary value problems. The solution in this case is

$$u(x, y) = \begin{cases} 0 & , \text{if } x^2 + y^2 \leq 1 \\ (x^2+y^2-1)\dfrac{\exp \arc \sin \dfrac{|x|}{\sqrt{x^2+y^2}} + \exp \arc \cos \dfrac{|y|}{\sqrt{x^2+y^2}}}{\exp \arc \sin \dfrac{1}{\sqrt{x^2+y^2}} + \exp \arc \cos \dfrac{1}{\sqrt{x^2+y^2}}} & \\ , \text{if } x^2+y^2 > 1, \; -1 < x < 1, \; -1 < y < 1. & \end{cases} \tag{3.40}$$

The solution is identically equal to zero in the largest circle interior to the square and it differs from zero in the corners only.

In the case of this particular example the boundary data and the solution at the corners merge smoothly into the solution $u=0$, at the core, $x^2+y^2 \leqslant 1$. This is because the prescribed data tend to zero as we approach the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ which are the points of contact of the square boundary and the largest circle inscribed in it. If, however, the data were prescribed arbitrarily on the square boundary (as is the case in a Dirichlet problem), so that the boundary data do not tend to zero as we approach the four points $(\pm 1, 0)$, $(0, \pm 1)$, then the solution would breakdown in the neighbourhood of these points, that is, it would no longer be a genuine solution. The Dirichlet problem is then not well posed.

EXERCISE 3.1

1. Show that the solution $u(x, t)$ of the equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+xu$$

for $-\infty < x < \infty$, $t > 0$ satisfying the initial condition

$$u(x, 0)=f(x)$$

is

$$u(x, t)=\frac{1}{2\sqrt{\pi t}} \exp\left(\frac{1}{3}t^3-xt\right)\int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{1}{4}\,\frac{(x-\xi-t^2)^2}{t}\right) d\xi.$$

2. Prove that the solution of the diffusion equation in $t \geqslant 0$ satisfying the conditions

$$u(x, 0)=1,\ x > 0;\ u(x, 0)=-1,\ x < 0$$

with discontinuity at $t=0$, is

$$u(x, t)=erf\ (x\sqrt{4t})$$

where the error function, $f(\xi)$, is defined by

$$erf\,(\xi)=(2/\sqrt{\pi})\int_0^{\xi} \exp(-\eta^2)\, d\eta.$$

§3.2 Initial-boundary Value Problems for a Semi-infinite Domain

These problems would arise in the flow of heat in a semi-infinite rod. This requires the specification of an initial condition, a boundary condition at the finite end $x=0$ and an order condition at infinity. The initial condition on the temperature distribution $u(x, t)$ can be taken to be

$$u(x, 0)=u_0(x), \quad 0 < x < \infty$$

where $|\, u_0(x)\, | < Me^{Ax^2}$ with M and A as constants. There are various boundary conditions that can be preseribed at the end $x=0$:

(i) The temperature is prescribed at $x=0$ for all time, i.e.

$$u(0, t)=f(t). \tag{3.41}$$

(ii) The flux of heat across $x=0$ is prescribed for all time, i.e.

$$u_x(0, t)=g(t). \tag{3.42}$$

(iii) The flux of heat across $x=0$ is proportional to the difference between the temperature at $x=0$ and the surrounding medium, i.e.

$$u_x(0, t)+\alpha u(0, t)=\text{constant}. \tag{3.43}$$

The order condition at infinity is eimilar to that in the theorem 3.2.

We define a function $h(x, t)$, called the derived singularity function:

$$h(x, t)=\frac{x}{t}K(x, t)=-2K_x(x, t), \quad t>0, x \geqslant 0. \tag{3.44}$$

The function $h(x, t)$ has the following properties:

1. It satisfies the heat equation for $t>0$ since
$$h_{xx}-h_t=-2\frac{\partial}{\partial x}(K_{xx}-K_t)=0 \tag{3.45}$$
2. $h>0$ for $t>0$
3. $h(x, t) \in C^\infty$ for $t>0$ and $h(x, 0+)=h(x, +\infty)=0$ for $x>0$.
4. $\int_0^\infty h(x, t)\, dt=1.$ (3.46)
5. $\lim_{x\to 0+} \int_0^c h(x, t)\, dt=1,\ c>0.$ (3.47)

Properties (4) and (5) follow from the relation

$$\int_0^c h(x, t)\, dt=erfc(x\sqrt{4c})=\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4c}}^\infty e^{-y^2}\, dy$$

where the complementary error function, $erfc\ (\xi)$, is defined by

$$erfc\ (\xi)=1-erf\ (\xi).$$

Theorem 3.5

If $f(t) \in C$ for $0<t<\infty$ and $f(0)=0$, then

$$u(x, t)=\int_0^t h(x, t-y)f(y)\, dy \tag{3.48}$$

satisfies the heat equation in $0<x<\infty, 0<t<\infty$

and (i) $\lim_{x\to 0+} u(x, t)=f(t), \quad 0 \leqslant t \leqslant c$
(ii) $\lim_{t\to 0+} u(x, t)=0, \quad 0 \leqslant x<\infty$ } uniformly

where c is a constant.

Proof Because $h(x, t)$ satisfies the heat equation for $t>0$, so does $u(x, t)$, since differentiation under the integral sign is permissible. We shall first prove (ii). Given ϵ we can determine a δ such that

$$|f(t)|<\epsilon, 0<t<\delta.$$

Hence, for $0 < t < \delta$,

$$| u(x, t) | = \left| \int_0^t h(x, t-y)f(y)\, dy \right| \leqslant \int_0^t h(x, \xi) \, | f(t-\xi) | \, d\xi$$

$$< \epsilon \int_0^\infty h(x, \xi)\, d\xi = \epsilon. \tag{3.49}$$

For a specific value of δ, (3.49) holds for all x. Thus (ii) is proved.

To prove (i) we have to determine η independent of t on $0 \leqslant t \leqslant c$ such that when $0 < x < \eta$

$$| u(x, t) - f(t) | < \epsilon.$$

For a given ϵ, let δ_1 be such that

$$| f(\xi) | < \epsilon/2 \tag{3.50}$$

whenever $| \xi | < \delta_1$. Due to the uniform continuity of $f(t)$ in $0 \leqslant t \leqslant c$, we can choose δ_1, so that we also have

$$| f(t-\xi) - f(t) | < \epsilon/2 \text{ whenever } | \xi | < \delta_1, \; 0 \leqslant t \leqslant c. \tag{3.51}$$

Subdivide $0 \leqslant t \leqslant c$ into two subintervals:

$0 \leqslant t \leqslant \delta_1$ and $\delta_1 \leqslant t \leqslant c$. In $0 \leqslant t \leqslant \delta_1$ using (3.49) and (3.50), we have

$$| u(x, t) - f(t) | < \epsilon \text{ for all } x > 0. \tag{3.52}$$

In $\delta_1 \leqslant t \leqslant c$,

$$\begin{aligned}
u(x, t) - f(t) &= \int_0^t h(x, t-y) f(y)\, dy - \int_0^\infty h(x, y) f(t)\, dy \\
&= \int_0^t h(x, \xi) f(t-\xi)\, d\xi - \int_0^\infty h(x, y) f(t) dy \\
&= \int_0^t h(x, y)[f(t-y) - f(t)]\, dy - f(t) \int_t^\infty h(x, y)\, dy \\
&= \int_0^{\delta_1} h(x, y)[f(t-y) - f(t)]\, dy \\
&\quad + \int_{\delta_1}^t h(x, y)[f(t-y) - f(t)]\, dy - f(t) \int_t^\infty h(x, y)\, dy.
\end{aligned}$$

Therefore

$$| u(x, t) - f(t) | < \epsilon/2 \int_0^\infty h(x, y)\, dy + 3M \int_{\delta_1}^\infty h(x, y)\, dy \tag{3.53}$$

where $M = \sup\limits_{0 \leqslant t \leqslant c} | f(t) |$.

Since from (3.47)

$$\lim_{x \to 0+} \int_0^c h(x, y)\, dy = 1, \text{ we have } \left| \int_{\delta_1}^\infty h(x, y)\, dy \right| < \frac{\epsilon}{6M} \tag{3.54}$$

for sufficiently small x, i.e. $0 < x < \eta$, which is independent of t. Combining (3.52)-(3.54), we have the required result

$$|u(x, t)-f(t)| < \epsilon \quad \text{for } 0 < x < \eta \text{ and } 0 \leqslant t \leqslant c. \tag{3.55}$$

In the case when $f(0) \neq 0$, the initial and boundary conditions are still satisfied by the solution (3.48). We still have

$$u(0+, t)=f(t), \quad 0 < t \leqslant c$$
$$u(x, 0+)=0, \quad 0 < x < \infty$$

but these limits are not attained uniformly in x and t.

The solution (3.48), namely

$$u(x, t)=\int_0^t h(x, t-y)f(y)\, dy$$

can be interpreted as the solution of problem (i) namely the problem of finding the temperature of a semi-infinite bar, whose initial temperature at $t=0$ is everywhere zero and whose temperature at the finite end $x=0$ is prescribed for all time t as $f(t)$.

The second problem (ii) regarding the temperature of a semi-infinite bar arises when the initial temperature at $t=0$ is everywhere zero and the heat flux at the finite end $x=0$ is prescribed for all time t as $g(t)$, $g(0)=0$. In this scase

$$u(x, t)=-2\int_0^t K(x, t-y)g(y)\, dy \tag{3.56}$$

is the solution for $0 < x < \infty$, $0 < t < c$, where c is a constant.

As in the preceding theorem, because K satisfies the heat equation, so does u. As in (3.49) we have

$$\lim_{t\to 0+} u(x, 0+)=0 \quad \text{for } 0 \leqslant x < \infty. \tag{3.57}$$

Also

$$u_x(x, t)=-2\int_0^t K_x(x, t-y)g(y)\, dy$$
$$=\int_0^t h(x, t-y)g(y)\, dy.$$

From (3.55),

$$u_x(x, t) \to g(t) \text{ as } x \to 0+, 0 < t \leqslant c. \tag{3.58}$$

For solution of problem (iii), see Exercise 3.2, Problem 1.

In problems (i), (ii) and (iii), so far the initial temperature distribution is taken to be zero. We now consider the case of non-zero initial distribution and zero boundary conditions at $x=0$, namely,

$$u(x, 0)=u_0(x) \quad 0 \leqslant x < \infty$$

with

$$|u_0(x)| < Me^{Ax^2}$$

and one of the following boundary conditions:

(iv) $u(0, t) = 0, \quad 0 < t < \infty$

(v) $u_x(0, t) = 0, \quad 0 < t < \infty$

(vi) $u_x(0, t) + \alpha u(0, t) = 0, \quad 0 < t < \infty.$

In problems (iv) to (vi), we attempt to extend $u_0(x)$ for $x < 0$ in such a way that the function $u(x, t)$ has the correct boundary values.

In problem (iv), we extend $u_0(x)$ as an odd function:

$$u_0(x) = -u_0(-x) \text{ for } x < 0.$$

Then the solution of the initial value problem in $-\infty < x < \infty$ is given by (3.16) i.e.

$$u(x, t) = \int_{-\infty}^{\infty} K(x-y, t)u_0(y)\, dy$$

$$= \int_0^{\infty} [K(x-y, t) - K(x+y, t)]u_0(y)\, dy. \tag{3.59}$$

Since $K(0-y, t) - K(0+y, t) = 0$, the boundary condition $u(0+, t) = 0$ is automatically satisfied. We may take the limit as $x \to 0$, under the integral sign, since it is uniformly convergent in x and t for $0 < t \leqslant c$, with c arbitrary.

In problem (v) we extend $u_0(x)$ as an even function of x for $x < 0$: $u_0(x) = u_0(-x)$, $x < 0$. Then the solution is given by

$$u(x, t) = \int_0^{\infty} [K(x-y, t) + K(x+y, t)]u_0(y)\, dy. \tag{3.60}$$

Here $K_x(0-y, t) + K_x(0+y, t) = 0$ so that

$$u_x(0+, t) = 0 \text{ for } 0 < t < \infty.$$

In problem (vi) we extend $u_0(x)$ as follows:

$$u_0(-x) = u_0(x) + 2\alpha e^{\alpha x} \int_0^x e^{-\alpha\xi} u_0(\xi)\, d\xi \text{ for } x \geqslant 0. \tag{3.61}$$

in order to satisfy the boundary condition

$$u_x(0+, t) + \alpha u(0+, t) = 0, \quad 0 < t < \infty.$$

The solution in this case is given by

$$u(x, t) = \int_0^{\infty} u_0(y)\{K(x-y, t) + K(x+y, t)\}\, dy$$

$$+ 2\alpha \int_0^{\infty} K(x+y, t) e^{\alpha y} \int_0^y e^{-\alpha\xi} u_0(\xi)\, dy\, d\xi. \tag{3.62}$$

A linear combination or superposition of solutions of one of the problems (i), (ii), (iii) with $u_0(x) = 0$ and one of (iv), (v), (vi) lead us to the general mixed initial-boundary value problems for the heat equation for semi-infinite bar.

Consider the two problems associated with the heat equation for the semi-infinite bar:

$$\text{(vii)}\qquad \begin{aligned} &u_{xx}-u_t=0, && 0<x,t<\infty \\ &u(x,0)=u_1(x), && 0<x<\infty \\ &u(0+,t)=f(t), && 0\leqslant t\leqslant c. \end{aligned} \tag{3.63}$$

This is a combination of problems (i) and (iv) and due to the linear nature of the equation and the initial-boundary values, the solution of problem (vii) is a linear combination of the solutions of problems (i) and (iv):

$$u(x,t)=\int_0^\infty [K(x-y,t)-K(x+y,t)]u_0(y)\,dy+\int_0^t h(x,t-y)f(y)\,dy,$$
$$0<x<\infty,\quad 0\leqslant t\leqslant c. \tag{3.64}$$

Similarly, we consider the case where the initial distribution and the flux is prescribed at $x=0$

$$\text{(viii)}\qquad \begin{aligned} &u(x,0)=u_0(x) && 0<x<\infty \\ &u_x(0+,t)=g(t) && 0\leqslant t\leqslant c. \end{aligned} \tag{3.65}$$

This is a combination of problems (ii) and (v) and the solution is

$$u(x,t)=\int_0^\infty [K(x-y,t)+K(x+y,t)]u_0(y)\,dy-2\int_0^t K(x,t-y)g(y)\,dy,$$
$$0<x<\infty,\ 0\leqslant t\leqslant c. \tag{3.66}$$

EXERCISE 3.2

1. $u(x, t)$ is the solution of the diffusion equation satisfying the conditions

$$u(x,0)=0,\ x\geqslant 0;\quad u_x(0,t)-\alpha u(0,t)=\varphi(t),\ t\geqslant 0$$

where α is a positive constant and $\varphi(t)$ is continuous. Prove that when $x>0$, $t>0$

$$\frac{\partial u}{\partial x}-\alpha u=\int_0^t \varphi(\tau)h(x,t-\tau)\,d\tau.$$

Hence show that

$$u(x,t)=-\int_0^\infty e^{-\alpha\xi}\int_0^t \frac{x+\xi}{\tau}\varphi(t-\tau)K(x+\xi,\tau)\,d\xi\,d\tau.$$

2. Prove that the solution of the diffusion equation in $x\geqslant 0$, $t\geqslant 0$ satisfying the conditions

$$u(x,0)=0(x>0),\quad u(0,t)=1\quad (0<t<T),\quad u(0,t)=0(t>T)$$

is $$u(x,t)=\frac{2}{\sqrt{\pi}}\,erfc\,\frac{x}{\sqrt{t}}$$

$$=\frac{2}{\sqrt{\pi}}\left[erfc\,\frac{x}{2\sqrt{t}}-erfc\,\frac{x}{2\sqrt{(t-T)}}\right],\ t>T.$$

§3.3 Initial-boundary Value Problem for Heat Conduction in a Finite Bar

The case of the heat conduction in a finite bar can be studied using the previously derived results by inducing in the problem a certain periodicity in x over the whole space. Consider a finite bar of length π, with two ends at $x=0$, $x=\pi$. An initial-boundary value problem associated with it is the following

(ix) $$u_{xx}-u_t=0, \qquad 0<x<\pi, \quad 0<t\leqslant c$$

with

$$u(x,0)=u_0(x), \qquad 0<x<\pi$$

and

$$\left.\begin{aligned} u(0+,t)&=f_1(t)\\ u(\pi-,t)&=f_2(t)\end{aligned}\right\} \quad 0<t\leqslant c. \tag{3.67}$$

To study this we introduce two theta functions of Jacobi

$$\left.\begin{aligned}\theta(x,t)&=\sum_{-\infty}^{\infty}K(x+2n\pi,t)\\ \phi(x,t)&=\sum_{-\infty}^{\infty}h(x+2n\pi,t).\end{aligned}\right\} t>0 \tag{3.68}$$

These are sums of the values of the singularity function and the derived singularity function, respectively. These series converge for all x and t and their sums satisfy the diffusion equation. Both θ and φ are periodic in x and of period 2π.

Theorem 3.6 If $u_0(x)\in C$ on $0\leqslant x\leqslant\pi$, $f_1(t), f_2(t)\in C$ on $0\leqslant t<\infty$ and $f_1(0)=u_0(0)$, $f_2(0)=u_0(\pi)$, then the solution of problem (ix) is

$$u(x,t)=\int_0^{\pi}[\theta(x-y,t)-\theta(x+y,t)]u_0(y)\,dy+\int_0^t\varphi(x,t-y)f_1(y)\,dy$$
$$+\int_0^t\varphi(\pi-x,t-y)f_2(y)\,dy. \tag{3.69}$$

The existence of the above solution and necessary convergence properties are discussed in Widder (1975). To establish uniqueness is simple in this case. Let, if possible, u_1, u_2 be two solutions with the same initial and boundary values. Set $u=u_1-u_2$.

Let $$I(t)=\frac{1}{2}\int_0^{\pi}u^2(x,t)\,dx. \tag{3.70}$$

Then

$$I'(t) = \int_0^{\pi} u u_t \, dx = \int_0^{\pi} u u_{xx} \, dx$$

$$= [u\, u_x]_0^{\pi} - \int_0^{\pi} u_x^2 \, dx.$$

Since the boundary values are the same for both u_1 and u_2, the first term on the right vanishes and

$$I'(t) = -\int_0^{\pi} u_x^2 \, dx.$$

For $t \geqslant 0$, $$I'(t) \leqslant 0.$$

Now $I(0) = \frac{1}{2}\int_0^{\pi} \{u(x, 0)\}^2 \, dx = 0$, since $u(x, 0) = 0$ in $0 \leqslant x \leqslant \pi$. Further $I'(t) \leqslant 0$. this implies that for every $t \geqslant 0$, $I(t) \leqslant 0$. But $I(t)$ has a positive integrand and necessarily $I(t) \geqslant 0$. Therefore

$$I(t) = 0 \tag{3.71}$$

which requires

$$u = 0, \quad \text{for} \quad 0 \leqslant x \leqslant \pi, \quad 0 \leqslant t \leqslant c. \tag{3.72}$$

The uniqueness can be proved by using the maximum-minimum principle as well.

EXERCISE 3.3

1. Find the solution of the diffusion equation for $0 < x < \pi$, $t > 0$ such that

   $u(x, 0) = 0$, $0 < x < \pi$; $u(0, t) = 1 - e^{-t}$, $t > 0$, $u(\pi, t) = 0$, $t > 0$.

   What happens to this solution as $t \to \infty$?
2. $u(x, t)$ satisfies the diffusion equation in $0 \leqslant x \leqslant a$, $t \geqslant 0$ under the conditions

   $$u(x, 0) = f(x), \; 0 < x < a$$
   $$u_x(0, t) = u_x(a, t) = 0, \; t > 0.$$

   Find $u(x, t)$ using the Fourier half range cosine series for $f(x)$.

*§3.4 Maximum-Minimum Principle for the Heat Equation and for Some Parabolic Equations

Let ∂D_1 denote an interval on the line $t = T$ excluding the end points A and B (Fig. 3.2). ∂D_1 is open. Let ∂D_2 be a curve joining A and B lying in the region, below $t=T$. ∂D_2 includes the points A and B and is closed. Let D be the region bounded by $\partial D_1+\partial D_2$.

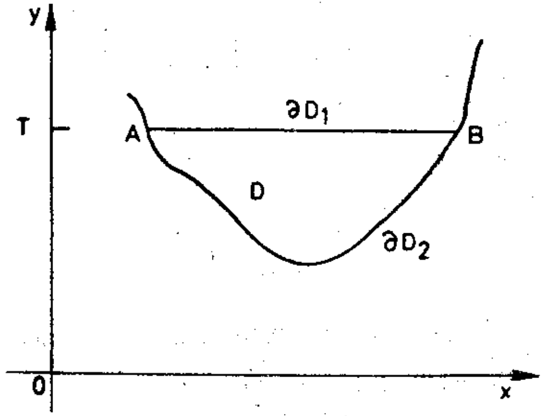

Fig. 3.2 D is a region bounded by ∂D_1 and ∂D_2 with $t=T$ on ∂D_1

Theorem 3.7 Consider a solution $u(x, t)$ of the diffusion equation $u_{xx}=u_t$ in the domain D. Let $u(x, t) \in C^0$ in $D+\partial D_1+\partial D_2$ with u_{xx}, $u_t \in C^0$ in $D+\partial D_1$. Then $u(x, t)$ assumes its maximum and minimum on ∂D_2.

Proof For arbitrary $\epsilon > 0$, consider

$$v(x, t)=u(x, t)-\epsilon t \tag{3.73}$$

v is continuous in the closed region $D+\partial D_1+\partial D_2$ and so must assume its maximum and minimum in the region. Suppose the maximum of v is attained at $(x_0, t_0) \in D+\partial D_1$. For sufficiently small $\delta > 0$, the points (x, t_0) with $x_0-\delta \leqslant x \leqslant x_0+\delta$ lie in $D+\partial D_1$. At (x_0, t_0), $v_{xx}(x_0, t_0) \leqslant 0$ and

$$v_{xx}-v_t=u_{xx}-u_t+\epsilon=\epsilon, \text{ i.e. } v_t(x_0, t_0) \leqslant -\epsilon. \tag{3.74}$$

Because of the continuity of v_t in $D+\partial D_1$, a number $\delta_1 > 0$ can be chosen so small that $v_t(x_0, t) < -\epsilon/2$ for $t_0-\delta_1 \leqslant t \leqslant t_0$.

$$v(x_0, t_0)-v(x_0, t_0-\delta_1)=\int_{t_0-\delta_1}^{t_0} v_t(x_0, t)\, dt \leqslant -\frac{\epsilon}{2}\delta_1 < 0. \tag{3.75}$$

Equations (3.75) contradicts our assumption that v attains its maximum at (x_0, t_0). The maximum of v must lie on ∂D_2. Since ϵ is arbitrarily small, the maximum of u must also lie on ∂D_2 (from (3.73)). Similarly we can show that the minimum of u lies on ∂D_2. This proves the theorem.

We now consider the case when the boundary ∂D_2 consists of the three sides of a rectangle: $t=0$, $a \leqslant x \leqslant b$; $x=a$ and $x=b$, $0 \leqslant t \leqslant T$. For points on $x=a$ and $x=b$, the same argument as above can be applied and we can show that the maximum and minimum of u must occur only on $t=0$, $a \leqslant x \leqslant b$, i.e. in the initial data.

Consider a more general parabolic equation

$$u_{xx}-u_t+e(x, t)u=f(x, t) \tag{3.76}$$

in the two independent variables x and t in the domain $D+\partial D_1+\partial D_2$ as defined in Theorem 3.7.

Theorem 3.8 If $e<0$, or $f\leqslant 0$ (or $f\geqslant 0$) in $D+\partial D_1+\partial D_2$ domain, then if a nonconstant solution possesses a negative minimum (or positive maximum), it must attain this value on ∂D_2.

Proof Suppose a negative minimum of u is assumed at $(x_0, t_0)\in D$. Then

$$u(x_0, t_0)<0$$

$$u_x(x_0, t_0)=u_t(x_0, t_0)=0,\ u_{xx}(x_0, t_0)\geqslant 0.$$

According to the equation

$$u_{xx}(x_0, t_0)=u_t(x_0, t_0)-e(x_0, t_0)u(x_0, t_0)+f(x_0, t_0)<0.$$

This is a contradiction. Hence it cannot be attained in D. Now suppose that a negative minimum is attained at $(x_0, T)\in\partial D_1$. Then $u(x_0, T)<0$, $u_x(x_0, T)=0$, $u_t(x_0, T)\leqslant 0$, $u_{xx}(x_0, T)\geqslant 0$. This also leads to a contradiction. Hence the negative minimum must occur on ∂D_2. Similarly the theorem can be proved for a positive maximum if $e<0, f\geqslant 0$.

EXERCISE 3.4

1. Let u be a solution of the mixed initial-boundary value problem:

$$u_{xx}-u_t=0,\quad 0<x<l, 0<t<T$$

$$u(x, 0)=f(x),\ 0\leqslant x\leqslant l$$

$$u(0, t)=u(l, t)=0,\ 0\leqslant t\leqslant T$$

describing the distribution of temperature in a rod of length l. Show that u attains its maximum and minimum values at $t=0$ and use this result to establish a uniqueness theorem for this problem.

§4 THE WAVE EQUATION

In the earlier discussion of elliptic and parabolic equations the following properties were noted:

1. All properties derived for the Laplace equation in two space variables or the diffusion equation in one space variable and a time variable, also hold in the case of many space variables and can be easily extended to latter case. There is no decisive dependence on dimension of the space.

2. Smoothness properties of the boundary and initial values are enhanced in the region where the solution exists. For example, in a Dirichlet problem

if the boundary data $\in C^0$ on the boundary ∂D, it belongs to C^∞ in D. Also in the diffusion equation for an infinite bar, even if the initial data has certain finite jump discontinuities, the solution belongs to C^∞ for $t > 0$.

3. Confined perturbations of initial and boundary values do not remain confined. They are felt everywhere where the solution exists.

All these properties are violated in the case of hyperbolic equations of which the wave equation is an example. There is a distinct difference in the behaviour of the solution of the wave equation in one, two and three space dimensions. Certain permitted singularities of the boundary and initial data will be carried into the solution, which can support discontinuities as well. Besides, confined initial disturbances remain confined in space for all time.

These properties merit a detailed study of the wave equation. In this section, we shall restrict our attention to the wave equation in one, two and three space dimensions.

§4.1 The One-dimensional Wave Equation

The wave equation in one space dimension is of the form

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0, \quad c^2 = \text{constant} > 0. \tag{4.1}$$

This can be reduced to the characteristic form with the help of the transformation

$$\xi = x + ct, \ \eta = x - ct. \tag{4.2}$$

We then have

$$u_{\xi\eta} = 0.$$

The general solution of (4.1) is

$$u = f(x+ct) + g(x-ct) \tag{4.4}$$

where f and g are arbitrary c^2 functions of their arguments. If x and t denote a space coordinate and time, respectively, then the term $f(x+ct)$ represents a wave moving with constant speed c in the negative x direction and the term $g(x-ct)$, a wave moving with the same speed in the positive x-direction.

(a) *Initial value problem for the wave equation*

An initial value problem or a Cauchy problem for the wave equation consists in finding a solution in the upper half of the (x, t)-plane satisfying the conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad -\infty < x < \infty.$$

Theorem 4.1 If $u_0(x) \in C^2$, and $u_1(x) \in C^1$ in $-\infty < x < \infty$, then the function

$$u(x, t) = M(t)u_1 + \frac{\partial}{\partial t}[M(t)u_0] \tag{4.5}$$

where

$$M(t)u_i=\frac{1}{2c}\int_{x-ct}^{x+ct} u_i(\tau)\, d\tau,\quad i=0,1 \tag{4.6}$$

belongs to C^2 in $-\infty < x < \infty, 0 < t < \infty$ and is a solution of the Cauchy problem

$$u_{xx}-\frac{1}{c^2}u_{tt}=0,\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x). \tag{4.7}$$

Proof Using the general solution (4.4), we have

$$u_0(x)=f(x)+g(x)$$

$$u_1(x)=cf'(x)-cg'(x).$$

This gives

$$f(x)=\frac{u_0(x)}{2}+\frac{1}{2c}\int_0^x u_1(\tau)\, d\tau+\delta_1$$

$$g(x)=\frac{u_0(x)}{2}-\frac{1}{2c}\int_0^x u_1(\tau)\, d\tau+\delta_2$$

where $\delta_1+\delta_2=f(x)+g(x)-u_0(x)=0.$

Hence

$$u(x,t)=\frac{1}{2}\left[u_0(x+ct)+u_0(x-ct)+\frac{1}{c}\int_{x-ct}^{x+ct} u_1(\tau)\, d\tau\right]. \tag{4.8}$$

(4.8) is precisely (4.5), which is written in terms of mean values $M(t)u_1$ and $M(t)u_0$ of u_1 and u_0, respectively, on the interval $(x-ct, x+ct)$. (4.8) is called *D'*Alembert's solution of the one-dimensional wave equation.

We next examine the uniqueness and the stability of the solution. It is unique, because we have singled out this solution from the general solution using the initial conditions. Given $\epsilon > 0$, if we change $u_i(x)$ in the interval $a \leqslant x \leqslant b$ to $\bar{u}_i(x)$, so that

$$|\bar{u}_0-u_0|\leq\frac{\varepsilon}{4},\qquad |\bar{u}_1-u_0|\leq\frac{\varepsilon c}{2+(b-a)} \tag{4.9}$$

then the solution $\bar{u}(x, t)$ corresponding to initial values $\bar{u}_i(x)$ differs from $u(x, t)$ by

$$|\ \bar{u}(x,t)-u(x,t)\ | < \epsilon, \text{ for all } t. \tag{4.10}$$

So the solution (4.5) exists, is unique and depends continuously on the initial data.

b/(a) Domains of dependence and influence

The solution at a point (x_0, t_0) does not depend on the global behaviour of the initial values $u_i(x)$ but only on those values of x which lie in the interval $[x_0-ct_0, x_0+ct_0]$. The set $G(x_0, t_0)=\{x: |\ x-x_0\ | \leqslant c\ |\ t_0\ |\ \}$ is called the *domain of dependence* of a given point (x_0, t_0). If we fix an arbitrary interval $I: |\ x-x_0\ | \leqslant r$ of the x-axis, then the set of points

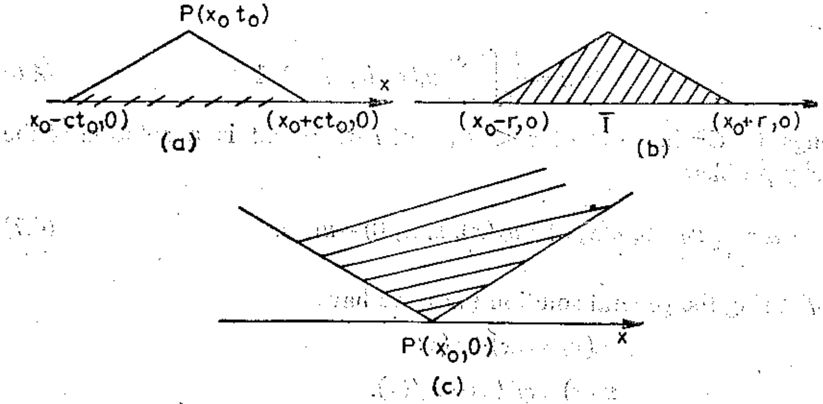

Fig. 4.1 (a) Domain of dependence of P
(b) Domain of determinacy of I
(c) Domain of influence of P

(x, t) in the upper half plane at which $u(x, t)$ depends only on the initial values $u_i(x)$ in $\bar{I}$ is called the *domain of determinancy* $B(\bar{I})$, say and is given by

$$B(\bar{I})=\{(x, t) : |x-x_0| \leqslant r-ct\}.$$

Corresponding to any point P on the intial line $t=0$, there is a set of points (x, t) at which $u(x, t)$ is affected by the initial values $u_i(P)$. This set ($D(x_0)$, say) of points constitutes the *domain of influence* of the point P:

$$D(x_0)=\{(x, t) : |x-x_0| < ct\}.$$

(c) A special initial-boundary value problem

Suppose the initial values $u_i(x)$ are given in $0 \leqslant x \leqslant l$, together with boundary conditions, for example $u(0, t)=u(l, t)=0$ then we try to convert it into an initial value problem by continuing $u_i(x)$ in the interval $-\infty < x < \infty$ so that $u_0(x) \in C^2$ and $u_1(x) \in C^1$ for all x. If this is possible so that the boundary conditions are automatically satisfied, then (4.5) is the solution of the initail-boundary value problem. Now the functions $u_0(x) \in C^2$ and $u_1(x) \in C^1$ in $0 \leqslant x \leqslant l$ are continued as odd periodic functions of period $2l$, so that

$$u_i(x)=-u_i(-x), \quad u_i(x+2l)=u_i(x) \tag{4.11}$$

Then

$$u(0, t)=\frac{1}{2}\left[u_0(ct)+u_0(-ct)+\int_0^{ct} u_1(\tau)\, d\tau+\int_{-ct}^{0} u_1(\tau)\, d\tau\right]$$

$$=\frac{1}{2}\int_0^{ct} [u_1(\tau)+u_1(-\tau)]\, d\tau=0 \tag{4.12}$$

$$u(l, t)=\frac{1}{2}\left[u_0(l+ct)+u_0(l-ct)+\int_0^{l+ct} u_1(\tau)\, d\tau+\int_{l-ct}^{0} u_1(\tau)\, d\tau\right]$$

$$=\frac{1}{2}\left[u_0(l+ct)-u_0(-l+ct)+\int_0^{l+ct} u_1(\tau)\, d\tau+\int_0^{-l+ct} u_1(-\tau)\, d\tau\right].$$

Setting $u_1(-\tau) = -u_1(\tau)$ and using the result $\int_y^{y+2l} u_1(\tau)\, d\tau = 0$ for all y, we get after taking $y = -l + ct$

$$u(l, t) = \frac{1}{2}\left[u_0(l+ct) - u_0(-l+ct+2l) + \frac{1}{c}\int_0^{l+ct} u_1(\tau)\, d\tau - \frac{1}{c}\int_0^{-l+ct+2l} u_1(\tau)\, d\tau\right] = 0. \tag{4.13}$$

(4.5) is the required solution with $u_i(x)$ defined for all x by (4.11).

(d) Generalised or weak solution: Even though $u_0(x)$ and $u_1(x)$ may not satisfy the required conditions specified in Theorem 4.1 [i.e. $u_0(x) \in C^2$, $u_1(x) \in C^1$ in $-\infty < x < \infty$], function (4.5) may still be meaningful as a solution. We shall now explain what is meant by a *generalised* or *weak solution* of the initial value problem.

Example 4.1

Consider an example of the initial value problem (4.7) with

$$u_0(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x < 0, \\ 0 & \text{for } x = 0 \end{cases} \quad u_1(x) = 0 \tag{4.14}$$

In this case $u_0(x) \notin C^2$ in $-\infty < x < \infty$. $u_0(x) \in C^2$ everywhere, except at $x = 0$, where its second derivative is discontinuous. Here it has a jump discontinuity from left to right* denoted by $[u_0''(0)] = u_0''(+0) - u_0''(-0) = 4$. The expression (4.5) when evaluated, is given by

$$u(x, t) = \begin{cases} x^2 + c^2t^2, & x > ct \\ 2ctx, & -ct < x < ct \\ -(x^2 + c^2t^2), & -ct < x. \end{cases} \tag{4.15}$$

$u(x, t) \in C^2$ everywhere except along the two characteristics passing through the point of initial discontinuity, $x = 0$: $[u_{xx}(x - ct, t)] = 2 = [u_{xx}(x + ct, t]$. The initial discontinuity in u_{xx} breaks into two discontinuities each of magnitude equal to the half of the original one and these two parts propagate in opposite directions with velocities $+c$ and $-c$, respectively. Introducing characteristic variables ξ and η, we get

$$u = \frac{1}{2}\begin{cases} (\xi^2 + \eta^2), & \eta > 0 \quad \text{in Region I} \\ (\xi^2 - \eta^2), & \eta < 0 < \xi \text{ in Region II} \\ -(\xi^2 + \eta^2), & \xi < 0 \quad \text{in Region III.} \end{cases} \tag{4.16}$$

*With respect to a reader for whom the positive direction of the x-axis points from left to right. [ ] on a quantity represents the jump in the quantity as one moves towards right.

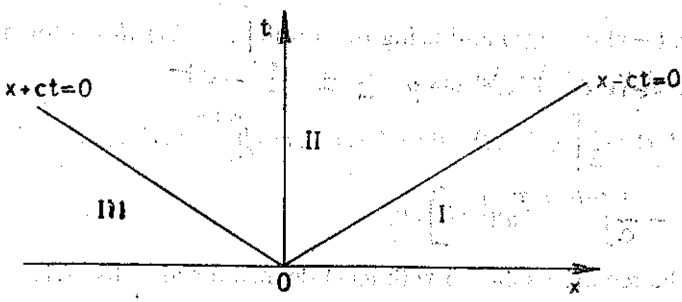

Fig. 4.2 Regions, I, II and III

Then

$$[u_{\eta\eta}]=2,\ [u_{\xi\xi}]=[u_{\xi\eta}]=0 \quad \text{on } \eta=0$$

$$[u_{\xi\xi}]=2,\ [u_{\eta\eta}]=[u_{\xi\eta}]=0 \quad \text{on } \xi=0$$

showing that only the 'exterior' second order derivatives are discontinuous along the two characteristics. $u(x, t)$ given by (4.15) is a 'weak' solution of the problem (4.7), where the solution $\in C^2$ everywhere in the upper half of the (x, t) plane except along the characteristics.

The following theorem helps us to define 'weak' solutions of the wave equation, even when the derivatives may not exist in the ordinary sense.

Theorem 4.2 A function u with continuous second derivatives is a solution of the wave equation

$$u_{xx}-\frac{1}{c^2}u_{tt}=0$$

if and only if u satisfies the difference equation

$$u(A)+u(C)=u(B)+u(D) \tag{4.17}$$

where A, B, C, D are the vertices of any parallelogram, whose sides are characteristics curves (see Fig. 4.3).

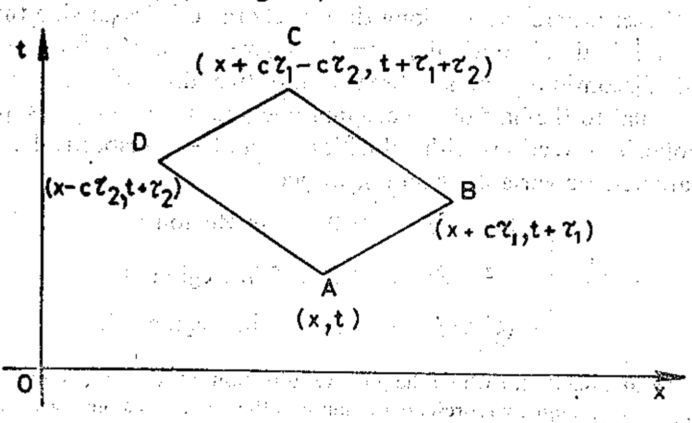

Fig. 4.3 *ABCD* is a parallelogram whose sides are characteristic curves

Proof Suppose u is a solution of the wave equation, then u can be written as

$$u(x, t) = f(x+ct) + g(x-ct).$$

Hence

$$f(A) + f(C) = f(x+ct) + f(x+ct+2c\tau_1)$$
$$= f(D) + f(B).$$

Similarly for g. Therefore u satisfies the difference equation (4.17). Conversely, suppose (4.17) is satisfied and second order derivatives are continuous. Then

$$u(x, t) + u(x+c\tau_1 - c\tau_2, t+\tau_1+\tau_2) - u(x+c\tau_1, t+\tau_1) - u(x-c\tau_2, t+\tau_2) = 0$$

Using a finite Taylor expansion for small τ_1, τ_2 we get

$$u_{xx}(x, t) = \frac{1}{c^2} u_{tt}(x, t).$$

Any function (not necessarily $\in C^2$) satisfying the difference equation (4.17) can be viewed as a *weak solution* of the wave equation.

We shall use Theorem 4.2 to solve an initial-boundary value problem for the wave equation in the domain $0 < x < L$, $t > 0$ of the x, t plane satisfying:

initial conditions: $u(x, 0) = u_0(x)$

$$u_t(x, 0) = u_1(x), \quad 0 < x < L \tag{4.18}$$

and

boundary conditions: $u(0, t) = h_1(t)$ (4.19)

$$u(L, t) = h_2(t), \; t \geq 0.$$

We divide the strip into a number of regions by the characteristics through the corners and through the points of intersections with the boundaries as shown in Fig. 4.4. In region I, u is determined from the initial data by (4.5). At a point A of region II we form the characteristic parallelogram $ABCD$, with B on the boundary and C, D lying in the region I, where the solution

$$u(A) = -u(C) + u(B) + u(D)$$

is already known from (4.17) with $u(B)$ known from (4.19) and $u(C)$, $u(D)$ known from region I. In this way, the solution can be found everywhere in the strip. If the solution is to belong to C^2, then the data have to fit together in the corners, so that u and its first and second derivatives are the same when computed from both (4.18) and (4.19). This leads to the compatibility conditions:

$$u_0(0+) = h_1(0), \; u_1(0+) = h_1'(0), \; c^2 u_0''(0+) = h_1''(0)$$
$$u_0(L-) = h_2(0), \; u_1(L-) = h_2'(0), \; c^2 u_0''(L-) = h_2''(0) \tag{4.20}$$

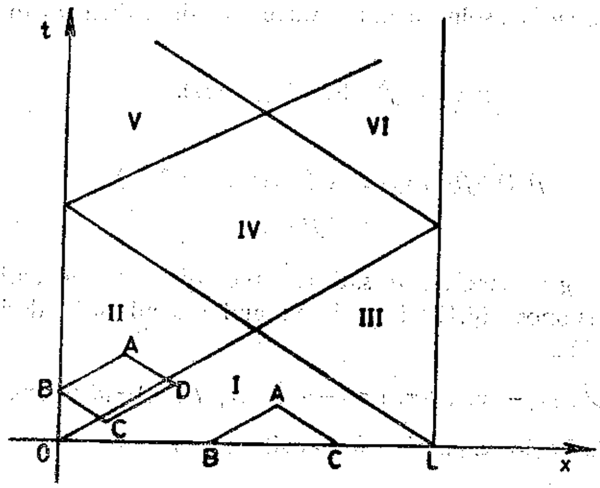

Fig. 4.4 Initial-boundary value problem for the wave equation in $0 \leqslant x \leqslant L, t \geqslant 0$

These conditions are also sufficient, when u_0, h_1, $h_2 \in C^2$ and $u_1 \in C^1$, to make $u \in C^2$. If for example $u_0(0) \neq h_1(0)$, u will have a jump along the characteristic through the corner $x=0$, $t=0$.

EXERCISE 4.1

1. In the theory of acoustics (linearised theory of sound with small disturbances about an equilibrium state) the velocity components u and v, pressure p and density ρ satisfy the following equations:

$$\frac{\partial \rho}{\partial t}+\rho_0\frac{\partial u}{\partial x}+\rho_0\frac{\partial v}{\partial y}=0$$

$$\rho_0\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}=0,\ \rho_0\frac{\partial v}{\partial t}+\frac{\partial p}{\partial y}=0,\ p-p_0=a_0^2(\rho-\rho_0)$$

where the subscript 0 denotes the equilibrium state. Show that each of the quantities u, v, p, ρ satisfy the wave equation with velocity of propagation a_0. Find the solution in the one-dimensional case (when $v=0$ and all quantities are independent of y) given that initially $u(x, 0)=f(x), p(x, 0)=0$.

2. Using an energy integral $I(t)=\int_a^b (u_x^2+u_t^2)\,dx$, show that the solution to the mixed initial and boundary value problem of the one-dimensional wave equation

$$\frac{1}{c^2}u_{tt}-u_{xx}=0,\ a<x<b,\ t<0$$

with

$u(x, 0)=\varphi(x), u_t(x, 0)=\psi(x), u(a, t)=\chi(t), u(b, t)=\lambda(t)$ is unique,

3. If $u(x, t)$ satisfies the wave equation

$$u_{xx}-\frac{1}{c^2}u_{tt}=0$$

show that

$$\int_C \{u_\tau(\xi,\tau)\,d\xi+c^2u_\xi(\xi,\tau)\,d\tau\}$$

a round any simple closed curve is zero. Deduce that the solution which satisfies the initial conditions $u=u_0(x), u_t=u_1(x)$ when $t=0$, is given by (4.8)

4. Find the solution of the equation $u_{tt}-c^2u_{xx}=0$, given that on $t=0$, $u=\sin \pi x/c$ when $0 \leqslant x \leqslant c$, $u=0$ when $x > c$ and $x < 0$, $u_t=0$ for all x. Examine the continuity of u and its derivatives.

5. A flexible string of length l is fastened at the ends $x=0$ and $x=l$ and is in equilibrium under a uniform tension. It is displaced at $x=\frac{1}{2}l$ to an elevation h and then released. Find subsequent displacement for all times, assuming that the motion is governed by the wave equation.

6. Find the deflection $u(x, t)$ of a taut string which was at rest at time $t=0$, if it is fastened at the end point $x=l$ and subjected at the other end point $x=0$ to a motion represented by $u(0, t)=f(t)$.

7. Determine the generalised or weak solution of the equation

$$u_{xx}-\frac{1}{c^2}u_{tt}=0$$

given that

$$u(x,0)=\begin{cases} x^2+5, & x>0\\ 5, & x=0\\ -x^2+5, & x<0\end{cases}$$

$u_t(x,0)=0.$

Verify that the discontinuity in the second order derivatives will propagate along characteristics.

8. Determine the weak solution of the equation

$$u_{xx}=\frac{1}{c^2}u_{tt}+\frac{2}{c}u_t-u_x=0$$

given that $u_t(x,0)=0$ and

$$u(x,0)=\begin{cases} x^2, & x>0\\ 0, & x=0\\ -x^2, & x<0.\end{cases}$$

Examine how the discontinuity in the second order derivatives will propagate.

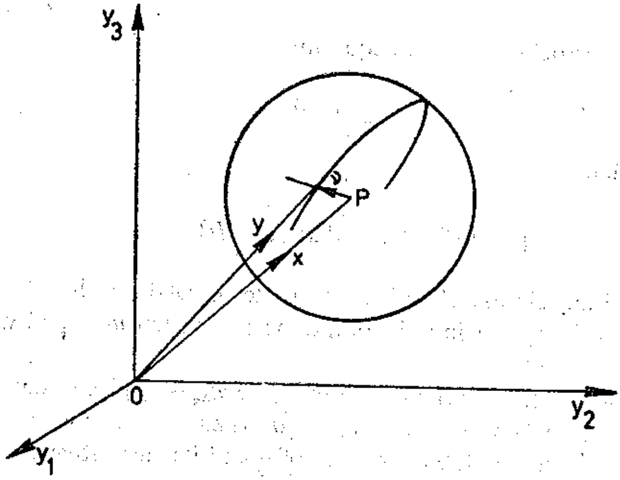

Fig. 4.5 Sphere, centre, x, radius ct, in three dimensional space

§4.2 The Three-dimensional Wave Equation

Consider the equation

$$\sum_{\alpha=1}^{3} u_{x_\alpha x_\alpha} - \frac{1}{c^2} u_{tt} = 0.$$

We look for a solution of the Cauchy problem for this equation.

Theorem 4.3 If $u_0(\mathbf{x}) \in C^3$ and $u_1(\mathbf{x}) \in C^2$ in $-\infty < x_\alpha < \infty$, $\alpha = 1, 2, 3$, then the function

$$u(\mathbf{x}, t) = tM(t)u_1 + \frac{\partial}{\partial t}[tM(t)u_0] \tag{4.21}$$

where $M(t)u_i \equiv (M(t)u_i)(\mathbf{x}, t)$

$$M(t)u_i = \frac{1}{4\pi}\int_{|\nu|=1} u_i(\mathbf{x}+\nu ct)\,d\omega, \quad i = 0, 1 \tag{4.22}$$

belongs to C^2 in $-\infty < x_\alpha < \infty$, $0 < t < \infty$, and is a solution of the Cauchy problem

$$\Delta_3 u - \frac{1}{c^2} u_{tt} = 0$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}),\ u_t(\mathbf{x}, 0) = u_1(\mathbf{x}).$$

$M(t)u_i$ denotes the mean value of u_i over a sphere with centre at $\mathbf{x}$ and radius ct in three-dimensional space.*

Proof Assuming (4.21) to hold, we verify first whether the initial conditions are satisfied:

$$u(\mathbf{x}, 0) = M(0)u_0$$
$$= \frac{1}{4\pi}\int_{|\nu|=1} u_0(\mathbf{x})\,d\omega = u_0(x).$$

*ν is a unit vector and $d\omega$ is an infinitesimal surface area on the unit sphere.

$$u_t(\mathbf{x}, 0) = \lim_{t\to 0} [M(t)u_1 + 2M_t(t)u_0]$$
$$= M(0)u_1 = u_1(\mathbf{x})$$

where $M_t(t)u_i \equiv \frac{\partial}{\partial t}\{M(t)u_i\}$

so that $M_t(0)u_0 = \frac{1}{4\pi} c \sum_{\alpha=1}^{3} \int_{|\nu|=1} u_{0x_\alpha}(\mathbf{x})\nu_\alpha \, d\omega$

$$= \frac{1}{4\pi} c \sum_{\alpha=1}^{3} u_{0x_\alpha}(\mathbf{x}) \int_{|\nu|=1} \nu_\alpha \, d\omega$$

$= 0$ from Gauss Integral Theorem.

We will now show that an expression of the form

$$v(\mathbf{x}, t) = tM(t)\xi \tag{4.23}$$

satisfies the wave equation, where ξ is any function of x_α, we get

$$v_{tt} = 2M_t(t)\xi + tM_{tt}(t)\xi.$$

Now

$$M_t(t)\xi = \frac{c}{4\pi} \sum_{\alpha=1}^{3} \int_{|\nu|=1} \nu_\alpha \xi_{y_\alpha}(\mathbf{x} + \boldsymbol{\nu}ct) \, d\omega$$
$$= \frac{c}{4\pi c^2 t^2} \sum_{\alpha=1}^{3} \int_{|\mathbf{y}-\mathbf{x}|=ct} \nu_\alpha \xi_{y_\alpha}(\mathbf{y}) \, dS, \; y_\beta = x_\beta + \nu_\beta ct$$
$$= \frac{1}{4\pi c t^2} \int_{|\mathbf{y}-\mathbf{x}|\leqslant ct} \Delta_3 \xi \, d\mathbf{y} \text{ by Gauss Theorem.}$$

Therefore

$$M_t(t)\xi = \frac{1}{4\pi c t^2} \int_0^{ct} d\rho \int_{|\mathbf{y}-\mathbf{x}|=\rho} \Delta_3 \xi \, dS \tag{4.24}$$

$$M_{tt}(t)\xi = -\frac{1}{2\pi c t^3} \int_{|\mathbf{y}-\mathbf{x}|\leqslant ct} \Delta_3 \xi \, d\mathbf{y} + \frac{c}{4\pi c t^2} \int_{|\mathbf{y}-\mathbf{x}|=ct} \Delta_3 \xi \, dS \tag{4.25}$$

so that

$$v_{tt} = \frac{1}{4\pi t} \int_{|\mathbf{y}-\mathbf{x}|=ct} \Delta_3 \xi \, dS.$$

Further,

$$\Delta_3 v = \frac{t}{4\pi} \int_{|\nu|=1} \Delta_3 \xi(\mathbf{x} + \boldsymbol{\nu}ct) \, d\omega$$
$$= \frac{1}{4\pi c^2 t} \int_{|\mathbf{y}-\mathbf{x}|=ct} \Delta_3 \xi \, dS.$$

Therefore, $\Delta_3 v - \frac{1}{c^2} v_{tt} = 0.$

It follows that both $tM(t)u_1$ and $tM(t)u_0$ satisfy the wave equation, hence also $\partial/\partial t(tM(t)u_0)$. (4.21) is therefore a solution of the three-dimensional wave equation satisfying the given initial conditions.

EXERCISE 4.2

1. Find the solution of the wave equation

$$\frac{1}{c^2}u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0$$

given that

$$\left.\begin{array}{l} u = 0 \\ u_t = x^2 + xy + z^2 \end{array}\right\} \text{ when } t = 0.$$

2. Find the solution of the three-dimensional wave equation given that at $t = 0$, $u = 0$ and

$$\begin{aligned} u_t &= 1 \quad \text{for } x^2 + y^2 + z^2 \leqslant a^2 \\ &= 0 \quad \text{for } x^2 + y^2 + z^2 > a^2. \end{aligned}$$

Examine the nature of the discontinuities of the solution.

3. Find the solution of the three-dimensional wave equation with the following boundary and initial values.

$$u = 0 \text{ for } x = 0,\ t > 0\ \forall\ y, z$$

$$u = 0 \text{ for } x > 0,\ t = 0\ \forall\ y, z$$

$$u_t = \begin{cases} 1 \text{ for } (x-1)^2 + y^2 + z^2 < \frac{1}{8},\ t = 0,\ x > 0 \\ 0 \text{ for } (x-1)^2 + y^2 + z^2 > \frac{1}{8},\ t = 0,\ x > 0. \end{cases}$$

4. Find the spherically symmetric solution $u(x, t)$ of the three-dimensional wave equation with initial data $u = \varphi(r)$, $u_t = 0$.

5. u, a solution of the three-dimensional wave equation, is of the form

$$u(x, y, z, t) = v(r, t) \cos \theta$$

where $\quad r^2 = x^2 + y^2 + z^2$, $\tan \theta = \sqrt{(x^2 + y^2)}/z$. Show that

$$u = \cos \theta \frac{\partial}{\partial r}\left\{\frac{\varphi(r-t) + \psi(r+t)}{r}\right\}$$

where φ and ψ are arbitrary functions.

6. The velocity potential φ of sound waves of small amplitude due to a source at the origin expanding into the gas at rest, satisfies the wave equation with spherical symmetry. If $Q(t)$ is the volume flux of the fluid across a sphere of radius r at time t, show that

$$\varphi(r, t) = -\frac{1}{4\pi}\ \frac{Q(t - r/a_0)}{r}$$

where a_0 is the sound velocity at rest.

7. In the ballon problem in acoustics, the pressure p inside a region of radius R_0 is $P + p_0$ while the pressure outside is p_0. The gas is initially

at rest and the ballon is burst at time $t=0$. If the velocity potential φ which satisfies the wave equation with spherical symmetry, be such that the radial velocity $u=-\frac{\partial\varphi}{\partial r}$ and the pressure $p=\frac{1}{\rho_0}\frac{\partial\varphi}{\partial t}$, determine the pressure distribution for subsequent times.

8. Find the solution u of the wave equation in three-dimensions given that for $t=0$ $u=0$ and

$$u_1=1 \text{ for } x^2+y^2+z^2 \leqslant a^2$$
$$=0 \text{ for } x^2+y^2+z^2 > a^2.$$

§4.3 Method of Spherical Means

The notion of spherical means can be used to find a formula for the solution of the Cauchy problem in higher dimensions. Let $u(\mathbf{x}, t)$ denote the function $u(x_1, x_2, \cdots, x_m, t)$ in m-space variables and time. We associate with $u(\mathbf{x}, t)$, a function $\tilde{M}u(\mathbf{x}, r, t)$ which is its average on a sphere of radius r and with centre $\mathbf{x}$:

$$\tilde{M}u(\mathbf{x}, r, t)=\frac{1}{\omega_m}\int_{|\boldsymbol{\nu}|=1} u(\mathbf{x}+\boldsymbol{\nu}r, t)\, d\omega. \tag{4.26}$$

Then

$$\frac{\partial}{\partial r}(\tilde{M}u)=\frac{1}{r^{m-1}\omega_m}\sum_{\alpha=1}^{m}\int_{|\mathbf{y}-\mathbf{x}|=r} \nu_\alpha u_{y_\alpha}(\mathbf{y}, t)\, dS$$
$$=\frac{1}{r^{m-1}\omega_m}\int_{|\mathbf{y}-\mathbf{x}|\leqslant r} \Delta_m u\, dy$$

by Gauss' Theorem. If u satisfies the wave equation, then

$$\frac{\partial}{\partial r}(\tilde{M}u)=\frac{1}{\omega_m r^{m-1}c^2}\int_{|\mathbf{y}-\mathbf{x}|\leqslant r} u_{tt}\, dy$$
$$=\frac{1}{\omega_m r^{m-1}c^2}\int_0^r d\rho\int_{|\mathbf{y}-\mathbf{x}|=\rho} u_{tt}\, dS$$

i.e. $$r^{m-1}\frac{\partial}{\partial r}(\tilde{M}u)=\frac{1}{\omega_m c^2}\int_0^r d\rho\int_{|\mathbf{y}-\mathbf{x}|=\rho} u_{tt}\, dS. \tag{4.27}$$

Differentiating (4.27) with respect to r, we get

$$\frac{\partial}{\partial r}\left[r^{m-1}\frac{\partial}{\partial r}(\tilde{M}u)\right]=\frac{1}{\omega_m c^2}\int_{|\mathbf{y}-\mathbf{x}|=r} u_{tt}\, dS$$
$$=\frac{r^{m-1}}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{1}{\omega_m r^{m-1}}\int_{|\mathbf{y}-\mathbf{x}|=r} u\, dS\right]$$
$$=\frac{r^{m-1}}{c^2}(\tilde{M}u)_{tt}.$$

Therefore, $\tilde{M}u$ satisfies the partial differential equation

$$\frac{\partial}{\partial r}\left[r^{m-1}\frac{\partial}{\partial r}(\tilde{M}u)\right]=\frac{r^{m-1}}{c^2}(\tilde{M}u)_{tt}$$

or

$$\frac{\partial^2}{\partial r^2}(\tilde{M}u) + \frac{m-1}{r}\,\frac{\partial}{\partial r}(\tilde{M}u) - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}(\tilde{M}u) = 0. \tag{4.28}$$

This equation is known as the Darboux equation. When m is an odd integer, it can be reduced to the wave equation. For m even, the problem is much more difficult. In the case $m=3$, (4.28) reduces to

$$\frac{\partial^2}{\partial r^2}(\tilde{M}u) + \frac{2}{r}\,\frac{\partial}{\partial r}(\tilde{M}u) - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}(\tilde{M}u) = 0$$

i.e.

$$\frac{\partial^2}{\partial r^2}(r\,\tilde{M}u) - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}(r\,\tilde{M}u) = 0. \tag{4.29}$$

Therefore, $r(\tilde{M}u)$ satisfies the one-dimensional wave equation. The initial conditions to be satisfied by $r(\tilde{M}u)$ can be determined from those to be satisfied by u. They are

$$\left.\begin{aligned} r(\tilde{M}u)(\mathbf{x}, r, 0) &= rM\left(\frac{r}{c}\right) u_0(\mathbf{x}, r) = \frac{r}{4\pi}\int_{|\nu|=1} u_0(\mathbf{x}+\nu r)\,d\omega \\ \text{and}\quad & \\ [r(\tilde{M}u)]_t(\mathbf{x}, r, 0) &= rM\left(\frac{r}{c}\right) u_1(\mathbf{x}, r) = \frac{r}{4\pi}\int_{|\nu|=1} u_1(\mathbf{x}-\nu r)\,d\omega \end{aligned}\right\} \tag{4.30}$$

in the notation (4.22). The D'Alembert solution of the Cauchy problem (4.29) with (4.30) is given by (4.8) as

$$\tilde{M}u(\mathbf{x}, r, t) = \frac{1}{2r}\left[(r+ct)M\left(\frac{r+ct}{c}\right) u_0 + (r-ct)M\left(\frac{r-ct}{c}\right) u_0 + \frac{1}{c}\int_{r-ct}^{r+ct} \xi M\left(\frac{\xi}{c}\right) u_1\, d\xi\right].$$

Since Mu_0 and Mu_1 are even in t in (4.7)

$$\tilde{M}u(\mathbf{x}, r, t) = \frac{1}{2r}\left[(ct+r)M\left(\frac{ct+r}{c}\right) u_0 - (ct-r)M\left(\frac{ct-r}{c}\right) u_0 + \frac{1}{c}\int_{ct-r}^{ct+r} \xi M\left(\frac{\xi}{c}\right) u_1\, d\xi\right]. \tag{4.31}$$

From (4.26)

$$\lim_{r\to 0} \tilde{M}u(\mathbf{x}, r, t) = u(\mathbf{x}, t).$$

Going to the limit as $r \to 0$ in (4.31) we have

$$\begin{aligned} u(\mathbf{x}, t) &= \frac{\partial}{\partial(ct)}\left\{ct\, M\left(\frac{ct}{c}\right) u_0\right\} + tM(t)u_1 \\ &= \frac{\partial}{\partial t}\{tM(t)u_0\} + tM(t)u_1 \end{aligned} \tag{4.32}$$

which is the same as (4.21)

Now by the method of spherical means, we have shown that any solution of the initial value problem coincides with (4.32). The solution is thus unique.

To prove the stability of the solution consider an initial value problem for the wave equation, where the initial values $\bar{u}_i(\mathbf{x})$ differ slightly from $u_i(\mathbf{x})$. Denote the solution of this problem by $\bar{u}(\mathbf{x}, t)$. Let $\epsilon > 0$ be given. We can then find $\delta(\epsilon) > 0$, so that whenever

$$| \bar{u}_i(\mathbf{x}) - u_i(\mathbf{x}) | < \delta(\epsilon), \text{ and } | \bar{u}_{0x_\alpha}(\mathbf{x}) - u_{0x_\alpha}(\mathbf{x}) | < \delta(\epsilon)$$
$$i = 0, 1; \alpha = 1, 2, 3$$

Hence from (4.32) we get

$$| \bar{u}(\mathbf{x}, t) - u(\mathbf{x}, t) | < \epsilon, \text{ for all } \mathbf{x}, t.$$

So all the three requirements for wellposedness of the initial value problem of the three dimensional wave equation are satisfied.

The solution of the three dimensional wave equation given by (4.21) is of class C^2 for $t \geqslant 0$ when $u_0 \in C^3(R^3)$ and $u_1 \in C^2(R^3)$. Therefore, the solution can be less smooth than the data. There is a possible loss of one derivative. This loss of smoothness is due to focussing which can happen only for $m > 1$. For $m = 1$ the solution is as smooth for all t as the initial data at $t = 0$.

At a point $(\mathbf{x}_0, t_0)$, $u(\mathbf{x}_0, t_0)$ depends only on the value of $u_1(\mathbf{x})$ on the surface of the sphere with centre $\mathbf{x}_0$ and radius ct_0, and on the values of $u_0(\mathbf{x})$ in a neighbourhood of this surface. The domain of dependence of (x_0, t_0), denoted by $\bar{\mathcal{G}}(\mathbf{x}_0, t_0)$, is given by

$$\bar{\mathcal{G}}(\mathbf{x}_0, t_0) = \{(\mathbf{x}_0, t_0) : | \mathbf{x} - \mathbf{x}_0 | = ct_0\}. \tag{4.33}$$

If we fix a ball $\bar{S} : | \mathbf{x} - \mathbf{x}_0 | \leqslant r$, then the set of points $(\mathbf{x}, t)$ at which the solution $u(\mathbf{x}, t)$ depends only on the initial values $u_i(\mathbf{x})$ in $\bar{S}$ is called the domain of determinacy of $\bar{S}$ and is denoted by $\bar{B}(\bar{S})$. A point $(\mathbf{x}, t) \in \bar{B}(\bar{S})$ if and only if for any unit vector $\boldsymbol{\nu}$, $\mathbf{x} + \boldsymbol{\nu} ct \in \bar{S}$, i.e. iff

$$| \mathbf{x} + \boldsymbol{\nu} ct - \mathbf{x}_0 | \leqslant r \text{ or } | \mathbf{x} - \mathbf{x}_0 | \leqslant r - ct.$$

This represents a cone in space-time with vertex at $(\mathbf{x}_0, r/c)$. The lateral surfaces of the cone is a characteristic manifold

$$| \mathbf{x} - \mathbf{x}_0 | = r - ct. \tag{4.34}$$

EXERCISE 4.3

1. Show that the Darboux equation (4.28) is reducible to the form

$$\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{N}{\xi + \eta}\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) = 0$$

by suitably choosing ξ, η, N and v.

2. Show that $(1 - \xi w)^{-N}(1 + \eta w)^{-N}$ satisfies the equation in the previous problem where w is an arbitrary constant. By considering its expansion

in ascending powers of w, prove that the equation has homogeneous polynomial solutions

$$F_n(\xi, \eta) = \sum_{p+q=n} \frac{(N)_p(N)_q}{p!q!} \xi^p(-\eta)^q$$

where $(N)_p = N(N+1) \dots (N+p-1)$.

§4.4 The Two-dimensional Wave Equation: Hadamard's Method of Descent

We shall derive a formula for the solution of the Cauchy problem for the two dimensional wave equation by treating it as a special case of the three-dimensional one. This is an example of a general method introduced by Hadamard, called the method of descent, whereby one "steps down" from solutions to equations in n dimensions to solutions for certain equations in $n-1$ dimensions. In the following discussion $\bar{x} = (x_1, x_2)$ is a point in two dimensional space.

Theorem 4.4 If $u_0(\bar{x}) \in C^3$, $u_1(\bar{x}) \in C^2$ in $-\infty < x_1, x_2 < \infty$, $\bar{x} = (x_1, x_2)$ then the function

$$u(\bar{x}, t) = \overline{M}(t)u_1 + \frac{\partial}{\partial t}\overline{M}(t)u_0 \tag{4.35}$$

where

$$\overline{M}(t)u_i = \frac{1}{2\pi c}\int_{|\bar{y}-\bar{x}| \leqslant ct} \frac{u_i(\bar{y})}{\sqrt{c^2t^2 - |\bar{y}-\bar{x}|^2}}\, d\bar{y} \tag{4.36}$$

belongs to C^2 on $-\infty < x_1, x_2 < \infty$, and is a solution of the problem

$$\Delta_2 u - \frac{1}{c^2}u_{tt} = 0,\ u(\bar{x}, 0) = u_0(\bar{x}),\ u_t(\bar{x}, 0) = u_1(\bar{x}).$$

Proof If in the solution (4.21) for the three-dimensional wave equation, we assume that the initial values $u_i(\bar{x})$ do not depend on x_3 then the solution $u = u(\bar{x}, t)$ depends on x_1, x_2 and t only, and is independent of x_3.

This gives for $t > 0$

$$tM(t)u_i = \frac{1}{4\pi}\int_{|\mathbf{v}|=1} u_i(\mathbf{x} + \mathbf{v}ct)\, d\omega$$

$$= \frac{1}{4\pi c^2 t}\int_{|\mathbf{y}-\mathbf{x}|=ct} u_i(\mathbf{y})\, dS.$$

Here the integration is performed over the three-dimensional sphere $|\mathbf{y}-\mathbf{x}| = ct$ with centre at $\mathbf{x}$. Since the integrand is a function of only two variables y_1 and y_2, we can reduce the integration to one over a two-dimensional area in (y_1, y_2)-plane as shown below.

Solving $|\mathbf{y}-\mathbf{x}| = ct$ for y_3 gives

$$y_3 = x_3 \pm \sqrt{c^2t^2 - |\bar{y}-\bar{x}|^2},\ \bar{y} = (y_1 y_2).$$

On the upper hemisphere

$$\frac{\sqrt{c^2t^2 - |\bar{y}-\bar{x}|^2}}{ct}\, dS = dy_1,\, dy_2$$

i.e.

$$dS = \frac{ct\, dy_1\, dy_2}{\sqrt{c^2t^2 - |\bar{y}-\bar{x}|^2}}. \tag{4.37}$$

As we move on the surface of the upper hemisphere of $|\mathbf{y}-\mathbf{x}| = ct$, y_1 and y_2 in (4.37) take all values for which $|\bar{y}-\bar{x}| \leqslant ct$. Taking into account the lower hemisphere as well

$$tM(t)u_i = 2\,\frac{1}{4\pi c}\int_{|\bar{y}-\bar{x}| \leqslant ct} \frac{u_i(\bar{y})}{\sqrt{c^2t^2 - |\bar{y}-\bar{x}|^2}}\, d\bar{y}$$
$$\equiv \bar{M}(t)u_i.$$

Therefore the solution (4.21) can be put in the form

$$u(\mathbf{x}, t) = \bar{M}(t)u_1 + \frac{\partial}{\partial t}(\bar{M}(t)u_0)$$

which is the required solution of the initial value problem of the two-dimensional wave equation.

The domain of dependence of a point $(\bar{x}_0, t_0)$ in this case is the set of all points in the interior as well as on the boundary of the circle $|\bar{x}-\bar{x}_0| = ct_0$, that is

$$\bar{\mathscr{G}}(\bar{x}_0, t_0) : |\bar{x}-\bar{x}_0| \leqslant ct_0. \tag{4.38}$$

All $(\bar{x}, t)$ at which $u(\bar{x}, t)$ depends solely on the initial values $u_i(\bar{x})$ in a circular disc $\bar{C}_1 : |\bar{x}-\bar{x}_0| \leqslant r$ gives the domain of determinacy of $\bar{C}_1$, i.e.

$$\bar{B}(\bar{C}_1) : |\bar{x}-\bar{x}_0| \leqslant r - ct. \tag{4.39}$$

This is a cone in $(\bar{x}, t)$ space with vertex at $(\bar{x}_0, r/c)$.

Singularity function for the wave equation in the two-dimensional case.

In §2.2, we discussed the singularity function of the Laplace equation $\Delta_m u = 0$. It is represented in the case $m = 3$ for a point $\mathbf{a} = (a_1, a_2, a_3)$ by the relation

$$s(\mathbf{a}, \mathbf{x}) = \frac{1}{4\pi\,|\mathbf{x}-\mathbf{a}|}$$
$$= \frac{1}{4\pi}[(x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)^2]^{-1/2}.$$

A transformation

$$x_1 = \sqrt{1-\alpha^2}\,\bar{x}_1,\ x_2 = \sqrt{1-\alpha^2}\,\bar{x}_2,\ x_3 = \bar{x}_3,\ 0 < \alpha < 1$$

reduces the equation $\Delta_3 u = 0$ to the equation

$$\frac{\partial^2 u}{\partial \bar{x}_1^2} + \frac{\partial^2 u}{\partial \bar{x}_2^2} + (1-\alpha^2)\,\frac{\partial^2 u}{\partial \bar{x}_3^2} = 0.$$

The corresponding singularity function of this equation for the point $\bar{a}$ is $s(\bar{a}, \bar{x})$ obtained by applying the above transformation to $s(a, x)$, namely

$$s(\bar{a}, \bar{x}) = \frac{1}{4\pi}[(\bar{x}_3 - \bar{a}_3)^2 + (1 - \alpha^2)\{(\bar{x}_1 - \bar{a}_1)^2 + (\bar{x}_2 - \bar{a}_2)^2\}]^{1/2}$$

This solution is said to have "source like" singularity at $\bar{x} = \bar{a}$. It is axisymmetric about the line passing through the point $\bar{a}$ and parallel to $\bar{x}_3$ axis.

The expression for $s(\bar{a}, \bar{x})$, the singularity function of the equation, has been obtained under the assumption $0 < \alpha < 1$. In spite of this, we can verify by substitution that it also holds in the case $\alpha > 1$. In this case the singularity is not confined to a single point $\bar{x} = \bar{a}$, but to the entire surface of a cone given by

$$(\bar{x}_3 - \bar{a}_3)^2 - (\alpha^2 - 1)\{(\bar{x}_1 - \bar{a}_1)^2 + (\bar{x}_2 - \bar{a}_2)^2\} = 0.$$

The expression for $s(\bar{a}, \bar{x})$ takes real values only within this cone and we set it equal to zero outside the cone. We conclude that for $\alpha > 1$,

$$s(\bar{a}, \bar{x}) = \begin{cases} \dfrac{1}{4\pi}[(\bar{x}_3 - \bar{a}_3)^2 - (\alpha^2 - 1)\{(\bar{x}_1 - \bar{a}_1)^2 + (\bar{x}_2 - a_2)^2\}]^{1/2} & \text{inside the cone} \\ 0 & \text{outside the cone} \end{cases}$$

is the singularity function of the equation

$$\frac{\partial^2 u}{\partial \bar{x}_1^2} + \frac{\partial^2 u}{\partial \bar{x}_2^2} - (\alpha^2 - 1)\frac{\partial^2 u}{\partial \bar{x}_3^2} = 0.$$

This is analogous to the wave equation

$$\frac{\partial^2 u}{\partial \bar{x}_1^2} + \frac{\partial^2 u}{\partial \bar{x}_2^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

where the spatial coordinate $\bar{x}_3$ is replaced by t and $(\alpha^2 - 1)$ by $1/c^2$. Thus the singularity function for the wave equation in the two-dimensional case is

$$s(\bar{a}, \tau; \bar{x}, t) = \begin{cases} \dfrac{1}{4\pi}[(t - \tau)^2 - \dfrac{1}{c^2}\{(\bar{x}_1 - \bar{a}_1)^2 + (\bar{x}_2 - \bar{a}_2)^2\}]^{-1/2} & \text{inside the cone} \\ 0 & \text{outside the cone.} \end{cases}$$

Using the theory of distributions, $s(\bar{a}, \tau, \bar{x}, t)$ can be shown to satisfy the nonhomogeneous wave equation

$$\frac{\partial^2 u}{\partial \bar{x}_1^2} + \frac{\partial^2 u}{\partial \bar{x}_2^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\delta_{\bar{a},\tau}$$

where $\delta_{\bar{a},\tau}$ is the Dirac delta function associated with the point $(\bar{a}, \tau)$.

In fluid mechanics, the singularity function has a special significance, Ward (1955). Let $\alpha = M_\infty$, the Mach number of the undisturbed flow. When $M_\infty < 1$, i.e. for subsonic flow, $s(\bar{a}, \bar{x})$ represents the velocity potential in a uniform three-dimensional subsonic flow induced by a source of unit strength at $\bar{x}=\bar{a}$. If the source is of strength Q, the velocity potential φ is given by

$$\varphi = Qs(\bar{a}, \bar{x}).$$

When $M_\infty > 1$ i.e. for supersonic flow, $s(\bar{a}, \bar{x})$ is singular not just at the point $\bar{x}=\bar{a}$, but on the entire surface of a cone, which is called the Mach cone. The Mach cone represents the boundary of the domain in the fluid influenced by a three-dimensional source at the vertex in a uniform supersonic flow. Besides, the velocity potential φ is real only within the Mach cone. Disturbances from the source are not felt outside the cone and φ is set equal to zero there.

The results of three-dimensional steady supersonic flow can be carried over to two dimensional unsteady flow governed by the wave equation. For the wave equation, the singularity function is

$$s(\bar{a}, \tau; \bar{x}\ t) = \frac{1}{4\pi\sqrt{(t-\tau)^2 - R^2/c^2}}, \quad R = [(\bar{x}_1 - \bar{a}_1)^2 + (\bar{x}_2 - \bar{a}_2)^2]^{1/2}.$$

This can be used to define a velocity potential, for two-dimensional unsteady flow,

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{t-R/c} \frac{Q(\tau)\, d\tau}{\sqrt{(t-\tau)^2 - R^2/c^2}}$$

where the source strength is time-dependent.

If an axisymmetric body is placed in a flow field with its axis along the axis of z, it can be simulated by introducing sources into the flow field along the z-axis, i.e. the axis of symmetry. It can be shown that the strength $Q(z)$ of the source at a distance z from the origin on the z-axis is related to the profile $R=\pi\bar{R}^2(z)$ in a cylindrical coordinate system (R, θ, z) by the relation

$$Q(z) = 4\pi\bar{R}\frac{d\bar{R}}{dz}.$$

EXERCISE 4.4

1. Show that the velocity potential φ of a line source in a fluid along the z-axis with uniform strength $q(t)$ per unit length is given by

$$\varphi(R, t) = -\frac{1}{2\pi R}\int_{-\infty}^{t-R/a_0} \frac{q(\eta)\, d\eta}{\sqrt{(t-\eta)^2 - R^2/a_0^2}}$$

where $R=\sqrt{x^2+y^2}$, where a_0 is the sound velocity.

2. For a fixed value of R, determine how the solution in the previous problem decays to zero asymptotically as $t \to \infty$.

3. Show that the solution in the case of supersonic flow past a body of revolution is given by

$$\varphi = -\frac{U}{2\pi}\int_0^{z-BR} \frac{S'(\eta)\,d\eta}{\sqrt{(z-\eta)^2 - B^2R^2}}, \quad z - BR > 0$$

where U is the velocity of the fluid at infinity, $B=\sqrt{M^2-1}$, $M=U/a_0$ is the Mach number and $S(z)=\pi R^2(z)$ is the cross sectional area of the body at a distance z from the nose.

§4.5 Propagation of Confined Initial Disturbances

The solution $u(x, t)$ of the three-dimensional wave equation depends on the values of u_0 and u_1 and the first derivatives of u_0 on the surface of the sphere of centre **x** and radius ct. Conversely, the values of u_0 and u_1 at a point **y** in the plane $t=0$ influence only the value of $u(\mathbf{x}, t)$ at time t at points **x** near the sphere $S_{\mathbf{y},ct}$: $|\mathbf{x}-\mathbf{y}| = ct$. If u_0 and u_1 have support in a closed bounded region Ω of R^3 i.e. if they are both zero outside Ω, then at $t > 0$ $u(\mathbf{x}, t) \neq 0$ atmost at those points **x** which lie on a sphere of radius ct and centred at a point $y \in \Omega$ i.e. $\mathbf{x} \in S_{y,ct}$ for some $y \in \Omega$. The union of all such spheres $S_{y,ct}$ for $y \in \Omega$ contains the support of $u(\mathbf{x}, t)$ at time t. This gives the Huygen's construction of a wave front for a disturbance confined originally to Ω.

Consider the case where the supports of u_0 and u_1 are contained in a ball of radius δ about the origin. The spheres having their centres in this ball and radius ct constitute a spherical shell with centre at origin and radii $ct-\delta$ and $ct+\delta$, whenever $ct > \delta$.

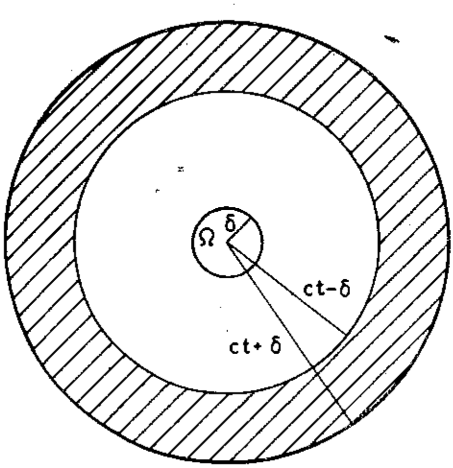

Fig. 4.6 Propagation of an initially confined disturbance for the three dimensional wave equation

For a fixed value of $\mathbf{x}$, $|\mathbf{x}| > \delta$, we have $u(\mathbf{x}, t) \neq 0$ only for a time interval of length $2\delta/c$, from $t = \frac{|\mathbf{x}| - \delta}{c}$ to $t = \frac{|\mathbf{x}| + \delta}{c}$. This permits the transmission of sharp signals in the form of waves with sharp leading and trailing fronts. After the wave has passed, the medium returns to absolute rest instantaneously. This contrasts with the behaviour of the solution of the two-dimensional wave equation, which we discuss below.

The solution of the two-dimensional wave equation at $(\bar{x}, t)$ depends on the values of u_0, u_1 and the first derivatives of u_0 in the entire disc of centre $\bar{x}$ and radius ct (and not only on the circumference, as in the three dimensional case). A disturbance initially created at some point Q, reaches an initially undisturbed point P after a finite time but continues to disturb the point P with diminishing intensity for the rest of the time. There is no trailing front to bring the medium to rest at P. If, for example, the supports of u_0 and u_1 are contained in a disc of radius δ about the origin, then the disturbance reaches a point $\bar{x}$, $|\bar{x}| > \delta$, at time $t_1 = \frac{|\bar{x}| - \delta}{c}$ and thereafter for $t > t_1$ the disturbance remains and decays to zero as t tends to infinity. This is called reverberation.

A solution of the wave equation in one space variable exhibits a behaviour intermediate between those in two and three space variables. If the supports of u_0 and u_1 are bounded in x, say $|x| < \delta$, the effect of u_0 vanishes at a given point x after a time $t = \frac{|x| + \delta}{c}$ (as in the three-dimensional case), but the effect of u_1 is felt for all time $t > \frac{|x| - \delta}{c}$ (as in the two-dimensional case) because of the presence of the integral term

$$\frac{1}{2c} \int_{x-ct}^{x+ct} u_1(\tau)\, d\tau.$$

When the number of space variables is greater than three, the solution of the wave equation in an n-dimensional space, with n even, shows the existence of the phenomenon of reverberation as for $n = 2$. In the case when n is odd, the solution shows that disturbances of finite extent are bounded by sharply defined leading and trailing fronts as for $n = 3$.

EXERCISE 4.5

1. A disturbance represented by $u(x, y, t)$ is governed by the two-dimensional wave equation and is initially given by

$$u(x, y, 0) = \begin{cases} 1 - x^2 - y^2 & \text{for } x^2 + y^2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$u_t(x, y, 0) = 0.$

Note, here $u_0 \notin C^3$.

Determine when the disturbance will reach a point (x, y) outside the unit circle and find how u will decay as $t \to \infty$.

2. In the balloon problem (Exercise 4.2, Problem 7) determine at what times a pressure discontinuity occurs at a fixed point when (i) $R > R_0$ (ii) $R < R_0$.

§4.6 Continuable Initial Conditions

We have shown that the solution of the three-dimensional wave equation in terms of initial conditions is given by

$$\begin{pmatrix} u(\mathbf{x}, t) \\ u_t(\mathbf{x}, t) \end{pmatrix} = \begin{pmatrix} N_t(t) & N(t) \\ N_{tt}(t) & N_t(t) \end{pmatrix} \begin{pmatrix} u_0(\mathbf{x}) \\ u_1(\mathbf{x}) \end{pmatrix} \tag{4.40}$$

where $N(t)$ is a linear operator given by.

$$N(t) = tM(t).$$

(4.40) can be written in matrix notation as

$$v(\mathbf{x}, t) = T(t)v_0(\mathbf{x}).$$

$T(t)$ can be interpreted as a solution opetator, which transforms the state v_0 at a time $t=0$ into the state v at time t. If the initial conditions are given at time $t = t_0$, then, since T depends only on the difference $t - t_0$, the operator $T(t-t_0)$ maps the state at $t = t_0$, to that at time t.

It must also be possible to obtain the state v at time t by intermediate steps. Starting from v_0 at time $t=0$, we first determine the state at time t_1' i.e. $T(t_1')v_0$. At time t_2' later we find the state (by taking the state at time t_1' as initial conditions) as $T(t_2')T(t_1')v_0$. This state must be the same had we progressed by time $t_1'+t_2'$ from $t=0$. Therefore, we require the operator equality

$$T(t_1'+t_2') = T(t_2')\cdot T(t_1'). \tag{4.41}$$

(4.41) can be rewritten, by replacing t_2' by $t-t_1$ and t_1' by (t_1-t_0), in the form

$$T(t-t_0) = T(t-t_1)T(t_1-t_0),\ t \geqslant t_1 > t_0. \tag{4.42}$$

As this composition is associative the operators T form a *semigroup*.

The conditions of theorem 4.3 are too weak to satisfy the Hadamard requirement (4.41). Starting with $u \in C^3$, $u_t \in C^2$ in the three dimensional case, after the first step we have $u \in C^2$, $u_t \in C^1$, but these cannot be used as initial values again, as we need $u \in C^3$, $u_t \in C^2$ here also. These are non-continuable initial values.

Let us consider an example to show that smoothness properties of the initial values are lost later on. In this we shall start with $u \in C^\infty$, $u_t \in C^2$ at $t=0$. We shall show that the solution u for $t > 0$ ceases to have its earlier smoothness property at $t=0$.

Example 4.2

Consider the three dimensional wave equation

$$\frac{1}{c^2}u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0 \tag{4.43}$$

with initial values

$$u(0, x_1, x_2, x_3) = u_0(x_1, x_2, x_3) = 0$$

$$u_t(0, x_1, x_2, x_3) \equiv u_1(x_1, x_2, x_3) = \begin{cases} (1-r^2)^{5/2} & \text{for } r < 1 \\ 0 & \text{for } r \geqslant 1 \end{cases} \tag{4.44}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. These initial values are such that $u_0 \in C^\infty$ and $\overset{*}{u_1} \in C^2$.

The solution of the initial value problem is given along the t-axis by

$$u(t, 0, 0, 0) = \begin{cases} t(1-c^2t^2)^{5/2} & \text{for } ct \leqslant 1 \\ 0 & \text{for } ct > 1. \end{cases} \tag{4.45}$$

We then have

$$u_t = 0,\ u_{tt} = 0,\ u_{ttt} = 0 \text{ for } ct > 1,$$

while for $ct < 1$

$$\begin{aligned} u_t &= (1-c^2t^2)^{5/2} - 5c^2t^2(1-c^2t^2)^{3/2} \\ u_{tt} &= -15c^2t(1-c^2t^2)^{3/2} + 15c^4t^3(1-c^2t^2)^{1/2} \\ u_{ttt} &= -15c^2(1-c^2t^2)^{3/2} + 90c^4t^2(1-c^2t^2)^{1/2} - 15c^6t^4(1-c^2t^2)^{-1/2} \end{aligned} \tag{4.46}$$

u_{ttt} fails to be continuous and tends to infinity as t approaches $1/c$. Since $(u_t)_{tt}$ is related to second order spatial derivatives of u_t through (4.43), it follows that u_t is not a C^2 function of x_1, x_2, x_3 at the origin when $t = 1/c$.

We now look for a property, which when possessed by the initial values of a hyperbolic equation, continues to be possessed by the solution for all times $t > 0$. For this we shall now define the space $H_r(t)$ where r is a positive integer. Let $R(t)$ denote a domain depending on a parameter t. Let us denote "smooth functions" in $R(t)$ with continuous derivatives of order up to r with respect to the x variables, for which the r-norm is finite. The r-norm of u is defined as

$$\| u(t) \|_r^2 = \int_{R(t)} \sum_{|p| \leqslant r} | D^p u |^2 \, dx \tag{4.47}$$

$$p = (p_1, p_2, \cdots, p_m),\ D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_m^{p_m}},\ |p| = p_1 + p_2 \cdots + p_m$$

where the summation on the right extends over all partial derivatives $D^p u$, of order $|p| \leqslant r$ with respect to the x variables. The completion of the space of such functions u in $R(t)$ with the r-norm is a Hilbert space $H_r(t)$.

In studying initial value problems of a hyperbolic equation we choose a special form of $R(t)$. Let P be a point $(\mathbf{x}_0, t_0)$ in $(\mathbf{x}, t)$-space with $t_0 > 0$. Let

$R(h)$ denote the domain of dependence of P in the hyperplane $t=h$. If u are functions belonging to $C^r(R(h))$ such that r-norm is finite in $R(h)$, we define "energy integrals" of order r as the r-norm of u, as in (4.47) with t replaced by h.

The general existence theorem (Courant and Hilbert, Chapt. 6 §10, 1975) states: If $L(U)=0$ is a symmetric system of linear equations with $U(x, 0) = U_0(x)$, the initial value problem has a smooth solution in $R(t)$, $t > 0$, provided the operator L is hyperbolic and its coefficients, as well as the initial function U_0 are sufficiently differentiable. If U_0 belongs to H_r over $R(0)$ then U belongs to H_r in every section $R(t)$ of the conoid of dependence.

In particular, if the coefficients possess continuous derivatives up to order $r+1$, where $r > m/2+1$ (m being the number of spatial variables), and U_0 belongs to H_r, we can construct a solution U, which belongs to H_r. In case U is not sufficiently smooth, the solution in the generalised sense is defined with the help of the limit in the r-norm of sufficiently smooth functions. By reducing the wave equation to a symmetric hyperbolic system of first order equations, the existence theorem guarantees that if $U(\mathbf{x}, 0)$ belongs to H_r over the hyperplane $t=0$, then the solution of the wave equation is uniquely determined for all later times $t > 0$ and it belongs to $H_r(t)$.

A property P is said to be persistent, if whenever an initial function $U_0(\mathbf{x})$ has the property P, the corresponding solution $U(\mathbf{x}, t)$ at any other time also has the property P. Here, we have seen that the property of having finite r-norm is persistent. $U_0(\mathbf{x}) \in H_r(0)$ gives a set of continuable initial conditions, whereas the conditions of existence and continuity of derivatives of $U(\mathbf{x})$ are not persistent and may be lost under the transformation $T(t)$. This shows that physically relevant persistency conditions are the existence of energy integrals, rather than differentiability properties.

§4.7 Duhamel's Principle, Solution of the Inhomogeneous Wave Equation, Retarded Potential

The method of Duhamel is similar to the method of variation of parameters for ordinary differential equations, which enables one to find a solution of the inhomogeneous equation with the help of the general solution of the homogeneous equation. Consider the equation

$$u_{tt} - L(u) = g(\mathbf{x}, t), \text{ with } u(\mathbf{x}, 0) = 0,\ U_t(\mathbf{x}, 0) = 0 \tag{4.48}$$

where L is a linear differential operator in m spatial variables with constant coefficients and derivatives, with respect to t, of order not more than one. We attempt to write the solution in the form

$$u(\mathbf{x}, t) = \int_0^t v(\mathbf{x}, t, \tau)\, d\tau \tag{4.49}$$

where $v(\mathbf{x}, t, \tau)$ is a one parameter family of solutions of

$$v_{tt} - L(v) = 0 \tag{4.50}$$

for all values of the parameter τ.
At $t=\tau$, we assume that

$$v(\mathbf{x}, \tau, \tau)=0, \quad \text{for each } \tau. \tag{4.51}$$

Then

$$u_t=v(\mathbf{x}, t, t)+\int_0^t v_t(\mathbf{x}, t, \tau)\, d\tau$$

$$=\int_0^t v_t(\mathbf{x}, t, \tau)\, d\tau \text{ using (4.51)}$$

$$u_{tt}=v_t(\mathbf{x}, t, t)+\int_0^t v_{tt}\, d\tau$$

and

$$Lu=\int_0^t Lv\, d\tau.$$

Since u satisfies (4.48) and v satisfies (4.50), we have

$$v_t(\mathbf{x}, \tau, \tau)=g(\mathbf{x}, \tau). \tag{4.52}$$

Therefore if v satisfies the equation

$$v_{tt}-L(v)=0$$

with initial conditions at $t=\tau$

$$v(\mathbf{x}, \tau, \tau)=0, \quad v_t(\mathbf{x}, \tau, \tau)=g(\mathbf{x}, \tau) \tag{4.53}$$

u defined by (4.49) then satisfies the given equation and initial conditions, and the problem for the solution of v can be reduced to one in which the initial conditions are prescribed at $t=0$ instead of at $t=\tau$. For this we find a solution $v^*(\mathrm{x}, t, \tau)$ which satisfies the equation (4.50) together with initial conditions at $t=0$

$$v^*(\mathbf{x}, 0, \tau)=0, \; v_t^*(\mathbf{x}, 0, \tau)=g(\mathbf{x}, \tau)$$

then

$$v(\mathbf{x}, t, \tau)=v^*(\mathbf{x}, t-\tau, \tau). \tag{4.54}$$

Example 4.3

We use Duhamel's Principle to solve the nonhomogeneous three-dimensional wave equation

$$u_{tt}-c^2(u_{x_1x_1}+u_{x_2x_2}+u_{x_3x_3})=g(\mathbf{x}, t), \qquad \mathbf{x}=(x_1, x_2, x_3) \tag{4.55}$$

with zero initial conditions

$$u(\mathbf{x}, 0)=0, \; u_t(\mathbf{x}, 0)=0.$$

We first determine $v^*(\mathbf{x}, \tau)$, where $v^*(\mathbf{x}, \tau)$ satisfies the homogeneous three-dimensional wave equation with initial conditions at $t=0$,

$$v^*(\mathbf{x}, 0, \tau)=0, \; v_t^*(\mathbf{x}, 0, \tau)=g(\mathbf{x}, \tau). \tag{4.56}$$

From (4.21)

$$v^*(\mathbf{x}, t, \tau) = \frac{t}{4\pi}\int_{|\nu|=1} g(\mathbf{x}+\boldsymbol{\nu}ct, \tau)\, d\omega$$

$$= \frac{1}{4\pi tc^2}\int_{S_{\mathbf{x},ct}} g(\boldsymbol{\xi}, \tau)\, dS, \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$$

where $S_{\mathbf{x},ct}$ is the sphere $(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2 = c^2t^2$.

Now

$$v(\mathbf{x}, t, \tau) = \frac{1}{4\pi(t-\tau)c^2}\int_{S_{\mathbf{x},c(\tau-t)}} g(\boldsymbol{\xi}, \tau)\, dS$$

and

$$u(\mathbf{x}, t) = \frac{1}{4\pi c^2}\int_0^t d\tau \int_{S_{\mathbf{x},c(\tau-t)}} \frac{g(\boldsymbol{\xi}, \tau)}{t-\tau}\, dS. \tag{4.57}$$

The values of g in the integral in (4.57) are taken on

$$(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2 = c^2(t-\tau)^2,\ 0 \leqslant \tau \leqslant t$$

that is, on the characteristic cone with vertex at $(\mathbf{x}, t)$.

Let $\quad r = \sqrt{(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2}.$

Hence we have

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi c^2}\int_0^{ct}\frac{dr}{r}\int_{S_{x_\alpha,r}} g(\boldsymbol{\xi}, t-r/c)\, dS.$$

on the cone $r = c(t-\tau)$.

One can write the integral as a triple integral over the projection of the cone

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi c^2}\int_{r\leqslant ct}\frac{g(\boldsymbol{\xi}, t-r/c)}{r}\, d\boldsymbol{\xi}, \quad d\boldsymbol{\xi} = d\xi_1 d\xi_2 d\xi. \tag{4.58}$$

We can give a physical interpretation to this solution. In electrodynamics, there exists a scalar function $\varphi(x_1, x_2, x_3, t)$ which, in Gaussian units, satisfies the equation

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{4\pi}{K}\rho(x_1, x_2, x_3, t),\quad c = \frac{c_0}{\sqrt{K\mu}}$$

where c is the velocity of light in the medium of the dielectric constant K and magnetic permeability μ, and ρ is the electric charge density. In particular, if we consider a time-independent concentrated electric charge ρ_0 at the origin, the corresponding electrostatic potential φ_0 in free space satisfies the Poisson's equation obtained by dropping the term $\partial^2\phi/\partial t^2$ and is given by

$$\varphi_0(\mathbf{x}) = \frac{1}{K}\ \frac{\rho_0}{|\mathbf{x}|},\quad |\mathbf{x}| = \sqrt{x_1^2+x_2^2+x_3^2}.$$

For a time-dependent charge distribution in space with density $\rho(x_1, x_2, x_3, t)$, the electrodynamic field φ is a solution of the above mentioned equation. Assuming that initially $\varphi=0$, $\phi_t=0$, everywhere, we get from (4.58)

$$\varphi(x_1, x_2, x_3, t)=\frac{1}{K}\int_{r\leqslant ct}\frac{\rho(\xi_1, \xi_2, \xi_3, t-r/c)}{r}\,d\xi_1\,d\xi_2\,d\xi_3 \qquad (4.59)$$

$$r^2=(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2.$$

This shows that the contribution to the electrodynamic potential φ at a point (**x**) at time t comes from the potentials of all charges inside the sphere of radius ct about (**x**). However the charge at a point $\boldsymbol{\xi}$ inside this sphere is not to be taken at time t, but at an earlier time $t-r/c$, where the difference r/c is the time interval which a signal moving with speed c would need to traverse the distance between the points $\boldsymbol{\xi}$ and **x**. For this reason the expression (4.59) is called a retarded potential.

EXERCISE 4.7

1. Explain how to solve by Duhamel's principle the Cauchy problem for a linear hyperbolic equation $L(u)=0$ with the conditions $u=f(x, y, z)$, $u_t=g(x, y, z)$ given at $t=h(x, y, z)$.
2. $u(x, t)$ is the solution of $u_{tt}-u_{xx}=h(x, t)$ which satisfies the initial conditions $u=0$, $u_t=0$ when $t=0$. By considering the integral of $u_{tt}-u_{xx}$ over a suitable triangle prove that

$$u(x, t)=\frac{1}{2}\int_0^t d\tau\int_{x-\tau}^{x+\tau} h(\xi, t-\tau)\,d\xi.$$

Obtain this result by Duhamel's principle.

§4.8 Boundary Value Problem for the One-dimensional Wave Equation

The initial value problem is by far the more discussed problem for equations of the hyperbolic type when compared to the boundary value problem. However, we will find that in one of the important general classical theories the characteristic boundary value problem plays a leading role. In the 'Riemann method' it provides a resolvent function, which is used in representing solutions of other boundary value problems.

(i) *Characteristic boundary value problem*

A characteristic boundary value problem for the one-dimensional wave equation

$$u_{tt}-\frac{1}{c^2}u_{xx}=0$$

is to find a solution which takes up prescribed values on the two characteristic curves through a point, in the (x, t)-plane, say the origin, i.e. to find a solution $u(x, t)$ such that

$$u(x, t) = f(\alpha) \quad \text{on } x = \alpha, \quad t = \frac{\alpha}{c}, \; \alpha > 0$$

$$u(x, t) = g(\beta) \quad \text{on } x = \beta, \quad t = -\frac{\beta}{c}, \; \beta < 0,$$

where f and g are known functions. (4.60)

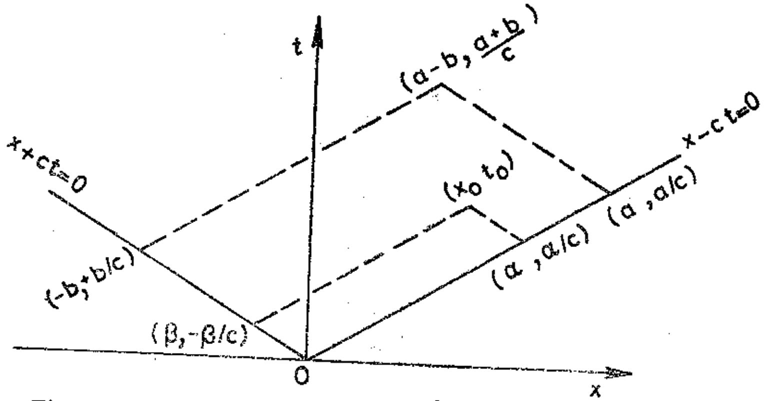

Fig. 4.7 Characteristic boundary value problem with data given on $x-ct=0$ and $x+ct=0$

The general solution of the wave equation is

$$u(x, t) = F(x+ct) + G(x-ct).$$

The functions F and G for which $u(x, t)$ takes up prescribed values (4.60) satisfy

$$F(2\alpha) + G(0) = f(\alpha)$$
$$F(0) + G(2\beta) = g(\beta).$$

Therefore,

$$F(x) = f(x/2) - G(0), \; G(x) = g(x/2) - F(0). \qquad (4.61)$$

The arbitrary functions F and G are determined, except for an additive constant. Thus

$$u(x, t) = f\left(\frac{x+ct}{2}\right) + g\left(\frac{x-ct}{2}\right) - (F(0) + G(0)).$$

But

$$u(0, 0) = f(0) = g(0) = F(0) + G(0).$$

Therefore the solution $u(x, t)$ is given by

$$u(x, t) = f\left(\frac{x+ct}{2}\right) + g\left(\frac{x-ct}{2}\right) - f(0) \qquad (4.62)$$

provided f and g satisfy the relation

$$f(0) = g(0).$$

At any point (x_0, t_0) in the region $(-ct < x < ct)$ bounded by the two characteristics $x+ct=0$, $x-ct=0$, the solution picks up the value of f at $\alpha=(x_0+ct_0)/2$, where $(\alpha, \alpha/c)$ is the point where $x-ct=0$ intersects the characteristic $x+ct=x_0+ct_0$ through (x_0, t_0). The solution takes the value of g at $\beta=(x_0-ct_0)/2$ where $(\beta, -\beta/c)$ is the point where $x+ct=0$ intersects the characteristic $x-ct=x_0-ct_0$.

The solution is unique, as the equation is linear and $u(x, t)=0$ if $f(x)=g(x)=0$. Region of determinacy, if $f(\alpha)$ and $g(\beta)$ are specified for $0 \leqslant \alpha \leqslant a$, $-b \leqslant \beta \leqslant 0$, respectively, is the parallelogram bounded by $x-ct=0$, $x+ct=0$, $x+ct=2a$, $x-ct=-2b$.

We have seen earlier that specification of Cauchy data (satisfying the compatibility condition) on a characteristic does not uniquely determine a solution. From the above problem we verify that the unique determination requires the specification of another piece of data on an intersecting line, in this case a characteristic of the other family.

(ii) *Mixed boundary value problem*

Suppose the data is given on one characteristic curve, say $x-ct=0$, and one non-characteristic curve, say $x=0$. This can be called a mixed boundary value problem—the combination of a characteristic boundary value problem, where the data on one of the characteristics is ignored and a Cauchy problem where the data on the derivative has been ignored. Given data is

$$u(x, 0)=f(x)$$
$$u(\alpha, \alpha/c)=g(\alpha). \tag{4.63}$$

The general solution of the wave equation is

$$u(x, t)=F(x+ct)+G(x-ct).$$

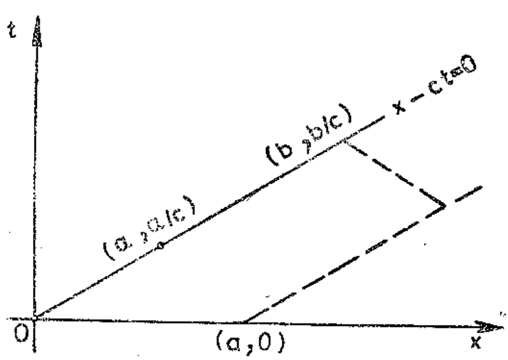

Fig. 4.8 Mixed boundary value problem with data given on $x-ct=0$, $x=0$

Substituting the general solution in (4.63) we get

$$f(x)=F(x)+G(x)$$
$$g(\alpha)=F(2\alpha)+G(0).$$

This leads to

$$F(x)=g(x/2)-G(0),\ G(x)=f(x)-g(x/2)+G(0). \tag{4.64}$$

Therefore

$$u(x,t)=g\left(\frac{x+ct}{2}\right)-g\left(\frac{x-ct}{2}\right)+f(x-ct) \tag{4.65}$$

where the continuity of u at the origin requires

$$f(0)=g(0).$$

The solution is unique as $u=0$ if $f=0$, $g=0$. The region of determinacy, if $f(x)$ and $g(\alpha)$ are specified for $0 \leqslant x \leqslant a$, $0 \leqslant \alpha \leqslant b$, is the region bounded by the characteristic segments through $(a, 0)$ and $(b, b/c)$ and the given segments on $t=0$ and $x=ct$, as shown in Fig. 4.9.

(iii) *The Goursat problem*

Here the data is specified on two intersecting non-characteristic curves strictly contained in an angle between two characteristics passing through the point of intersection of the curves. Without loss of generality, we can take the point of their intersection to be the origin. The intersecting curves are, in this case, taken to be straight lines:

$$x=0 \text{ and } \beta x=ct, \qquad 0<\beta<1.$$

Given data consists of

$$u(x,0)=f(x)$$
$$u\left(\alpha, \frac{\beta\alpha}{c}\right)=g(\alpha),\ 0<\beta<1 \tag{4.66}$$

and $f(0)=g(0)$ for continuity of u at the origin.

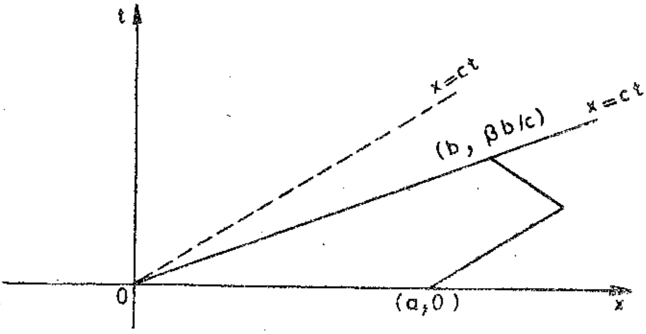

Fig. 4.9 Goursat problem in the region bounded by $x=ct/\beta$, $t=0$ $0<\beta<1$

The general solution is

$$u(x,t)=F(x+ct)+G(x-ct).$$

Now we require that

$$f(x)=F(x)+G(x) \tag{4.67}$$
$$g(\alpha)=F(\alpha(1+\beta))+G(\alpha(1-\beta)). \tag{4.68}$$

It is not possible to determine the functions F and G easily from these relations, as could be done in cases (i) and (ii). From (4.67) and (4.68)

$$G\{(1+\beta)x\}-G\{(1-\beta)x\}=f\{(1+\beta)x\}-g(x).$$

Let

$$x=\frac{X}{1+\beta},\ 0<\delta=\frac{1-\beta}{1+\beta}<1,\ p(X)=g\left(\frac{X}{1+\beta}\right)-f(x).$$

Then

$$p(X)=-G(X)+G(\delta X).$$

It follows that

$$p(\delta X)=-G(\delta X)+G(\delta^2 X)$$

and so on. Therefore

$$\sum_{i=0}^{n} p(\delta^i X)=-G(X)+G(\delta^{n+1}X).$$

Since G is continuous and $0<\delta<1$, letting n tend to infinity we get

$$G(X)=G(0)-\sum_{i=0}^{\infty} p(\delta^i X)$$

provided $\sum_{i=0}^{\infty} p(\delta^i X)$ exists. Using (4.67) we can get $F(x)$. Hence the solution of the problem is given by

$$u(x,t)=f(x+t)+\sum_{i=0}^{\infty} p\{\delta^i(x+ct)-\sum_{i=0}^{\infty} p\{\delta^i(x-ct)\}. \tag{4.69}$$

The functions f and g must be such that $\sum_{i=0}^{\infty} p(\delta^i X)$ converges in order that the solution (4.69) is valid. Here also the solution is unique. The region of determinacy when $f(x)$ and $g(\alpha)$ are specified for $0\leqslant x\leqslant a$ and $0<\alpha<b$, respectively, is the region bounded by the characteristics through $(a, 0)$ and $(b, b\beta/c)$ and the given segments on $t=0$ and $\beta x=ct$.

In all the three cases (i)-(iii) the problems are wellposed. The solution u is given in terms of f and g, which when equal to zero, imply that u is also zero. If f and g are arbitrarily small, then u is also arbitrarily small. Hence the solutions are unique and stable and, therefore, the problem is wellposed.

(iv) *The Dirichlet problem for the wave equaiton*

While the Dirichlet problem is basic for the Laplace equation and other equations of elliptic type, we examine here its wellposedness for an equation of hyperbolic type.

Suppose in a rectangle bounded by $x=0, x=l, t=0, t=T$, we look for a solution of the one dimensional wave equation

$$u_{xx}-\frac{1}{c^2}u_{tt}=0$$

satisfying the boundary condition

$$u = 0 \text{ on the sides of the rectangle.}$$

Let the solution be given by

$$u = \varphi(x)\ \psi(t).$$

Then we requre

$$\varphi'' + k^2\varphi = 0 \qquad \psi'' + k^2c^2\psi = 0.$$

Therefore
$$\varphi = A \sin kx + B \cos kx \tag{4.70}$$
$$\psi = C \sin kct + D \cos kct.$$

Since $\varphi(0) = \varphi(l) = 0$ we have $B = 0$, $kl = n\pi$, $n = 1, 2\ldots$

Also $\psi(0) = \psi(T) = 0$ implies $D = 0$, $ckT = m\pi$, $m = 1, 2, \ldots$

This requires $\dfrac{n\pi}{l} = \dfrac{m\pi}{cT}$ for specific m and n

i.e.
$$\frac{cT}{l} = \frac{m}{n}. \tag{4.71}$$

If the ratio of c times height to length of base of the rectangle is rational, the solution of the Dirichlet problem is nonunique. This is because, if m_1, n_1, are possible values of m and n, respectively, so are $2m_1$ and $2n_1$ also. Besides, no nontrivial separable solution is possible if the ratio cT to l is irrational.

The solution to the Dirichlet problem for the wave equation on a rectangle is unique if the ratio of lengths of its sides is irrational; otherwise the solution is nonunique. If the boundary is slightly perturbed so that cT/l changes from an irrational to a neighbouring rational number arbitrarily close to it, the zero solution is changed to a nonunique non-zero solution. Therefore the solution ceases to be stable. Thus the Dirichlet problem is not wellposed for the wave equation.

EXERCISE 4.8

1. Find the solution $u(x, t)$ of the characteristic boundary value problem

$$u_{xt} = 0$$

with $\quad u(x, 0) = 3x^2 + 5, \quad u(0, t) = -t^3 + 5.$

2. Find the solution $u(x, t)$ of the following Goursat problem

$$u_{xx} - u_{tt} = 0$$

$$u(x, 0) = \sin x,\ u(x, \tfrac{2}{3}x) = x.$$

§4.9 Riemann's Method for Linear Hyperbolic Equation in x, t

Consider a Cauchy problem associated with a linear hyperbolic equation, in two independent variables in the canonical form,

$$u_{xt}+du_x+eu_t+fu=g \tag{4.72}$$

where d, e, f, g are functions of x and t in a domain and have continuous second derivatives there in. u and $\partial u/\partial \nu$ are prescribed on a noncharacteristic curve C, where $\partial/\partial\nu$ denotes differentiation in the direction of the normal to C.

To solve this problem we consider a function $v \in C^2$ and use the identity

$$vLu-u\bar{L}v=\frac{\partial}{\partial x}E+\frac{\partial}{\partial t}F \tag{4.73}$$

where

$$Lu \equiv u_{xt}+du_x+eu_t+fu$$
$$\bar{L}v \equiv v_{xt}-dv_x-ev_t+(f-d_x-e_t)v$$
$$E \equiv \frac{1}{2}(vu_t-uv_t)+duv$$
$$F \equiv \frac{1}{2}(vu_x-uv_x)+euv.$$

$\bar{L}$ is the adjoint operator of L. Applying Green's theorem to a surface intergal of (4.73), we get

$$\iint_D (vLu-u\bar{L}v)\,dx\,dt=\int_{\partial D} -F\,dx+E\,dt \tag{4.74}$$

where ∂D is the closed curve, which is the boundary of domain D.

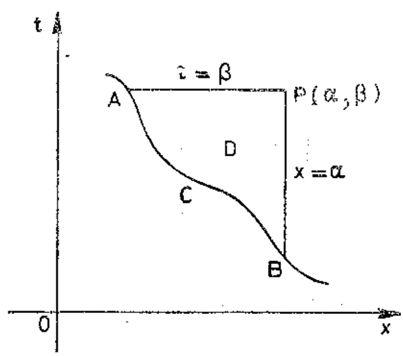

Fig. 4.10 Riemann method to determine solution of (4.72) at P for datum curve C

We now apply the identity (4.74) to a domain D bounded by the datum curve C and two characteristics $x=\alpha$ and $t=\beta$ passing through a point $P(\alpha, \beta)$.

Here

$$\int_{\partial D} -F\,dx+E\,dt=\int_{AB} -F\,dx+E\,dt$$
$$+\int_{t_B}^{\beta} E\,dt-\int_{\alpha}^{x_A} F\,dx$$

where x_A is the x coordinate of A and t_B is the t coordinate of B.

$$\int_{x_A}^{\alpha} F\,dx=\int_{x_A}^{\alpha}\left[\frac{1}{2}(vu_x-uv_x)+euv\right]dx$$
$$=\int_{x_A}^{\alpha}\left[\frac{1}{2}(uv)_x-uv_x+euv\right]dx$$
$$=\frac{1}{2}(uv)_P-\tfrac{1}{2}(uv)_A-\int_{x_A}^{\alpha} u(v_x-ev)\,dx$$

and

$$\int_{t_B}^{\beta} E\,dt=\int_{t_B}^{\beta}\left[\frac{1}{2}(vu_t-uv_t)+duv\right]dt$$
$$=\frac{1}{2}(uv)_P-\frac{1}{2}(uv)_B-\int_{t_B}^{\beta} u(v_t-dv)\,dt.$$

We chose v to satisfy the equation

$$\bar{L}v=0 \text{ in } D \tag{4.75}$$

together with characteristic boundary conditions

$$\begin{aligned} v_x&=ev \quad \text{on} \quad t=\beta \\ v_t&=dv \quad \text{on} \quad x=\alpha. \end{aligned} \tag{4.76}$$

The characteristic boundary conditions are satisfied if

$$\left.\begin{aligned} v(x,\beta)&=\exp\int_{\alpha}^{x} e(\xi,\beta)\,d\xi \\ \text{and}\quad v(\alpha,t)&=\exp\int_{\beta}^{t} d(\alpha,\eta)\,d\eta \end{aligned}\right\} \tag{4.76a}$$

where we choose v so that

$$v(\alpha,\beta)=1.$$

The solution to this characteristic boundary value problem (4.75), (4.76a) is called the Riemann-Green function (or simply Riemann function) with respect to (α,β) and is denoted by $v(x,t;\alpha,\beta)$. Using this function, (4.74) reduces to

$$\iint_D v(x,t,x_0,y_0)g(x,t)\,dx\,dt=(uv)_P-\frac{1}{2}(uv)_A$$
$$-\frac{1}{2}(uv)_B+\int_{AB} -F\,dx+E\,dt. \tag{4.77}$$

Here

$$\int_{AB} -F\,dx+E\,dt=\int_{AB}\left[\left(-\frac{1}{2}vu_x+\frac{1}{2}uv_x-euv\right)dx\right.$$
$$\left.+\left(\frac{1}{2}vu_t-\frac{1}{2}uv_t+duv\right)dt\right]$$
$$=\int_{AB}\left[\frac{1}{2}v\frac{du}{d\lambda}-\frac{1}{2}u\frac{dv}{d\lambda}-euv\frac{dx}{ds}+duv\frac{dt}{ds}\right]ds$$

where d/ds denotes differentiation along the curve C and

$$\frac{d}{d\lambda}\equiv-\frac{dx}{ds}\frac{\partial}{\partial x}+\frac{dt}{ds}\frac{\partial}{\partial t}.$$

Since u and $\partial u/\partial v$ are prescribed on AB, the integrand in (4.77) is completely known, as all derivatives of u on AB can be calculated from the given data. Using $v(\alpha,\beta)=1$ in (4.77) we have

$$(u)_P=\frac{1}{2}(uv)_A+\frac{1}{2}(uv)_B-\int_{AB}\left[\frac{1}{2}v\frac{du}{d\lambda}\right.$$
$$\left.-\frac{1}{2}u\frac{dv}{d\lambda}-euv\frac{dx}{ds}+duv\frac{dt}{ds}\right]ds+\iint_D vg\,dx\,dt. \qquad (4.78)$$

This is the required form of the solution of the Cauchy problem. It gives the value of u at P in terms of the Riemann function and the Cauchy data for u along C.

Thus we have reduced the problem of solving of a Cauchy or initial value problem to that of solving a corresponding characteristic boundary value problem. Unlike the Green's function introduced in §2.2, the Riemann function does not depend in any way on the arc carrying the Cauchy data and it is regular in D, i.e. it is not required to have a singularity in the domain D.

Symmetry of Riemann's Function

Let D be the region enclosed by a rectangle $PQRS$ whose sides are characteristic curves and let the initial data be given on a curve passing through Q and S. Also let $g=0$.

Then if v is the Riemann function of $Lu=0$ with respect to P, we have from (4.78)

$$u_P=\frac{1}{2}(uv)_Q+\frac{1}{2}(uv)_S+\int_{x_R}^{x_S}\left(-\frac{1}{2}vu_x+\frac{1}{2}uv_x-euv\right)dx$$
$$+\int_{t_S}^{t_R}\left(\frac{1}{2}vu_t-\frac{1}{2}uv_t+duv\right)dt.$$
$$=\frac{1}{2}(uv)_Q+\frac{1}{2}(uv)_S-\left(\frac{1}{2}uv\right)_S+\left(\frac{1}{2}uv\right)_R+\int_{x_R}^{x_S}v(u_x+eu)\,dx$$
$$-\left(\frac{1}{2}uv\right)_Q+\frac{1}{2}(uv)_R+\int_{t_R}^{t_Q}v(u_t+du)\,dt$$

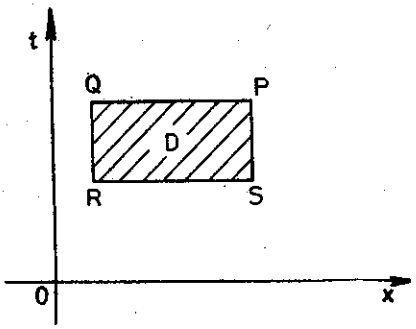

Fig. 4.11 *PQRS* is a rectangle whose sides are characteristic curves

where the integral over a curve joining Q and S has been replaced by integrals over QR and RS, since in view of (4.74) the integral over the closed curve $QRSQ$ vanishes. Therefore,

$$u_P = (uv)_R + \int_{x_R}^{x_S} v(u_x + eu)\, dx + \int_{t_R}^{t_Q} v(u_t + du)\, dt. \tag{4.79}$$

Choose w to be the Riemann function of the adjoint equation

$$\bar{L}v \equiv v_{xy} - dv_x - ev_t + (f - d_x - e_t)v = 0$$

with respect to the point R. Then w must be determined by the equation

$$\bar{\bar{L}}w = Lw = 0 \tag{4.80}$$

and boundary data

$$\begin{aligned} w_x + ew &= 0 \quad \text{on } RS, \\ w_t + dw &= 0 \quad \text{on } RQ, \\ w(R) &= 1. \end{aligned} \tag{4.81}$$

Since w satisfies $Lw = 0$, its solution must be given by (4.79). Using boundary data (4.81) in (4.79) for w, we get

$$w_P = v_R$$

i.e. $$w(x_P, t_P;\, x_R,\, t_R) = v(x_R,\, t_R;\, x_P,\, t_P). \tag{4.82}$$

In other words, the Riemann's functions v of an operator L becomes the Riemann's function w of its adjoint operator $\bar{L}$ when one interchanges the two sets of variables in its argument, i.e.

$$w(x_2,\, t_2;\, x_1,\, t_1) = v(x_1,\, t_1;\, x_2,\, t_2).$$

This is called the symmetric property of the Riemann function.

To give a physical meaning to the Riemann function, consider a sequence $\{g_k\}$ of functions with the properties:

(i) $g_k \geqslant 0$ is different from zero only in a neighbourhood N_k of a fixed point $Q=(\alpha, \beta)$

(ii) $\iint_{N_k} g_k \, dx \, dt = 1$ for each k, and

(iii) The neighbourhoods N_k shrink to the point Q as $k \to \infty$.

If we denote by y_k, the solution of $Ly = g_k$ which vanishes along with its first derivatives along the initial curve C, then its Riemann's representation (4.78) shows that since

$$\lim_{k\to\infty} \iint_D v g_k \, dx \, dt = v(Q, P),$$

$$\lim_{k\to\infty} y_k = y \text{ exists and}$$

$$y(P) = v(Q, P) \tag{4.83}$$

where $v(Q, P)$ is the Riemann function of the operator L and y satisfies the limiting equation

$$L(y) = \delta(x-\alpha, t-\beta) \tag{4.84}$$

with zero initial conditions.

The result
$$y(x, t) = v(\alpha, \beta; x, t)$$
can be interpreted physically as the intensity at point x at time t of radiation emitted from a source of unit strength at the point $Q(\alpha, \beta)$ in space-time.

Example 4.4 The Riemann function for the wave equation

$$u_{xt} = 0$$

is given by

$$v(x, t; \alpha, \beta) = 1$$

and hence the solution, in this case, is given by

$$u(p) = \frac{1}{2}[u(A) + u(B)] + \frac{1}{2}\int_{AB} (u_x \, dx - u_t \, dt). \tag{4.85}$$

Example 4.5 The telegraph equation, with c constant, is

$$u_{xt} + cu = g(x, t). \tag{4.86}$$

This equation is self-adjoint. The Riemann function for the point $Q(\alpha, \beta)$ remains constant along the straight lines parallel to the coordinate axes through Q and so can be taken to be a function of $(x-\alpha)(t-\beta)$. Therefore, for the point $Q(\alpha, \beta)$, the Riemann's function is of the form

$$v(x, t; \alpha, \beta) = f(z)$$

where $$z=(x-\alpha)(t-\beta).$$

From the equation satisfied by v, we get

$$zf''+f'+cf=0$$

wherein if we set $\lambda=\sqrt{4cz}$, it becomes

$$\frac{d^2f}{d\lambda^2}+\frac{1}{\lambda}\frac{df}{d\lambda}+f=0.$$

The solution which is regular at (α, β) is

$$v(x, t; \alpha, \beta)=J_0(\sqrt{4c(x-\alpha)(t-\beta)}\,) \tag{4.87}$$

where J_0 is the Bessel function of order zero.

Example 4.6 Consider the equation

$$u_{xt}-\frac{n}{x+t}(u_x+u_t)=0 \tag{4.88}$$

where n is a constant.

Then v satisfies the adjoint equation

$$v_{xt}+\frac{n}{x+t}(v_x+v_t)-\frac{2n}{(x+t)^2}v=0 \tag{4.89}$$

with

$$\left.\begin{aligned} v_x(x, \beta; \alpha, \beta) &= -\frac{n}{x+\beta}v(x, \beta; \alpha, \beta) \\ v_t(\alpha, t; \alpha, \beta) &= -\frac{n}{t+\alpha}v(\alpha, t; \alpha, \beta)\end{aligned}\right\} \tag{4.90}$$

and $$v(\alpha, \beta; \alpha, \beta)=1.$$

Integrating (4.90) along the characteristics, we get

$$v(x, \beta; \alpha, \beta)=\left(\frac{x+\beta}{\alpha+\beta}\right)^{-n}, \quad v(\alpha, t; \alpha, \beta)=\left(\frac{t+\alpha}{\alpha+\beta}\right)^{-n}. \tag{4.91}$$

As a solution of (4.89) we try the polynomial $\psi(w)$ in the expression for v, i.e.

$$v(x, t; \alpha, \beta)=\frac{(\alpha+\beta)^n}{(x+t)^n}\psi(w) \tag{4.92}$$

where

$$w=-\frac{(\alpha-x)(\beta-t)}{(\alpha+\beta)(x+t)} \tag{4.93}$$

and

$$\psi(w)=1+a_1w+\dots+a_nw^n.$$

Then ψ satisfies the equation

$$w(w-1)\psi''+(2w-1)\psi'-n(n+1)\psi=0. \tag{4.94}$$

This has as solution the hypergeometric function:

$$\psi = F\left(1+n,\ -n;\ 1;\ -\frac{(\alpha-x)(\beta-t)}{(x+y)(\alpha+\beta)}\right). \tag{4.95}$$

Therefore,

$$v(x,\ t;\ \alpha,\ \beta)=\left(\frac{\alpha+\beta}{x+t}\right)^n F\left(1+n,\ -n;\ 1;-\frac{(\alpha-x)(\beta-t)}{(\alpha+\beta)(x+t)}\right).$$

EXERCISE 4.9

1. Verify that the Riemann-Green function for

$$u_{xt}-\frac{2}{(x+t)^2}u=0$$

is

$$v(x,\ t;\ \alpha,\ \beta)=\frac{(x+\beta)(t+\alpha)+(x-\alpha)(t-\beta)}{(x+t)(\alpha+\beta)}.$$

Use Riemann's method to show that the solution which satisfies $u=0,\ u_t=t^2$ on $x=t$ is $u=\frac{1}{4}(t-x)(t+x)^2$.

2. Show that the Riemann-Green function for

$$u_{xx}-u_{tt}-\frac{2}{x}u_x=0$$

is

$$v(x,\ t;\ \alpha,\ \beta)=\frac{x^2+\alpha^2-(t-\beta)^2}{2x^2}.$$

Find the solution which satisfies the conditions

$$u=f(x),\ u_t=g(x) \text{ on } t=0.$$

3. Prove that the Riemann-Green function for

$$u_{xt}+\frac{a}{x}u_x+\frac{b}{x}u_t=0$$

where a and b are constants is

$$v(x,\ t;\ \alpha,\ \beta)=\left(\frac{x}{\alpha}\right)^b \exp\left\{\frac{a(t-\beta)}{\alpha}\right\}{}_1F_1\left(1-\beta;\ 1;\ \frac{-aZ}{x\alpha}\right)$$

where $Z=(x-\alpha)(t-\beta)$

and

$${}_1F_1(C_1;\ C_2;\ y)=\frac{\Gamma(C_2)}{\Gamma(C_1)}\sum_{n=0}^{\infty}\frac{\Gamma(C_1+n)}{\Gamma(C_2+n)}\frac{y^n}{n!}.$$

Γ denotes the Gamma function.

CHAPTER 3

Hyperbolic Partial Differential Equations

1. INTRODUCTION

In the previous chapter, we took up a study of the general second order linear partial differential equation and noted that the equations could be classified systematically. Problems associated with the different classes of equations become well posed provided different types of initial and boundary values are prescribed properly. Also, the properties of the solutions of the equations of different classes are basically different. The scope of Chapt. 2 restricted us from pointing out and showing that the solutions of hyperbolic equations show a variety of properties not observed in the solutions of equations of other classes.

A physical phenomenon, which is governed by hyperbolic equation(s) can always be described in the language of wave theory with finite speed of propagation. In this case, if the solution *initially* vanishes outside a closed bounded domain, then at any future time there exists another closed bounded domain outside which it continues to vanish. This property of the solutions of a hyperbolic equation is one of the most striking of its properties. In fact, starting with the assumption of the finiteness of the signal propagation velocity in a physical system it is possible to prove a very general theorem which asserts that the motion of the physical system is governed by a hyperbolic equation of order n, where n is the number of state variables of the system. Apart from the finiteness of the signal propagation velocity, the other principal assumption in the theorem is that the motion of the system is uniquely determined by n arbitrarily prescribed initial conditions (Lax, 1963).

Unlike in the case of elliptic equations, the hyperbolic equations in two independent variables differ very significantly in their properties and in their methods of solution from those in more than two independent variables. For example, the smoothness properties of the *initial* data for linear hyperbolic equations in two independent variables is preserved by the solutions but this is, in general, not the case for equations in three or more independent variables. Amongst the other striking properties of hyperbolic equations, which we have discussed for the wave equation are (i) the propagation of discontinuities in initial data along the characteristic curves in the one-dimensional case and (ii) well posedness of a non-characteristic Cauchy problem. We have also noted that the concepts of the domains of dependence, influence and determinacy are of relevance only for the wave equation and not for the Laplace and diffusion equations.

In this chapter, we shall systematically discuss those basic properties and methods of solution which are peculiar to hyperbolic equations. For hyperbolic equations, there is one *time-like* independent variable which plays a role very different from other independent variables specially when the total number of such variables is greater than or equal to three. Therefore, in the present chapter it will be more convenient to denote one independent variable by the symbol t and others by $x_1, x_2, \cdots, x_m$ or simply by x when the total number of independent variables is two.

Part A: EQUATIONS IN TWO INDEPENDENT VARIABLES

§2. FIRST ORDER HYPERBOLIC SYSTEM

In Chapt. 1 we discussed in great detail the simplest examples of hyperbolic equations, namely the equations of the first order. We started with a discussion of the first order semilinear equation in two independent variables:

$$a(x, t)u_t + b(x, t)u_x = c(x, t, u).$$

The theory of this equation was based on the fact that we could find a one-parameter family of characteristic curves in the (x, t)-plane such that along each one of these curves the above equation reduced to an ordinary differential equation in u which, therefore, became a compatibility condition. In the following, the treatment of a general hyperbolic system of first order equations is based exactly on the same principle. We can find families of curves such that along each one of these a linear combination of the equations of the system reduces to an ordinary differential equation leading to a compatibility condition.

§2.1 Definition of a First Order Hyperbolic System

A first order system of n partial differential equations can be written in the form

$$\left.\begin{aligned} &A_{ij}\frac{\partial u_j}{\partial t} + B_{ij}\frac{\partial u_j}{\partial x} + C_i = 0, \; i = 1, 2, \cdots, n \\ \text{or} \\ &A\frac{\partial U}{\partial t} + B\frac{\partial u}{\partial x} + C = 0 \end{aligned}\right\} \tag{2.1}$$

where the n components $u_1, u_2, \ldots, u_n$ of the column vector U are the dependent variables; A and B are $n \times n$ matrices and C is a column vector. A repeated suffix i, j or k in a term will represent sum over the range $1, 2, \ldots, n$. In general, we consider a *quasilinear* system where the elements of A, B and C are real C^1 functions over a domain D_2 of (x, t, U)-space. We emphasise that the results of this section are valid for both linear and quasilinear systems; however, in the latter case it is assumed that a known solution $u_i(x, t)$ (or $U(x, t)$) has been substituted for the dependent variables in the coefficients. Therefore, for a quasilinear system, our results are valid only for the particular solution under consideration.

In the ith equation in (2.1), the jth dependent variable has been differentiated in a direction given by $dx/dt = B_{ij}/A_{ij}$ which is generally different for different j. We attempt to form a linear combination of n equations (2.1) such that in the resulting equation all dependent variables are differentiated in the same direction. This direction, if it exists, is called a *characteristic direction* of the system.

Let l be a row vector of n components; then a linear combination of (2.1) is

$$\left.\begin{aligned} & l_i A_{ij}\frac{\partial u_j}{\partial t} + l_i B_{ij}\frac{\partial u_j}{\partial x} + l_i C_i = 0 \\ \text{or} \quad & \\ & lA\frac{\partial U}{\partial t} + lB\frac{\partial U}{\partial x} + lC = 0. \end{aligned}\right\} \tag{2.2}$$

If all dependent variables in (2.2) are differentiated in the same direction $dx/dt = \lambda$, we have

$$\lambda = (l_i B_{ij})/(l_i A_{ij}) \text{ or } l_i(B_{ij} - \lambda A_{ij}) = 0, \quad j = 1, 2, \ldots, n$$

or

$$l(B - \lambda A) = 0 \tag{2.3}$$

which shows that l is the left eigenvector of B with respect to the matrix A and λ is the corresponding eigenvalue satisfying

$$\det (B - \lambda A) = 0. \tag{2.4}$$

Let $\lambda = c$ be a real root with multiplicity p of the equation (2.4); then there exists a positive integer $s < p$ such that the rank of the matrix $B - cA$ is $n - s \leqslant n - 1$ and there exist s linearly independent left eigenvectors $l^{(\delta)}$, $\delta = 1, 2, \ldots, s$ corresponding to c. The corresponding characteristic direction in the (x, t)-plane is given by

$$\frac{dx}{dt} = c(x, t), \quad \left(\text{or } \frac{dx}{dt} = c(x, t, U(x, t))\right) \tag{2.5}$$

where c is a function of x and t even for a quasilinear system since we have substituted a known solution for the dependent variables. The number c is called a *characteristic root* or *characteristic velocity* of the system (2.1). The solution of the equation (2.5) gives a one-parameter family of curves in the (x, t)-plane, each member of which is called a *characteristic curve* corresponding to the root c.

Using (2.3) with $\lambda=c$ in (2.2) we get

$$l^{(\delta)}A\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)U+l^{(\delta)}C=0,\ \delta=1,2,\ldots,s. \tag{2.6}$$

We note that $\partial/\partial t+c\partial/\partial x$ represents an interior derivative with respect to the characteristic curve. Hence (2.6) gives s compatibility conditions along a characteristic curve. The compatibility conditions show that the values of the n variables $u_1, u_2, \ldots, u_n$ in a solution of (2.1) are constrained to satisfy s relations along a characteristic curve.

Before proceeding to the definition of a hyperbolic system, we assume without loss of any generality that at least one of the two matrices A and B is nonsingular or a linear combination of A and B is nonsingular at every point of a domain D_1 in the (x, t)-plane. In the second case, we can rotate the axes in the (x, t)-plane such that when the equations are written in terms of the new independent variables (x', t'), we have

$$\det A' \neq 0 \tag{2.7}$$

at every point of the domain D_1 where A' is the coefficient matrix of $\partial U/\partial t'$. The characteristic equation (after dropping dashes) (2.4) is now an nth degree polynomial equation in λ. Let the distinct roots (not necessarily real) of this equation be $c_1, c_2, \ldots, c_r (r \leqslant n)$ and the multiplicity of the root c_q be $p_q (1 \leqslant p_q \leqslant n)$ so that $\sum_{q=1}^{r} p_q = n$. Let the rank of $B-c_qA$ be $n-s_q \leqslant n-1$; then $s_q \leqslant p_q$. There will then be $\sum_{q=1}^{r} s_q$ linearly independent eigenvectors $l^{(q,\delta)} (\delta=1, 2, \ldots, s_q)$, not necessarily real. In general, $\sum_{q=1}^{r} s_q \leqslant n$.

When all the characteristic roots $c_1, c_2, \ldots, c_r$ are *real* and the *number of linearly independent eigenvectors corresponding to every characteristic velocity c_q is equal to multiplicity of c_q* (i.e. $s_q = p_q$ for $q=1, 2, \ldots, r$), the system (2.1) is called *hyperbolic*. In this case we get n independent compatibility conditions

$$l^{(q,\delta)}A\left(\frac{\partial}{\partial t}+c_q\frac{\partial}{\partial x}\right)U+l^{(q,\delta)}C=0$$
$$\delta=1, 2, \ldots, p_q;\ q=1, 2, \ldots, r \tag{2.8}$$

along the r distinct families of characteristic curves*. In particular, when all characteristic roots are *real* and *simple* $(r=n)$, $p_q=s_q=1$ for $q=1, 2, \ldots, n$, then to each root there corresponds a unique (except for a scalar multiplying factor) eigenvector and the system is *hyperbolic*.

If, in the extreme contrast to the hyperbolic case, the characteristic equation (2.4) does not have any real root, the system (2.1) is called elliptic.

*Throughout this book we shall assume that the multiplicity of each of the characteristic roots remains constant in the entire domain under consideration. This implies that if $c_i < c_j$ (for some i and j) at one point of the domain, $c_i < c_j$ everywhere in the domain.

Example 2.1 As an example of a hyperbolic system with one multiple characteristic, we shall consider a quasilinear system and show that this system is always hyperbolic irrespective of a particular solution U considered.

Consider the one-dimensional magnetohydrodynamic flow of a perfect gas through a channel of slowly varying cross section. Assuming that the conductivity is infinite and that the direction of the magnetic field is perpendicular to the axis of the channel, we can write the equations representing the conservation of mass, momentum and energy and the equation representing the interaction of the magnetic field with fluid flow as

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}+\frac{\rho u}{A}\frac{dA}{dx}=0,\ A=A(x) \tag{2.9}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{B}{\rho\mu}\frac{\partial B}{\partial x}=0 \tag{2.10}$$

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}-a^2\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)=0 \tag{2.11}$$

and

$$\frac{\partial B}{\partial t}+u\frac{\partial B}{\partial x}+B\frac{\partial u}{\partial x}=0 \tag{2.12}$$

where ρ is the mass density, p gas pressure, u particle velocity, B intensity of magnetic field, μ magnetic permeability, a the isentropic velocity of sound $=\sqrt{\gamma p/\rho}$ and A the cross-sectional area of the channel. A is a known function of x. The system can be written in the matrix form with

$$U=\begin{bmatrix}\rho\\ u\\ p\\ B\end{bmatrix},\quad \bar{A}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\-a^2&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\bar{B}=\begin{bmatrix}u&\rho&0&0\\0&u&\frac{1}{\rho}&\frac{B}{\mu\rho}\\-a^2u&0&u&0\\0&B&0&u\end{bmatrix},\quad \bar{C}=\begin{bmatrix}\frac{\rho u}{A}\frac{dA}{dx}\\0\\0\\0\end{bmatrix}. \tag{2.13}$$

Consider any solution of the system. All the characteristic velocities are real and given by

$$c_1=u+a_A,\ c_2=u,\ c_3=u-a_A \tag{2.14}$$

of which $c_2=u$ is of multiplicity two. a_A represents the Alfven velocity and is given by

$$a_A=\sqrt{B^2/(\rho\mu)+a^2}.$$

The rank of the matrix $\bar{B}-u\bar{A}$ is two showing that the system is hyperbolic. For $c_1=u-a_A$, we can choose

$$l^{(1)}=(a^2,\ \rho a_A,\ 1,\ B/\mu) \tag{2.15}$$

so that the compatibility condition becomes

$$\frac{\partial p}{\partial t}+(u+a_A)\frac{\partial p}{\partial x}+\rho a_A\left\{\frac{\partial u}{\partial t}+(u+a_A)\frac{\partial u}{\partial x}\right\}$$
$$+\frac{B}{\mu}\left\{\frac{\partial B}{\partial t}+(u+a_A)\frac{\partial B}{\partial x}\right\}=0. \qquad (2.16)$$

$+ \frac{\rho u a^2}{A}\frac{dA}{dx}$

For $c_2=u$, we can choose

$$l^{(2,1)}=(0,\ 0,\ 1,\ 0) \qquad (2.17)$$

which gives the compatibility condition

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}-a^2\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)=0 \qquad (2.18)$$

and

$$l^{(2,2)}=(-B,\ 0,\ 0,\ \rho) \qquad (2.19)$$

leading to the compatibility condition

$$B\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)-\rho\left(\frac{\partial B}{\partial t}+u\frac{\partial B}{\partial x}\right)+\frac{\rho B u}{A}\frac{dA}{dx}=0. \qquad (2.20)$$

For $c_3=u-a_A$, we get the following left eigenvector

$$l^{(3)}=\left(a^2,\ -\rho a_A,\ 1,\ \frac{B}{\mu}\right) \qquad (2.21)$$

and the compatibility condition

$$\frac{\partial p}{\partial t}+(u-a_A)\frac{\partial p}{\partial x}-\rho a_A\left\{\frac{\partial u}{\partial t}+(u-a_A)\frac{\partial u}{\partial x}\right\}$$
$$+\frac{B}{\mu}\left\{\frac{\partial B}{\partial t}+(u-a_A)\frac{\partial B}{\partial x}\right\}=0. \qquad (2.22)$$

$+ \frac{\rho u a^2}{A}\frac{dA}{dx}$

Comment on definition of hyperbolicity for a single higher order equation.

Let us consider here an nth order linear partial differential equation

$$Lu\equiv L^{(n)}u+L^{(n-1)}u+\ \dots\ +L^{(0)}u=f(x,\ t) \qquad (2.23)$$

where

$$L^{(i)}\equiv L^{(i)}\left(\frac{\partial}{\partial t},\ \frac{\partial}{\partial x}\right)=\sum_{j=0}^{i}a_j^{(i)}(x,\ t)\frac{\partial^i}{\partial t^j\partial x^{i-j}},\quad a_n^{(n)}\neq 0 \qquad (2.24)$$

is a linear homogeneous partial differential operator of order i.

The nth order equation (2.23) is defined to be hyperbolic if it can be reduced to a hyperbolic system of n first order equations.

With the help of the highest order derivative terms we define a characteristic equation for (2.23)

$$L^{(n)}(-\lambda,\ 1)=0 \qquad (2.25)$$

which has been obtained by replacing $\partial/\partial t$ by $-\lambda$, and $\partial/\partial x$ by 1 in the operator $L^{(n)}$. The characteristic equation is a polynomial in λ of degree n since $a_n^{(n)}\neq 0$ in the domain of the $(x,\ t)$-plane under consideration. If the characteristic equation (2.25) has n real and distinct roots for λ in a domain D_1 in (x, t)-plane, it is simple to show that (2.23) is hyperbolic in D_1. In the case of a multiple root, the fact that all the roots are real does not ensure the hyperbolicity of the equation, as in this case it may not be possible to reduce the equation to a first order hyperbolic system (see example 2.3 below). Consider now an equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)u+\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)u=0 \tag{2.26}$$

for which the characteristic equation has a double root. It can be reduced to a pair of equations:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)u=v \tag{2.27}$$

and

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)v=-v \tag{2.28}$$

which form a hyperbolic system in the canonical form (2.8). Therefore, the equation (2.26) is hyperbolic.

Example 2.2 Consider the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0 \tag{2.29}$$

where c is a constant not equal to zero.

The characteristic equation is $\lambda^2-c^2=0$ which has two distinct real roots $c, -c$ and hence the equation is hyperbolic.

Example 2.3 Consider the one dimensional diffusion equation

$$\frac{\partial u}{\partial t}-K\frac{\partial^2 u}{\partial x^2}=0 \tag{2.30}$$

where K is non-zero constant. This is an equation of second order and $a_2^{(2)}=0$. Therefore we consider the characteristic equation $L^{(2)}(-1, \mu)=0$ in terms of a variable $\mu=1/\lambda$. It has a double root. Reducing the partial differential equation to a first order system we get

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}u_1\\ u_2\end{bmatrix}+\begin{bmatrix}0 & -K\\ 1 & 0\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}u_1\\ u_2\end{bmatrix}+\begin{bmatrix}0\\ -u_2\end{bmatrix}=0 \tag{2.31}$$

where $u_1=u$ and $u_2=\dfrac{\partial u_1}{\partial x}$.

We note here that the matrix A is singular and B is nonsingular. In this case the degree of the characteristic equation, $\det(B-\lambda A)=0$, is less than the number of equations in the system. So we write the characteristic equation in terms of $\mu=1/\lambda$. This gives

$$\det(A-\mu B)=0 \quad\text{or}\quad \mu^2=0. \tag{2.32}$$

Therefore $\mu=0$ is a double root. However, the rank of the matrix $A-\mu B$, for $\mu=0$, is one showing that there exists only one eigenvector. The system (2.31) and consequently the equation (2.30) is not hyperbolic. The diffusion equation belongs to a class of equations which are *parabolic*.

§2.2 Linear and Semilinear Equations: Canonical Form, Numerical Solution, Domains of Dependence, Influence and Determinacy

Consider a semilinear hyperbolic system (2.1) where we assume that (i) the matrices A and B are functions of x and t only and (ii) the matrix A is non-singular in a domain D_1 of (x, t)-plane. The column vector C need not be a linear function of U, i.e. $C=C(x, t, U)$. The second assumption implies that a characteristic velocity is finite everywhere, and therefore a tangent to a characteristic curve of (2.1) is nowhere parallel to the x-axis. Let the n characteristic velocities of the system be denoted by

$$\lambda_1, \lambda_2, \dots, \lambda_n \tag{2.33}$$

where all λ's are not necessarily distinct. Note that the distinct characteristic velocities in this set are denoted by $c_1, c_2, \dots, c_r$ $(r \leqslant n)$. Since the system is hyperbolic, there exist n linearly independent left eigenvectors $l^{(i)}$ and n independent right eigenvectors $r^{(i)}$ satisfying

$$l^{(M)}(B-\lambda_M A)=0, \quad (B-\lambda_M A)r^{(M)}=0 \tag{2.34}$$

$$\text{no sum over } M, \ M=1, 2, \dots, n.$$

We note here that $l^{(M)}$ is a row vector and $r^{(M)}$ is a column vector and they correspond to the Mth characteristic velocity λ_M. Using the theory of linear algebra, we can show that it is possible to choose the left and right eigenvectors such that

$$l^{(i)}Ar^{(j)} \begin{cases} =0 \text{ for } i \neq j, \\ \neq 0 \text{ for } i = j, \end{cases} \tag{2.35}$$

at all points (x, t) of the domain D_1 (see problems 1, 2 and 3 Exercise 2.1).

Since the set of right eigenvectors is linearly independent, we change the dependent variables from the set $\{u_1, u_2, \dots, u_n\}$ to a new set $\{w_1, w_2, \dots, w_n\}$ by the transformation

$$\left.\begin{aligned} U &= \sum_{j=1}^{n} r^{(j)} w_j \\ &\text{or more explicitly} \\ u_i &= \sum_{j=1}^{n} r_i^{(j)} w_j \equiv r_{ij} w_j, \text{ say.} \end{aligned}\right\} \tag{2.36}$$

If R is the $n \times n$ matrix whose ith column is the ith right eigenvector $r^{(i)}$, i.e.

$$R=[r^{(1)}, r^{(2)}, \dots, r^{(n)}] \equiv [r_{ij} = r_i^{(j)}] \tag{2.37}$$

and W is the column vector whose components are $w_1, w_2, \cdots, w_n$, we can write (2.36) in the form

$$U = RW \tag{2.36'}$$

where R is nonsingular.

Substituting (2.36) in (2.1), premultiplying the resultant equation by $l^{(M)}$ and using (2.34) and (2.35) we get

$$l^{(M)}Ar^{(M)}\left[\frac{\partial w_M}{\partial t}+\lambda_M\frac{\partial w_M}{\partial x}\right]+l^{(M)}A\left\{\frac{\partial r^{(i)}}{\partial t}+\lambda_M\frac{\partial r^{(i)}}{\partial x}\right\}w_i+l^{(M)}C=0,$$

no sum over M, $M=1, 2, \cdots, n$.

Dividing by $l^{(M)}Ar^{(M)}$ we get

$$\frac{\partial w_M}{\partial t}+\lambda_M\frac{\partial w_M}{\partial x}+G_{Mi}w_i+\Gamma_M=0,\ M=1, 2, \cdots n \tag{2.38}$$

where

$$G_{Mi}=l^{(M)}A\left\{\frac{\partial r^{(i)}}{\partial t}+\lambda_M\frac{\partial r^{(i)}}{\partial x}\right\}\Big/(l^{(M)}Ar^{(M)}) \tag{2.39a}$$

and

$$\Gamma_M=l^{(M)}C/(l^{(M)}Ar^{(M)}). \tag{2.39b}$$

Let Λ be the diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_n$ and F be the column vector with elements $G_{Mi}w_i+\Gamma_M (M=1, 2, \cdots, n)$. The semilinear hyperbolic system (2.1) now reduces to the canonical form

$$\frac{\partial W}{\partial t}+\Lambda\frac{\partial W}{\partial x}+F=0. \tag{2.38'}$$

In the derivation of (2.38), we have also proved the following theorem:

Theorem 2.1 Any semilinear hyperbolic system in two independent variables is equivalent to a system $I\,\partial w/\partial t+\Lambda\partial w/\partial x+F=0$ in which the matrices I and Λ are diagonal and the matrix I is positive definite.

We note that the equation (2.38′) is a particular case of a symmetric hyperbolic system.

Definition: We call the variable w_i as the *characteristic variable* of the ith characteristic field.

The effectiveness of the canonical form (2.38) lies in the fact that the Mth equation gives the rate of change of the Mth characteristic variable along the Mth characteristic curve. Any solution of the semilinear hyperbolic system consists of n parts $w_1, w_2, \cdots, w_n$. The part w_M 'propagates' along the x-axis with the characteristic velocity λ_M. The mutual 'interaction' between these parts takes place only through the last term in (2.38), namely $G_{Mi}w_i+\Gamma_M$. Corresponding to a multiple characteristic velocity λ_q (which is repeated p_q times in $\lambda_1, \lambda_2, \cdots, \lambda_n$) the solution vector W has p_q components all of which move with the same velocity λ_q. Again, the mutual interaction between these p_q parts and interaction between all parts take place only through the third term in (2.38).

Numerical computation with the help of the canonical form

Let us briefly discuss here numerical solution of a Cauchy problem, a detailed formulation, existence, uniqueness and stability of which will be taken up only in §2.4. Here we briefly mention that the Cauchy problem for a system of first order equations consists of finding a solution $U(x, t)$ which takes up prescribed initial values $U_0(\eta)$ on a curve

$$\gamma : x = x_0(\eta),\ t = t_0(\eta).$$

We assume here that the curve γ is not a characteristic curve (see Fig. 2.1). On any one side of the curve γ, we draw a set of non-intersecting curves γ_1, $\gamma_2, \gamma_3, \ldots$ such that the normal distances between γ and γ_1, γ_1 and γ_2, γ_2 and γ_3 etc. are sufficiently small. Take a point $P(x, t)$ on the curve γ_1. From P draw characteristics corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct), to meet the curve γ at points $P_1, P_2, \ldots, P_{n-1}, P_n$ respectively. Let the coordinates of P_M be (x_M, t_M). Replacing the derivative in the characteristic direction on the left hand side of (2.38) by a first order forward difference scheme, we get

$$w_M(P) \simeq w_M(P_M) - F_M(P_M)\,(t - t_M),\ M = 1, 2, \ldots, n. \tag{2.40}$$

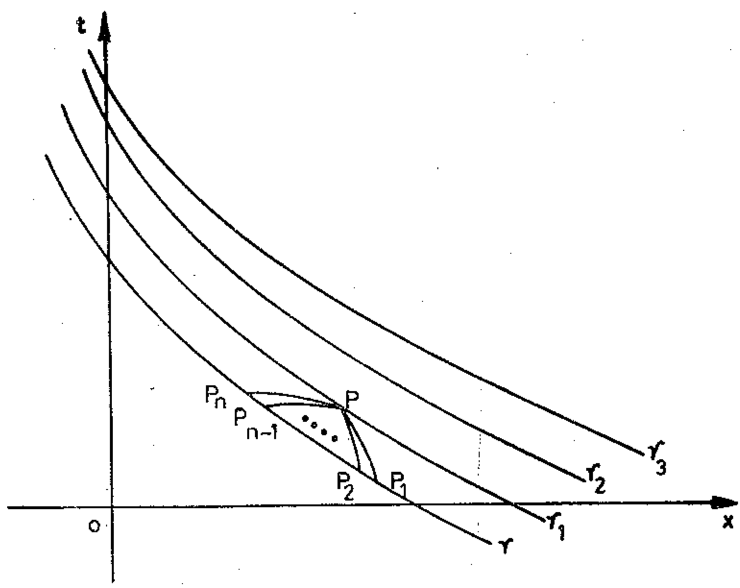

Fig. 2.1 Numerical solution of hyperbolic equations

Therefore, $w_M(M = 1, 2, \ldots, n)$ and hence the solution $U(x, t)$ is approximately determined at an arbitrary point P of the curve γ_1 from the knowledge of the Cauchy data on γ. Since the solution is now known on γ_1, we can adopt the same approximate scheme to calculate the solution on γ_2 and the procedure can be continued as long as the curves $\gamma, \gamma_1, \gamma_2, \ldots$ are nowhere parallel to the characteristic curves. Thus we have described a numerical scheme for the solution of a non-characteristic Cauchy problem. A scheme of this type has been extensively used for the solution of hyperbolic equations, especially quasilinear equations and has even been extended to hyperbolic equations in more than two independent variables (see Reddy, Tikekar and Prasad (1982) for further literature). This approximate method is also sufficient to conclude that a solution, of a noncharacteristic Cauchy problem, obtained in this way is stable for hyperbolic equations, i.e. a small change in the Cauchy data on γ leads only to small change in the solution at P on γ_1.

Domains of dependence, influence and determinacy

The proof of the existence, uniqueness and stability theorem of the §2.4 is very much similar to what we have described here. Let us consider now a particular Cauchy problem where the data has been prescribed on the line $t=0$. Through every point P in the upper half of the (x, t)-plane, there pass n characteristic curves (some of these curves may be multiple). For simplicity we assume that distinct characteristic velocities $c_1, c_2, \ldots, c_r$ satisfy the relation

$$c_1 < c_2 < \cdots < c_{r-1} < c_r$$

everywhere in D. Through the point $P(x, t)$ we draw the characteristics in t-decreasing direction till they meet the initial line $t=0$ at points P_r $(x=x_r)$, P_{r-1} $(x=x_{r-1})$, $\ldots$, P_1 $(x=x_1)$ such that

$$x_r < x_{r-1} < \cdots < x_1.$$

Then the outermost characteristics from P are C_r and C_1 as shown in the Fig. 2.2. From the above method of approximate solution or from the proof of the existence, uniqueness and stability theorem in §2.4, we conclude that the solution at P is determined only by the data in the closed interval (P_r, P_1) on the initial line $t=0$. The initial value can be arbitrarily changed outside the interval (P_r, P_1) without affecting the value of U at P.

Domain of dependence of a point P is the closed interval $\mathscr{I}_P$ of the initial curve between the points P_r and P_1. As remarked earlier, a change in the Cauchy data outside $\mathscr{I}_P$ does not affect the solution at P.

Domain of influence. Let J be a set of points on the initial curve $t=0$. Then the domain of influence of the set J consists of those points of the upper half plane whose domains of dependence contain points of J. If J is an interval between the points R and S $(R < S)$, then its domain of influence is bounded by the C_1 characteristic through the point R and C_r characteristic through the point S. These are the outermost characteristics starting from the points of J.

Domain of determinacy. The domain of determinacy of an *interval* J on the x-axis is the set of those points P with $t > 0$ whose domains of dependence $\mathscr{I}_P$ are entirely in J. This is bounded by the innermost characteristics pointing from the end points, i.e. the characteristic C_r from the left end R and the characteristic C_1 from the right end S.

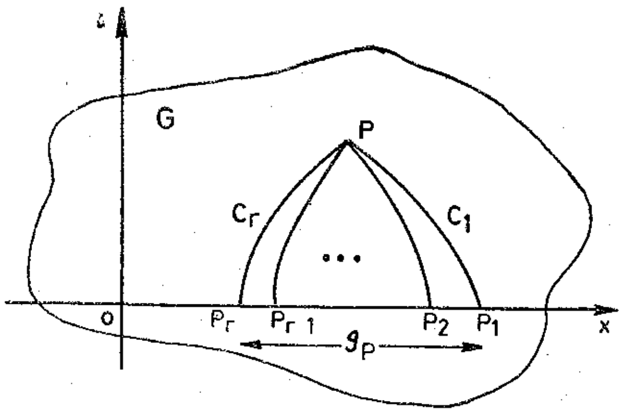

Fig. 2.2 Domain of dependence $\mathcal{J}_P$

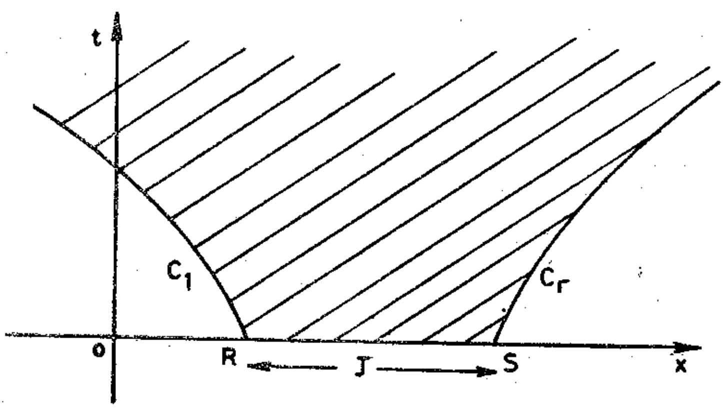

Fig. 2.3 Domain of influence when J is an interval

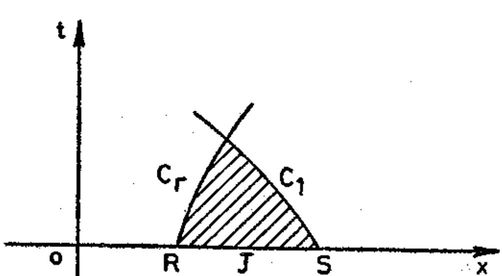

Fig. 2.4 Domain of determinacy of an interval J

EXERCISE 2.1

1. Let $R=[r^{(1)}, r^{(2)}, \ldots, r^{(n)}]$ be a matrix whose columns form a set of n linearly independent right eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. of a matrix T. Let $l^{(1)}, l^{(2)}, \ldots, l^{(n)}$ be the n rows of the matrix $L=R^{-1}$. Show that $l^{(1)}, l^{(2)}, \ldots, l^{(n)}$ are linearly independent left eigenvectors of T and

$$l^{(i)}r^{(j)}=\delta_{ij}.$$

2. Let A be the nonsingular matrix so that there exist two nonsingular matrices P and Q which reduce A to an identity matrix I i.e. $PAQ=I$. Let B be any other matrix, and $\bar{l}$ and $\bar{r}$ be the left and right eigenvectors respectively of the matrix PBQ in the usual sense, i.e. $\bar{l}\,(PBQ-\lambda I)=0$ and $(PBQ-\lambda I)\bar{r}=0$. Show that the corresponding generalised eigen vectors l and r of the matrix B relative to A (satisfying $l(B-\lambda A)=0$ and $(B-\lambda A)r=0$) are given by

$$l=\bar{l}P \text{ and } r=Q\bar{r}.$$

3. When the matrices A and B are such that there exist a set of n linearly independent left (and right) eigenvectors satisfying (2.34), prove the result (2.35). (Hint: Use the results of problems 1 and 2).
4. Find the region of the (x, t)-plane where each of the following system of equations is hyperbolic. Find the equations of the characteristic curves and obtain the compatibility conditions

(i) $u_x-xv_t=0,\ v_x-u_t=0$

(ii) $(x^2-1)u_x+xt(u_t+v_x)+(t^2-1)v_t+4xu+4tv=0,\ v_x-u_t=0$

(iii) $u_x+xtv_t=0,\ v_x-u_t=0.$

Show that the characteristic curves of (ii) are straight lines all tangent to the unit circle, which is their envelope.

5. Let u and v satisfy a system of quasilinear equations

$$u_t+uu_x+2vv_x=0$$
$$2v_t+2uv_x+vu_x=0.$$

If u and v are prescribed on a curve $C: x=x_0(t)$ in the (x, t)-plane, show that the partial derivatives of u and v at the points of C can be found uniquely if $x_0'(t)\neq u\pm v$.

6. The one-dimensional equations of nonlinear dynamic elasticity, when the medium is assumed to be isotropic, are

$$\frac{\partial^2 u}{\partial t^2}-v_d^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left\{v_1^2\left(\frac{\partial u}{\partial x}\right)^2+v_2^2\left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$\frac{\partial^2 v}{\partial t^2}-v_s^2\frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left(2v_2^2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right)$$

where v_d and v_s are the linear longitudinal and shear bulk wave velocities, v_1^2 and v_2^2 are constants and u and v are longitudinal and shear displacement components. Reduce the two equations to a system of first order equations. Show that the system is hyperbolic and find the characteristic equations and the compatibility conditions.

*§2.3 Propagation of Discontinuities Along a Characteristic Curve

The hyperbolic character of a system of first order equations exhibits itself in the fact that it is possible to have solutions whose derivatives are discontinuous and these discontinuities *propagate* along the characteristic curves. Let D be a domain in the (x, t)-plane and let D_1 and D_2 be the two portions of D separated by a curve C (see Fig. 2.5), D_1 being on the left* and D_2 on the right* of C. Let U_1 defined in the domain D_1 and U_2 in the domain D_2 be genuine solutions of (2.1). We assume that the limiting value U_l of U_1 as we approach a point P on C from the domain D_1 and the limiting value U_r as we approach P from the domain D_2 exist and are such that $U_l = U_r$ at every point of the curve C. Now, consider a function U defined in the domain D by

$$U = \begin{cases} U_1 \text{ in } D_1 \\ U_2 \text{ in } D_2. \end{cases} \tag{2.41}$$

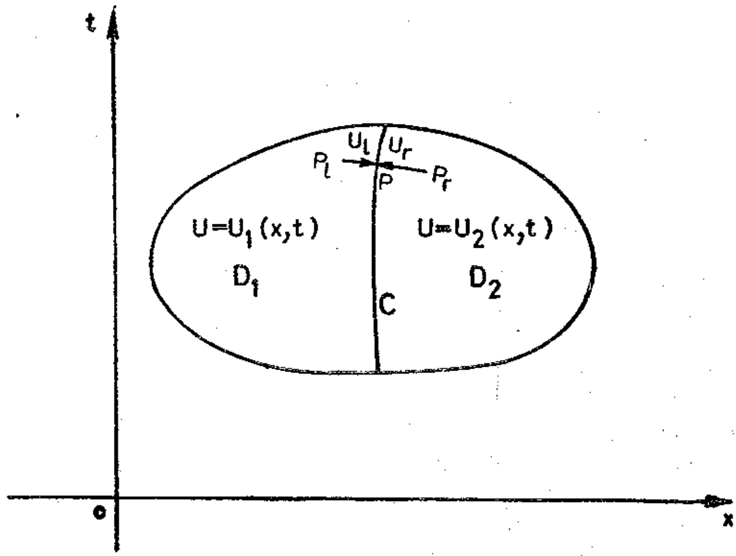

Fig. 2.5 $D = D_1 \cup D_2$

The function U is a genuine solution of (2.1) in D_1 and D_2 separately and is continuous in D but its derivatives may be discontinuous across C. We assume that the limiting values of the derivatives of U as we approach P from two domains D_1 and D_2 exist. Then these derivatives, if discontinuous across C, have only a finite jump across C.

*With respect to the reader (see footnote on page 115).

Let the equation of the curve C be $\varphi(x, t)=0$ and let $\eta(x, t)$ be any other function independent of φ such that φ and η are sufficiently smooth and the Jacobian $\partial(\varphi, \eta)/\partial(x, t) \neq 0$ in the domain D. If we can introduce a new set of independent variables (φ, η) instead of (x, t), U_φ respresents an exterior derivative and U_η is a tangential derivative along the curve C.

Let us now assume that the first order derivatives U_x, U_t are discontinuous across C. Since the function U itself is continuous across C, its tangential derivative U_η must be continuous across C. Therefore, from $U_x=U_\varphi\varphi_x+U_\eta\eta_x$ and $U_t=U_\varphi\varphi_t+U_\eta\eta_t$, it follows that across the curve C, the exterior derivative U_φ must be discontinuous*. In terms of the limiting values $(U_\varphi)_r$ and $(U_\varphi)_l$ of the exerior derivative at the point P on C, the jump $[U_\varphi]$ in U_φ across C is given by

$$[U_\varphi]=(U_\varphi)_r-(U_\varphi)_l. \tag{2.42}$$

The jumps in the first order derivatives U_x and U_t are related to $[U_\varphi]$ by the relation

$$[U_x]=[U_\varphi]\varphi_x \quad \text{and} \quad [U_t]=[U_\varphi]\varphi_t. \tag{2.43}$$

The quasilinear equation (2.1) is valid everywhere in D except at the points of C and all quantities appearing in it other than the first order derivatives are continuous across C. Taking the limit of (2.1) as we move from the region D_1 to P and again as we move from D_2 to P and subtracting the result we get

$$(A\varphi_t+B\varphi_x)_{\text{at P}}\,[U_\varphi]=0. \tag{2.44}$$

Since $[U_\varphi]$ is not a zero vector, the matrix $(A\varphi_t+B\varphi_x)$ is singular on C, i.e.

$$\left.\begin{aligned} &\det(A\varphi_t+B\varphi_x)=0 \\ \text{or} \quad &\det(-\lambda A+B)=0, \text{ where } \lambda=-\varphi_t/\varphi_x \end{aligned}\right\} \tag{2.45}$$

at every point of the curve C. This implies that C is a characteristic curve. Thus we have proved the following important result.

Theorem 2.1 If the first order partial derivatives of a continuous function $U(x, t)$, satisfying the quasilinear system (2.1) on both sides of a curve C in the (x, t)-plane, are discontinuous across C, the curve C must be a characteristic curve of the system of equations.

Starting with a system obtained by differentiating (2.1), we can show exactly in the same manner that if the derivatives of a solution U up to order $r (r \geqslant 1)$ are continuous across C and $(r+1)$th derivatives are discontinuous across C, then C is necessarily a characteristic curve.

Generalised or weak solution

It is now natural to ask what happens in the situation when we have a solution U which satisfies the differential equation in D_1 and D_2 separately

*When at least one component of the vector is U_φ is discontinuous we say that U_φ is discontinuous.

but U itself is discontinuous across C. For a general quasilinear system (2.1) we shall show in § 5.3 of this chapter that the curve of discontinuity is, in general, not a characteristic. However, for a linear hyperbolic system

$$A(x, t)U_t + B(x, t)U_x + \bar{H}(x, t)U + \check{J}(x, t) = 0 \tag{2.46}$$

where the elements of the matrices A, B and $\bar{H}$, and the vector $\check{J}$ are functions of x and t only, we shall show that C is a characteristic curve. Using the canonical form of the last section we shall further derive a system of linear homogeneous ordinary differential equations, called transport equations, governing the growth and decay of the discontinuity $[U]$. Discontinuities in the solution cannot be studied for every function U satisfying (2.46) in D_1 and D_2. It can be studied only for a "generalised" or "weak" solution. Here we shall define a weak solution in a manner similar to that in §4.1 of Chapt. 2. A more elegant definition of the weak solution will be given in §5 of this chapter. We first reduce the system (2.46) to the characteristic canonical form (2.38) and note that $\partial/\partial t + \lambda_M(x, t)\partial/\partial x$ represents the directional derivative along the characteristics of the Mth field. Integrating it from a point P_M to $P(\xi, \tau)$, both lying on a characteristic of the Mth field, we get (for details, see the derivation of (2.65) in §2.4)

$$\begin{aligned} w_M(\xi, \tau) = &-\int_{P_M}^{P} H_{Mi}(x_M(t; \xi, \tau)\, t)\, w_i(x_M(t; \xi, \tau), t)\, dt + (w_M)_{\text{at } P_M} \\ &- \int_{P_M}^{P} J_M(x_M(t; \xi, \tau), t)\, dt, \quad M = 1, 2, \dots, n \end{aligned} \tag{2.47}$$

where

$$H_{Mi} = \left[(l^{(M)}A)\left\{ \frac{\partial r^{(i)}}{\partial t} + \lambda_M \frac{\partial r^{(i)}}{\partial x} \right\} + (l^{(M)}\bar{H}r^{(i)}) \right] \Big/ (l^{(M)}Ar^{(M)}) \tag{2.48}$$

$$J_M = (l^{(M)}\check{J})/(l^{(M)}Ar^{(M)}) \tag{2.49}$$

with no sum over M in the expressions (2.47)-(2.49), and $x = x_M(t; \xi, \tau)$ is the characteristic of Mth field through the point P. We can give a very compact form to the expression for H_{Mi} in terms of the operator

$$\mathcal{L} \equiv A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + \bar{H}$$

appearing on the right hand side of (2.46). Using (2.34) we can write (2.48) as

$$H_{Mi} = (l^{(M)}\mathcal{L}r^{(i)})/(l^{(M)}Ar^{(M)}). \tag{2.48'}$$

We now define a *generalised or weak* solution of (2.46) to be a function $U(x, t)$ obtained from (2.36) wherein $w_1, w_2, \dots, w_n$ satisfy the system of *integral equations* (2.47).

Transport equation for a linear hyperbolic system

Now we consider a weak solution $U(x, t)$ of (2.46) which is continuous in a domain D except on the curve C and which is a genuine solution of (2.46) in domains D_1 and D_2. We further assume that the function U has a jump discontinuity across C. In this case, the integrands on the right hand side of (2.47) are continuous functions of t except for a finite jump across C. Therefore, on performing the integration in (2.47) along a characteristic curve of the Mth family, we find that the characteristic variable w_M is given by a continuous function on this curve. If a curve C is not tangential to a characteristic of the Mth family (see Fig. 2.6), w_M must be continuous across C. However, according to our assumption, U is discontinuous across C and hence at least one of $w_1, w_2, \cdots, w_n$ must be discontinuous across C. Therefore it follows that C, *the curve of discontinuity, must be a characteristic curve of Jth family, say, and the jump in all the characteristic variables* w_i, $i \neq J$ *must be zero across* C.

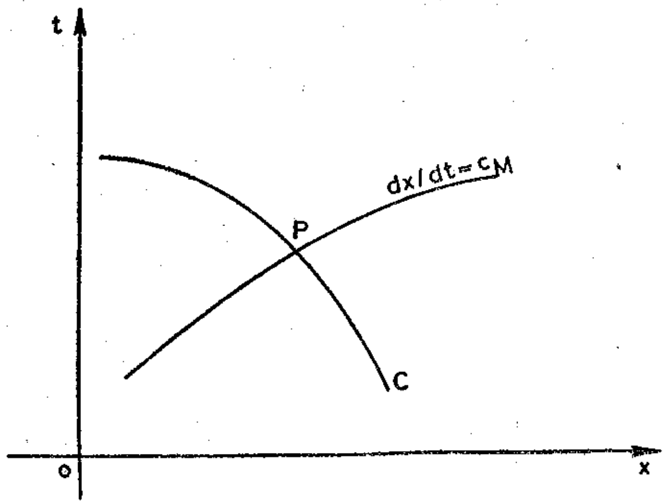

Fig. 2.6 w_M cannot be discontinuous across *a* curve C which is not a member of the characteristics of the Mth family

Let us assume now that the curve of discontinuity C is a characteristic curve of the Jth family. Then the jump $[w_i]$ in w_i satisfies

$$[w_i]=0, \qquad i\neq J \tag{2.50}$$

and

$$[w_J]\neq 0. \tag{2.51}$$

Therefore, from (2.36) we have

$$[U]=r^{(J)}[w_J], \text{ on sum over } J. \tag{2.52}$$

Writing the equation (2.38) for the system (2.46) at two points P_l and P_r on the two sides of C (see Fig. 2.5), taking the limit as both these points tend to a point P on C and subtracting, we get

$$\frac{d}{dt}[w_J]=-H_{JJ}[w_J], \text{ no sum over } J \tag{2.53}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\lambda_J\frac{\partial}{\partial x}.$$

This equation is called the *transport equation.* Along a given characteristic curve $x=x_J(t)$ of the Jth family, the function $H_{JJ}(x, t) \equiv H_{JJ}(x_J(t), t)$ is a function of t only. Therefore, the transport equation (2.53) is a linear homogeneous ordinary differential equation of first order and determines the variation of $[w_J]$ along a characteristic curve of the Jth family. From the properties of solutions of linear homogeneous ordinary differential equations, it follows that *if there is a discontinuity in U at some point of a characteristic curve C, the discontinuity in U remains nonzero at every point on the curve.* It also follows that for linear equations in two independent variables only discontinuous Cauchy data can lead to discontinuous solutions.

It is a simple matter to derive the transport equation for the discontinuities in the first (or higher order) derivatives of U. We just have to differentiate (2.38) once (or n times, $n > 1$) and proceed in exactly the same manner as we have done above for the jump in U (see also Sections 7.7 and 8.4).

EXERCISE 2.2

1. Consider the hyperbolic system

$$u_t+(x+t)v_x=0,\ (x+t)v_t+u_x=0.$$

Show that the variation in jump $[v]$ along the characteristic curve $x-t=c$ ($c=$constant) is given by

$$[v]=\frac{A}{(2t+c)^{1/2}},\ A=\text{constant}.$$

Derive also the transport equation for the propagation of discontinuities in the first order partial derivatives of u and v.

9. Consider a continuous solution of the equations

$$\rho_t+q\rho_x+\rho q_x+\frac{(\nu-1)\rho q}{x}=0$$

$$q_t+qq_x+\frac{a^2}{\rho}\rho_x=0$$

where $a^2=k^2\rho^{\gamma-1}$ and k, γ, ν are constants. The initial condition is prescribed for all values of x with the restriction that

$$\rho(x, 0)=\text{constant}=\rho_0,\ q(x, 0)=0 \text{ for } x \geqslant x_c=\text{constant}.$$

Derive the equation governing the propagation of the discontinuity $[q_t]$ in q_t starting from $x=x_c$ and separating the region of the constant solution from that of nonconstant solution in the form

$$\frac{d[q_t]}{dx}=-\frac{(\nu-1)}{2x}[q_t]+\frac{\gamma+1}{2a_0^2}[q_t]^2.$$

Find the time T when the solution ceases to be continuous in each of the cases $\nu=1$, $\nu=2$, $\nu=3$ and $\nu=4$.

3. When all the characteristic velocities λ_i are different from zero, show that the first order quasilinear hyperbolic system

$$A(x, t, U)\frac{\partial U}{\partial t}+B(x, t, U)\frac{\partial U}{\partial x}+C(x, t, U)=0$$

can be reduced to a diagonal canonical system of $2n$ equations

$$\frac{\partial U}{\partial t}-RW=0 \text{ and } \frac{\partial U}{\partial t}+\Lambda\frac{\partial W}{\partial x}+F=0$$

where the coefficients A, Λ and F are functions of x, t, U and W (Courant and Lax (1949)).

*§2.4 Existence, Uniqueness and Stability of the Cauchy Problem for a Linear Hyperbolic System

In the previous chapter we have shown that for the wave equation the appropriate problem, i.e. the problem which is *well posed* in the sense of Hadamard is the Cauchy problem. We shall prove here the same result for a hyperbolic system of first order equations, i.e. we shall prove that a solution of a Cauchy problem for such a system exists, is unique and depends continuously on the Cauchy data.

Consider a pair of scalar functions $x_0(\eta)$, $t_0(\eta)$ and a vector function $U_0(\eta)$ defined on an interval $\mathcal{J}$ of the real η-axis such that the functions $x_0(\eta)$, $t_0(\eta)$ are sufficiently smooth and $(x_0')^2+(t_0')^2\neq 0$ for $\eta\in\mathcal{J}$. We assume that when $\eta\in\mathcal{J}$, the point $(x_0(\eta), t_0(n), U_0(\eta))\in D_2$, where D_2 is a domain in the $n+2$ dimensional $(x, t, u_1, u_2, \cdots, u_n)$-space on which the coefficients A, B and C in the equation (2.1) are defined.

Cauchy problem for the first order system (2.1) is to find a domain D in the (x, t)-plane and a solution $U(x, t)$ of the system in the domain D such that

$$\begin{aligned}&(x_0(\eta), t_0(\eta))\in D \text{ when } \eta\in\mathcal{J}\\ &(x, t, U(x, t))\in D_2 \text{ when } (x, t)\in D\end{aligned} \qquad (2.54)$$

and

$$U(x_0(\eta), t_0(\eta))=U_0(\eta) \text{ for } \eta\in\mathcal{J}.$$

Therefore, the Cauchy problem for a system of first order equations consists of finding a solution which takes up prescribed values $U_0(\eta)$ on the curve

$$\gamma : x=x_0(\eta),\ t=t_0(\eta),\ \eta\in\mathcal{J} \qquad (2.55)$$

One of the oldest proofs for the existence and uniqueness of the solution of the Cauchy problem is due to Kowaleswskii (1875). Her proof assumes that the coefficients $A(x, t, U)$, $B(x, t, U)$, and $C(x, t, U)$ as well as the Cauchy data $U_0(x)$ are analytic functions of their arguments and she proves that in the class of analytic functions, a Cauchy problem has a unique solution provided the curve γ is not a characteristic curve. However, Kowaleswskii's theorem gives only an analytic solution for which the concept of the domains of dependence and influence, so important for hyperbolic equations, is meaningless. This is because the knowledge of an analytic function in any sub-domain determines it at all points in the (x, t)-plane where it can be analytically continued.

A very general proof of well posedness of the Cauchy problem, which brings out distinctly all important properties of the solutions of hyperbolic equations, is due to Courant and Lax (1949). Their proof uses the method of successive approximations after converting the problem to an equivalent system of "integral equations" and assumes only that the coefficients A, B, C, $\text{grad}_U C$; and the Cauchy data $U_0(\eta)$ are C^1 functions of their arguments. In order to simplify the proof we shall present it only for a linear hyperbolic system with more restrictive conditions on the coefficients, i.e. for

$$A(x, t)\frac{\partial U}{\partial t}+B(x, t)\frac{\partial U}{\partial x}+\bar{H}(x, t)U+\bar{J}(x, t)=0 \tag{2.56}$$

where

$$\left.\begin{aligned} &\det A(x, t)\neq 0,\\ &A,\ B\in C^2\\ &\bar{H},\ \bar{J}\in C^1\\ &U_0\in C^1.\end{aligned}\right\} \tag{2.57}$$

and

As in §2.2, we reduce the system (2.56) to its canonical form (2.38):

$$\frac{\partial w_M}{\partial t}+\lambda_M\frac{\partial w_M}{\partial x}+H_{Mi}w_i+J_M=0,\ M=1, 2, \ldots, n \tag{2.58}$$

where

$$H_{Mi}=(l^{(M)}\mathcal{L}r^{(i)})/(l^{(M)}Ar^{(M)}),\ \mathcal{L}\equiv A\frac{\partial}{\partial t}+B\frac{\partial}{\partial x}+\bar{H} \tag{2.59}$$

and

$$J_M=l^{(M)}\bar{J}/(l^{(M)}Ar^{(M)}), \tag{2.60}$$

with no summation when the repeated index is M.

The Cauchy data is prescribed on a non-characteristic curve $\gamma:\varphi(x, t)=0$. We can choose a new coordinate system (x', t') where $t'=\varphi(x, t)$ so that γ coincides with $t'=0$ (i.e. the x'-axis). Under such a transformation, the equation (2.56) transforms to an equation of the same form wherein the new coefficients also satisfy the conditions (2.57) provided φ has the required smoothness properties. Once this has been done, we can drop the dashes from x' and t'. Therefore, the Cauchy problem for (2.58) is now an initial value problem in which the data is prescribed on the line $t=0$ as

$$W(x, 0)=(R^{-1}U)_{t=0}=(R^{-1})_{t=0}U_0(x)\equiv W_0(x),\ x\in\mathcal{J} \tag{2.61}$$

where the components of the column vector W are $w_1, w_2, \ldots, w_n$ and $\mathcal{J}$ is an interval on the x-axis. The coefficients in the equations (2.58) and the initial data W_0 satisfy

$$\lambda_M(x, t)\in C^2(D_1), H_{Mi}(x, t)\in C^1(D_1),\ J_M(x, t)\in C^1(D_1),\ W_0(x)\in C^1(\mathcal{J}) \tag{2.62}$$

where D_1 is a domain in the (x, t)-plane.

As shown in the Fig. 2.7, let us consider a closed domain $D \in D_1$ in the (x, t)-plane such that all the n characteristics passing through an arbitrary point in D, when followed in decreasing t-direction, remain in D and meet the initial line $t=0$ at points in $\mathscr{I}$. The strip $0 \leqslant t \leqslant h$, where h is a suitably chosen positive number, within the closed domain D is denoted by D_h as shown in the Fig. 2.7. The characteristic C_k of the kth family through a point P with coordinates (ξ, τ) is obtained by solving the ordinary differential equation

$$\frac{dx}{dt} = \lambda_k(x, t),\ k=1, 2, \dots, n \tag{2.63}$$

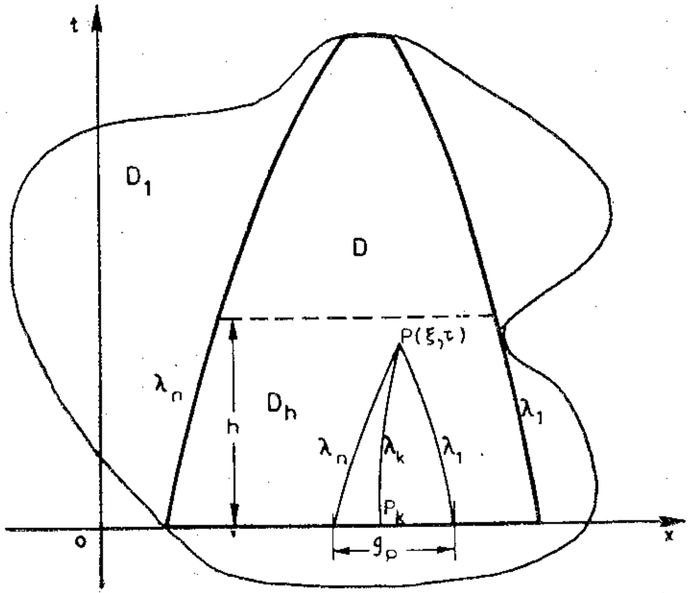

Fig. 2.7 The boundary of the domain D, wherein the solution is determined, has been shown by a thick line

with the condition $x=\xi$ when $t=\tau$, and is represented by

$$x = x_k(t;\ \xi, \tau). \tag{2.64}$$

Then $x_k(\tau;\ \xi, \tau)=\xi$. The point P_k where it intersects the initial line $t=0$ is given by $x=x_k(0;\ \xi, \tau)$. Let $\mathscr{I}_P$ be the closed interval on $t=0$ between the two extreme points amongst the points $P_1, P_2, \dots, P_n$. Now we prove.

Theorem 2.2 Under the conditions (2.62), the Cauchy problem (2.58) and (2.61) possesses a unique solution in D. Further, the solution depends continuously on the Cauchy data $W_0(x)$.

Proof We prove this theorem in five steps:

Step 1 Construction of an iterative scheme for the solution.

Assuming $W(x, t)$ to be any solution of the system of equations (2.58), integrating the Mth equation along the characteristic C_M from P_M to P we get

$$w_M(\xi, \tau) = T_M W + \chi_M,\ M = 1, 2, \ldots, n \tag{2.65}$$

where T_M is a linear operator defined by

$$T_M W = -\int_0^\tau H_{Mi}(x_M(t; \xi, \tau), t) w_i(x_M(t; \xi, \tau), t)\, dt \tag{2.66}$$

and χ_M is a known function given by

$$\chi_M(\xi, \tau) = w_{0M}(x_M(0; \xi, \tau)) - \int_0^\tau J_M(x_M(t; \xi, \tau), t)\, dt. \tag{2.67}$$

The n equations (2.65) constitute a system of n "integral equations" for the n components of the vector function W; however, it is important to note that in general, curve of integration is different for different components of W.

We write (2.65) in the form

$$W = TW + \chi \tag{2.68}$$

T is a mapping of the vector function W into a vector function whose components are $T_1W, T_2W, \ldots, T_nW$. Since $T_1, T_2, \ldots, T_n$ are linear integral operators, T is a linear operator. Though T is defined for more general functions, we consider it to act on the space S of functions V which are $C^1(D)$ and have common initial values $W_0(x)$. Under the condition (2.62), from the theorem concerning the existence of the derivatives of the solution of an initial value problem of an ordinary differential equation with respect to parameters (Coddington and Levinson, 1955, Chapt. 1 Section 7) we can show that the derivatives

$$\frac{\partial x_M(t; \xi, \tau)}{\partial \xi} \text{ and } \frac{\partial x_M(t; \xi, \tau)}{\partial \tau}$$

are continuous functions. Therefore, χ is $C^1(D)$ and hence if $V \in S$, then $\bar{V}$ defined by

$$\bar{V} = TV + \chi \equiv LV, \text{ say} \tag{2.69}$$

also $\in S$. The solution of the "integral equations" (2.65) is that element of S which is a fixed point of the transformation L taking V to $TV + \chi$.

We note that $W_0(x)$ belongs to S. Now we set up an iterative scheme for constructing a sequence of functions $W^{(r)}$ by

$$\left.\begin{aligned} W^{(0)} &= W_0(x) \\ W^{(r+1)} &= TW^{(r)} + \chi,\ r = 0, 1, 2, \ldots \end{aligned}\right\} \tag{2.70}$$

Step 2 Proof of the convergence of the sequence.

In order to prove the convergence of the sequence $\{W^{(r)}\}$, it is sufficient to work in a less restricted space viz. the space $C(D_h)$ of continuous and bounded vector functions on the closed bounded domain D_h. Due to the assumption (2.62) on H_{Mi}, it follows that there exists an upper bound μ such that in the closed domain D, and all the more in D_h

$$| H_{Mi} | \leqslant \mu. \tag{2.71}$$

Then for any two elements $V^{(1)}$ and $V^{(2)}$ of $C(D_h)$ and for $(\xi, \tau) \in D_h$, we get Mth component of (2.69) as

$$\bar{V}_M^{(1)} - \bar{V}_M^{(2)} = -\int_0^\tau H_{Mi}(x_M(t; \xi, \tau), t)\{v_i^{(1)}(x_M(t; \xi, \tau))$$
$$- v_i^{(2)}(x_M(t; \xi, \tau), t)\}\, dt.$$

Therefore

$$\| \bar{V}^{(1)} - \bar{V}^{(2)} \| \leqslant nh\mu \| V^{(1)} - V^{(2)} \| \tag{2.72}$$

where n is the number of components H_{Mi} of the vector H_M and we choose for the norm $\| \cdot \|$, the maximum norm, i.e.

$$\| V \| = \max_{i=1,2,\dots,n} \left\{ \max_{D_h} | v_i | \right\}. \tag{2.73}$$

Choosing h sufficiently small, we can make $nh\mu < 1$ which shows that the transformation (2.69) represents a contraction mapping. Since the norm used here is the maximum norm, the space $C(D_h)$ of continuous vector functions on the closed set D_h is complete. The sequence $\{W^{(r)}\}$ of functions defined by (2.70) converges uniformly to a continuous function $W(x, t)$ which satisfies the "integral equations" (2.65).

Step 3 Proof for the existence of the partial derivatives of W.

The Mth component $w_M(\xi, \tau)$ of the limit function W obtained in step 2 satisfies the "integral equations" (2.65). Differentiating the Mth integral equation in the direction of the Mth characteristic through P, we find that the derivative, $(d/d\tau)_M$, of w_M in the Mth characteristic direction exists and is continuous. But we note that when we move in the Mth characteristic direction through the point (ξ, τ)

$$(dx_M/d\tau)_M \equiv \frac{\partial x_M}{\partial \tau} + \lambda_M(\xi, \tau)\frac{\partial x_M}{\partial \xi} = 0$$

showing that along this characteristic of the Mth field, $x_M(t; \xi, \tau)$ is a function of t alone. Hence to get the derivative of W_M in the characteristic direction, i.e. to get $\left(\frac{dw_M(\xi, \tau)}{d\tau}\right)_M$, we differentiate (2.65) considering $x_M(t; \xi, \tau)$ to be a function of t alone. Using $x_M(\tau; \xi, \tau) = \xi$, we get (2.58) (with x and t replaced by ξ and τ), showing that the solution W of the "integral equations" is also a solution of the differential equations (2.58) in the characteristic canonical form.

However, the existence of the directional derivative of $w_M(x, t)$ in one direction (which is Mth characteristic direction) does not imply the existence of the partial derivatives, with respect to x and t. Therefore, we shall now prove the existence of the x-derivative $\partial w_M/\partial x$. This will imply the existence of the derivative $\partial w_M/\partial t$ also, since we have the relation

$$\frac{\partial w_M}{\partial t} = \left(\frac{dw_M}{dt}\right)_M - \lambda_M \frac{\partial w_M}{\partial x}. \tag{2.74}$$

Differentiating (2.69) with respect to ξ, we get

$$\frac{\partial \bar{v}_M}{\partial \xi} = -\int_0^\tau \left\{\frac{\partial H_{Mi}}{\partial x} v_i + H_{Mi}\frac{\partial v_i}{\partial x}\right\}\frac{\partial x_M(t;\xi,\tau)}{\partial \xi}dt + \frac{\partial \chi_M}{\partial \xi}. \tag{2.75}$$

From the assumptions (2.62), it follows that the derivatives $\frac{\partial H_{Mi}}{\partial x}$, $\frac{\partial x_M(t;\xi,\tau)}{\partial \xi}$ and $\frac{\partial \chi_M}{\partial \xi}$ are continuous functions so that the integrals on the right hand side of (2.75) exist if all $\partial v_i/\partial \xi$ are continuous on D_h and give $\partial \bar{v}_M/\partial \xi$ as continuous function of ξ and τ.

Since the characteristic derivative $(dw_M/dt)_M$ is continuous in D_h and $\partial w_M/\partial t$ is given by (2.74), we need not work in the space S, but in the complete space B of vectors $V(x, t)$ for which V and V_x are continuous and bounded in a closed domain D_h and in which we choose

$$|||V||| = \max(\|V\|, \|V_x\|). \tag{2.76}$$

as its norm from the above it follows that the transformation (2.69) maps B into B. Further for any two vectors $V^{(1)}$ and $V^{(2)}$, we get from (2.75)

$$\left\|\frac{\partial \bar{v}_M^{(1)}}{\partial \xi} - \frac{\partial \bar{v}_M^{(2)}}{\partial \xi}\right\| \leqslant nh[\tfrac{1}{2}\mu_1 + \tfrac{1}{2}\mu_2](|||V^{(1)} - V^{(2)})|||) \tag{2.77}$$

where $\frac{1}{2}\mu_1$ and $\frac{1}{2}\mu_2$ are $\sup_{i,M,D}\left|\frac{\partial H_{Mi}}{\partial x}\frac{\partial x_M(t;\xi,\tau)}{\partial \xi}\right|$ and $\sup_{i,M,D}\left|H_{Mi}\frac{\partial x_M(t;\xi,\tau)}{\partial \xi}\right|$ respectively. Denoting the largest of μ in (2.72) and μ_1 and μ_2 in (2.77) by $\bar{\mu}$, we get from (2.72), (2.76) and (2.77)

$$|||\bar{V}^{(1)} - \bar{V}^{(2)}||| \leqslant nh\bar{\mu}\ |||V^{(1)} - V^{(2)})|||. \tag{2.78}$$

This equation shows that for sufficiently small h, the transformation (2.69) represents a contraction mapping also on the complete metric space B and hence the limit function W obtained from the sequence (2.70) belongs to B, i.e. $W_x(x, t)$ exists on D_h (we note that convergence in B is uniform convergence).

Thus we have proved that the solution of the "integral equationss" (2.65) or equivalently of the initial value problem for the equations (2.58) is $C^1(D_h)$ for sufficiently small h.

Since the matrix R in (2.36) is nonsingular, the initial value problem for the equation (2.56) has a unique solution in D_h.

Step 4 Solution extended onto the whole domain D.

We note that the value of the solution $U(x, t)$ at $t = h$ has the same properties as the function $U_0(x)$ in (2.57), i.e. $U(x, h) \in C^1$. Therefore we can use the line $t = h$ as the new initial line and the value of the solution $U(x, h)$

as the new Cauchy data and solve the new Cauchy problem for the eq. (2.56) in the strip $h \leqslant t \leqslant 2h$. Since h is determined from the condition $nh\bar{\mu} < 1$ and $\bar{\mu}$ depends on the whole domain D, it follows that the width of the strip parallel to the x-axis, wherein the solution is determined, is at least h in the new problem also. In this way we can continue the solution step by step starting from the initial data on $t=2h, 3h, \ldots$. We finally get the solution of the original Cauchy problem in the whole domain D.

Step 5 The solution depends continuously on the Cauchy data.

Let $W^{(1)}(x, t)$ and $W^{(2)}(x, t)$ be two solutions corresponding to the initial data $W_0^{(1)}(x)$ and $W_0^{(2)}(x)$. Then the procedure of step 2 shows that for $(x, t) \in D$

$$\|W^{(1)} - W^{(2)}\| \leqslant \|W_0^{(1)} - W_0^{(2)}\| + nh\mu\|W^{(1)} - W^{(2)}\|$$

or

$$\|W^{(1)} - W^{(2)}\| \leqslant \frac{1}{1-nh\mu}\| W_0^{(1)} - W_0^{(2)}\|. \tag{2.79}$$

Putting $t=h$ in the above result, we get

$$\|W^{(1)}(x, h) - W^{(2)}(x, h)\| \leqslant \frac{1}{1-nh\mu}\| W^{(1)}(x, 0) - W^{(2)}(x, 0)\|.$$

Proceeding in the same way, we find that in the time interval $(r-1)h \leqslant t \leqslant rh$ we have the following inequality

$$\|W^{(1)}(x, t) - W^{(2)}(x, t)\| \leqslant \frac{1}{(1-nh\mu)^r}\|W_0^{(1)} - W_0^{(2)}\|. \tag{2.80}$$

Since the whole of D is covered by the union of a finite number of strips of the type $(r-1)h \leqslant t \leqslant rh$, it follows from (2.80) that

$$\text{as } \|W_0^{(1)} - W_0^{(2)}\| \to 0, \; \|W^{(1)}(x, t) - W^{(2)}(x, t)\| \to 0.$$

This proves that in the domain D, the solution of the Cauchy problem not only exists and is unique but also depends continuously on the initial data. The Cauchy problem is therefore well posed for hyperbolic equations.

*§2.5 Comments on Mixed Initial and Boundary Value Problems

Most physical phenomena governed by partial differential equations take place in a portion of the space bounded by stationary or moving boundaries. A simple example of this problem is the transverse vibration of a taut flexible string between two points, say $x=0$ and $x=a$. In this case one or both ends of the strings may be fixed or forced to slide in the transverse direction. The transverse displacement of the string satisfies one dimensional wave equation in the strip $0 < x < a$, $t > 0$ of the (x, t)-plane. To determine its motion we need to know not only the initial state of the string between $0 \leqslant x \leqslant a$ but also the boundary conditions at the two ends of the string. Thus we have a physical problem which is not a Cauchy problem for a hyperbolic equation. Such problems are called initial and boundary value problems. In this section we shall discuss briefly the nature of initial and boundary conditions which together with hyperbolic partial differential equations, form a well posed problem. Its detailed discussion is given by Courant and Hilbert (1962), pp. 471-475.

Consider a first order hyperbolic system of n equations with the n characteristic velocities λ_i arranged in the form

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{n-1} \leqslant \lambda_n. \tag{2.81}$$

Note that equality has been provided here in order to take into account the multiplicity of the characteristics. For simplicity we take the case when we need to solve the equations in the first quadrant of the (x, t)-plane, i.e. for $x > 0$, $t > 0$. Our aim here is to discuss the nature of the initial conditions imposed on the x-axis and the boundary conditions prescribed on the t-axis so that we get a well posed problem for determining the solution adjacent to the positive x-axis and the positive t-axis in the first quadrant. Let us assume that the conditions

$$0 < \lambda_{p+1} \leqslant \lambda_{p+2} \leqslant \ldots \leqslant \lambda_n \tag{2.82}$$

and

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_p \leqslant 0 \tag{2.83}$$

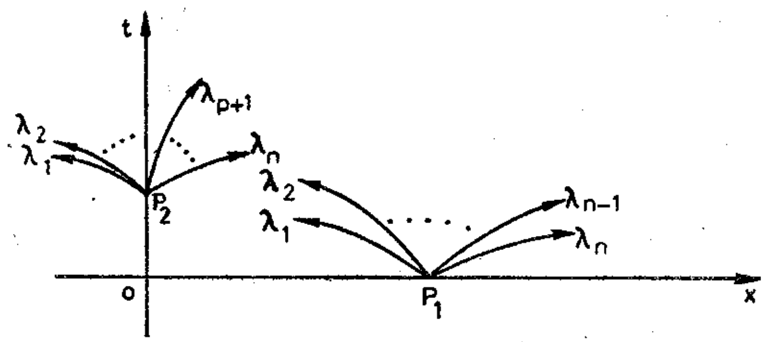

Fig. 2.8 Mixed initial-boundary value problem

are satisfied at every point of the t-axis. Then from an arbitrary point P_2 on $x=0$, $n-p$ characteristics enter into the first quadrant, where a characteristic is counted as many times as its multiplicity. The t-axis itself may or may not be a characteristic curve. If P_1 is a point on the initial line, the number of characteristics entering into the first quadrant from P_1 is obviously equal to n. If P_2 is a point on the t-axis, the number of characteristics entering into the first quadrant from P_2 is $n-p$. Let us note that when we say that a characteristic curve enters into a region from its boundary, we are associating definite forward and backward directions with the characteristic curve. Intuitively, we have assumed the forward direction (shown by arrows) as the t-increasing direction. This is meaningful in physical problems.

The following rule tells us about the nature of initial and boundary values: The number of functions which are to be prescribed on any curve as appropriate data on that curve is equal to the number of characteristics entering into the region (where the solution is required) from a point on the curve. According to this rule, we need to prescribe a Cauchy data on the initial line, i.e.

$$U(x, 0) = \Psi(x). \tag{2.84}$$

Or the boundary, we can prescribe $n-p$ components of $U(0, t)$, i.e.

$$u_i(0, t) = \varphi_i(t),\ i = 1, 2, \ldots, n-p, \text{ say} \tag{2.85}$$

or more generally we can prescribe $n-p$ linear combinations of $u_j(0, t)$, i.e.

$$\sum_{j=1}^{n} a_{ij} u_j(0, t) = \varphi_i(t),\ i = 1, 2, \ldots, n-p \tag{2.86}$$

in such a manner that the rank of the matrix a_{ij} is $n-p$. The n components of $\Psi(x)$ in (2.84) and the $n-p$ functions $\varphi_i(t)$, $i = 1, 2, \ldots, n-p$ in (2.85) can be arbitrarily chosen. If we wish to get a continuous solution in the neighbourhood of the origin of the (x, t)-plane, the initial and boundary values must satisfy the consistency condition

$$\sum_{j=1}^{n} a_{ij} \psi_j(0) = \varphi_j(0). \tag{2.87}$$

Similar consistency conditions can be obtained for the continuity of the partial derivatives of the solution.

If the initial and boundary conditions are prescribed as mentioned above, the problem becomes well posed. More precisely we state: Under the conditions prescribed above, the linear hyperbolic system has a solution which is unique and is stable in the region $x > 0$, $t > 0$ provided the data Ψ and φ_i and the coefficients A, B and C are sufficiently smooth.

Even in more complicated initial and boundary value problems, where the system may be confined between two fixed boundaries, say $x = 0$ and $x = a$ or between two boundaries, one or both of which are moving, the same rule of enumeration of number of characteristics entering into the region, determine the number of boundary conditions to be prescribed.

§3 HYPERBOLIC SYSTEM OF TWO FIRST ORDER QUASILINEAR EQUATIONS

A system of two first order equations

$$A_{11}u_t + A_{12}v_t + B_{11}u_x + B_{12}v_x + C_1 = 0 \tag{3.1}$$

and

$$A_{21}u_t + A_{22}v_t + B_{21}u_x + B_{22}v_x + C_2 = 0 \tag{3.2}$$

deserves a separate discussion. In this paragraph we briefly review the results of §2.1 for this system. This pair of equations is said to be hyperbolic if we can obtain two independent linear combinations of these equations such that in both the combinations the variables u and v are differentiated in the same direction in the (x, t)-plane. This requirement leads to a discussion of the roots of the characteristic equations (2.4), i.e. the roots of

$$\det(B-\lambda A) \equiv \det \begin{bmatrix} B_{11}-\lambda A_{11} & B_{12}-\lambda A_{12} \\ B_{21}-\lambda A_{21} & B_{22}-\lambda A_{22} \end{bmatrix} = 0 \tag{3.3}$$

where it is assumed that

$$\det A \neq 0. \tag{3.4}$$

When the two roots $\lambda = c_1, c_2$ are real and distinct, the pair (3.1), (3.2) is hyperbolic and the independent linear combinations can be taken to be $l_1^{(i)} \times (3.1) + l_2^{(i)} \times (3.2)$, where $l^{(i)} \equiv [l_1^{(i)}, l_2^{(i)}]$ is the left null vector of the matrix $B - c_i A$, $i = 1, 2$. When the equation (3.3) has equal real roots $\lambda = c$, the above pair is hyperbolic if there are two linearly independent left eigen vectors, which implies that the matrix B is a multiple of A, i.e. $B = cA$.

In this section, we shall discuss the properties of solutions of a pair of hyperbolic equations (3.1), (3.2). The system may be linear, semilinear or quasilinear. However, in the case of a quasilinear system, we shall assume that a known solution $u = u(x, t)$, $v = v(x, t)$ has been substituted in the coefficients, so that the coefficient matrices $A(x, t, u(x, t), v(x, t))$ and $B(x, t, u(x, t), v(x, t))$ can be regarded as functions of x and t only. In this case our results are true only for the given solution under consideration.

§3.1 Introduction of Characteristic Coordinates

We first assume that the characteristic roots c_1 and c_2 are real and distinct in a domain D_1 of the (x, t)-plane. The characteristic equations and their general solutions can be represented as

$$\frac{dx}{dt} = c_1 \Rightarrow \varphi(x, t) = \beta \tag{3.5}$$

and

$$\frac{dx}{dt} = c_2 \Rightarrow \psi(x, t) = \alpha \tag{3.6}$$

where β and α are constants giving the two families of characteristic curves. (3.5) gives us a one-parameter family of characteristic curves with parameter β and (3.6) gives another family of characteristics with parameter α. We call these families of characteristics as C_{I} and C_{II} families respectively. Along a member of C_{I} family the parameter β is constant and α varies. Similarly, along a member of C_{II} family the parameter α is constant and β varies. We remind the reader here that if the system (3.1)-(3.2) is linear or semi-linear, the characteristics are determined once for all in the (x, t)-plane, whereas, in the case of a quasilinear system they depend on a solution $u = u(x, t)$, $v = v(x, t)$ of the system.

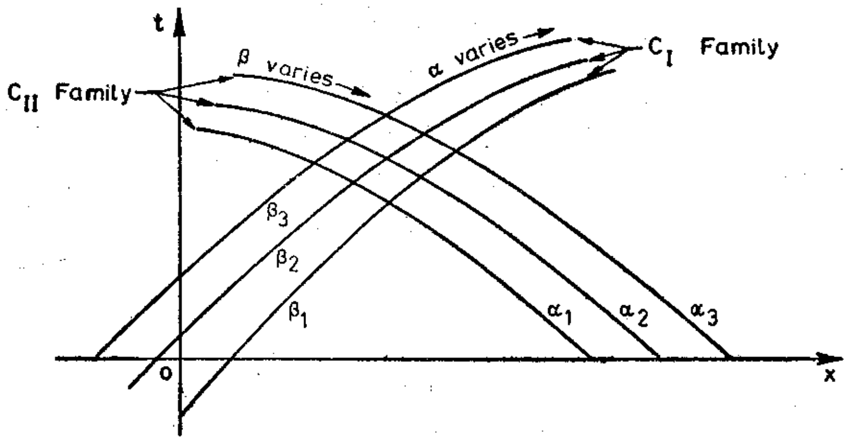

Fig. 3.1 The two families of characteristics

Using $c_1 = -\varphi_t/\varphi_x$, $c_2 = -\psi_t/\psi_x$, we get the relation

$$\frac{\partial(\varphi, \psi)}{\partial(x, t)} = \varphi_x \psi_x (c_1 - c_2). \tag{3.7}$$

For distinct and finite characteristic velocities, the Jacobian $\partial(\varphi, \psi)/\partial(x, t)$ does not vanish in the domain under consideration in the (x, t)-plane. Therefore we can solve (3.5) and (3.6) for x and t in terms of α and β:

$$x = x(\alpha, \beta), \; t = t(\alpha, \beta) \tag{3.8}$$

and then introduce α and β as independent variables.

The dependent variables can also be expressed as functions of α, β. Multiplying the hyperbolic system (3.1) and (3.2) by the components of the ith left eigenvectors $l^{(i)}$ and adding, we get

$$(l_1^{(i)} A_{11} + l_2^{(i)} A_{21})(u_t + c_i u_x) + (l_1^{(i)} A_{12} + l_2^{(i)} A_{22})\,(v_t + c_i v_x) \; + l_1^{(i)} C_1 + l_2^{(i)} C_2 = 0, \tag{3.9}$$

$i = 1, 2$, no sum over i.

The partial derivative x_α is evaluated by keeping β fixed, and therefore, for this we move along a member of C_I family, along which x_α and t_α are related by the relation $x_\alpha = c_1 t_\alpha$. Hence, we have

$$\frac{\partial}{\partial \alpha} = t_\alpha \left(\frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x} \right). \tag{3.10}$$

Similarly,

$$\frac{\partial}{\partial \beta} = t_\beta \left(\frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial x} \right). \tag{3.11}$$

Therefore we write the compatibility conditions (3.9) together with the characteristic equations, in terms of the independent variables α and β, as follows:

$$h_{11}(x, t, u, v) \frac{\partial u}{\partial \alpha} + h_{12}(x, t, u, v) \frac{\partial v}{\partial \alpha} = \gamma_1(x, t, u, v) t_\alpha \tag{3.12}$$

$$h_{21}(x, t, u, v)\frac{\partial u}{\partial \beta}+h_{22}(x, t, u, v)\frac{\partial v}{\partial \beta}=\gamma_2(x, t, u, v)t_\beta \tag{3.13}$$

$$x_\alpha=c_1(x, t, u, v)t_\alpha \tag{3.14}$$

and

$$x_\beta=c_2(x, t, u, v)t_\beta \tag{3.15}$$

The functions h_{ij} and γ_i can be easily expressed in terms of the components of A, B, C and $l^{(i)}$. We note here that the independent variables α and β do not appear explicitly in the coefficients of the equations (3.12)-(3.15).

Therefore, when the characteristics of the hyperbolic system of two equations are distinct, we have been able to reduce the system to an equivalent system of four (but simpler to deal with) equations (3.12) to (3.15). Though the number of equations increase, the new system is simpler, for numerical computation, compared to (3.1) and (3.2) due to the fact that the equations (3.12) and (3.14) involve differentiation with respect to only one variable α and hence can be treated as a coupled system of ordinary differential equations. Similarly, equations (3.13) and (3.15) can be treated as a coupled system of ordinary differential equations along a C_{II} characteristic. The characteristic canonical form (3.12)-(3.15) has been extensively used in the numerical solution of quasilinear partial differential equations, especially in gas dynamics (Shapiro, 1954).

We pointed out in the beginning of this section that a hyperbolic system of two equations deserves a separate discussion. This is due to the fact that in this case the number of characteristics through a point is equal to the number of independent variables and hence we can use the variables α, β as independent variables instead of the variables x, t. Such a transformation to characteristic coordinates is not suitable for a hyperbolic system of more than two equations.

Now, let us assume that the hyperbolic system (3.1)-(3.2) has a multiple characteristic velocity $\lambda=c$ (of course, of multiplicity two). Then the rank of the matrix $B-cA$ must be zero, i.e. we must have

$$B=cA. \tag{3.16}$$

Using (3.16) in (3.1) and (3.2) and premultiplying the resultant equations by the inverse of A we get an equivalent system of two equations:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\gamma_1 \tag{3.17}$$

and

$$\frac{\partial v}{\partial t}+c\frac{\partial v}{\partial x}=\gamma_2 \tag{3.18}$$

where γ_1 and γ_2 are the components of $A^{-1}C$. The compatibility conditions and the characteristic equation for these equations form a coupled system of three ordinary differential equations:

$$\frac{du}{dt}=\gamma_1(u, v, x, t) \tag{3.19}$$

$$\frac{dv}{dt} = \gamma_2(u, v, x, t) \tag{3.20}$$

and

$$\frac{dx}{dt} = c(u, v, x, t) \tag{3.21}$$

where d/dt represents temporal rate of change along a characteristic curve i.e.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

In this case, to solve a noncharacteristic Cauchy problem where the Cauchy data is prescribed in the form

$$\begin{gathered} u(x_0(\eta), t_0(\eta)) = u_0(\eta), \quad v(x_0(\eta), t_0(\eta)) = v_0(\eta) \\ x = x_0(\eta), \quad t = t_0(\eta) \end{gathered} \tag{3.22}$$

we solve (3.19) to (3.21) with the conditions (3.22) to get

$$u = u(\eta, t),\ v = v(\eta, t) \tag{3.23}$$

and

$$x = x(\eta, t). \tag{3.24}$$

Solving (3.24) for η in terms of x and t and substituting it in (3.23) we get the solution u and v of the Cauchy problem in terms of x and t.

We note that the method discussed above is applicable not only to a system of two equations but to a system of any number of equations when all characteristic velocities of the hyperbolic system are equal.

Example 3.1 The steady axi-symmetric irrotational motion of a gas is given by

$$\left.\begin{aligned} & u_y - v_x = 0 \\ \text{and} \quad & \\ & (a^2 - u^2)u_x - uv(u_y + v_x) + (a^2 - v^2)v_y + \frac{a^2 v}{y} = 0 \\ \text{where} \quad & \\ & a = a(q),\ q = \sqrt{u^2 + v^2} \end{aligned}\right\} \tag{3.25}$$

Replacing the role of t by y in the previous analysis, we get the characteristic roots of (3.25) as

$$c_{1,2} = \frac{1}{a^2 - v^2}[-uv \pm a\sqrt{(q^2 - a^2)}].$$

The pair (3.25) of equations form a hyperbolic system in the supersonic region $q^2 > a^2$. In this region, the characteristic canonical form is

$$(a^2 - u^2)u_\alpha + (a^2 - v^2)v_\alpha = -\frac{a^2 v}{y}y_\alpha$$

$$(a^2 - u^2)u_\beta + (a^2 - v^2)v_\beta = -\frac{a^2 v}{y}y_\beta$$

$$x_\alpha - c_1 y_\alpha = 0$$

and

$$x_\beta - c_2 y_\beta = 0.$$

EXERCISES 3.1

1. Find the solution of the Cauchy problem

$$u_t+u_x=2v^2+2uv,\ v_t+v_x=u^2-v^2$$
$$u(x,0)=0,\ v(x,0)=\sin x$$

and show that the solution is valid only in the strip $-\infty < x < \infty$, $-1 < t < 1$.

§3.2 Linearization of a Reducible System of Quasilinear Equations by Hodograph Transformation

We discuss here an important subclass of quasilinear equations in two independent variables. The system of equations (3.1) and (3.2) with $C_i=0$ reduces to

$$A_{11}u_t+A_{12}v_t+B_{11}u_x+B_{12}v_x=0 \tag{3.26}$$
$$A_{21}u_t+A_{22}v_t+B_{21}u_x+B_{22}v_x=0. \tag{3.27}$$

The homogeneous system (3.26) and (3.27) is said to be *reducible* if the coefficients A_{ij}, B_{ij} are fuctions of u and v only. We consider here only those solutions of the equations for which the Jacobian

$$j=u_xv_t-u_tv_x\neq 0. \tag{3.28}$$

In this case we can interchange the role of dependent and independent variables, i.e. we can consider x and t as functions of u and v.* Then the resultant equations for x and t as dependent variables and u, v as independent variables will be linear as shown below.

Solving for dx and dt from

$$du=u_x\,dx+u_t\,dt,\qquad dv=v_x\,dx+v_t\,dt \tag{3.29}$$

we get

$$dx=\frac{1}{j}(v_t\,du-u_t\,dv),\ dt=\frac{1}{j}(u_x\,dv-v_x\,du). \tag{3.30}$$

Therefore

$$u_x=j\,t_v,\quad u_t=-jx_v,\ v_x=-jt_u,\quad v_t=jx_u. \tag{3.31}$$

Substituting (3.31) in (3.26) and (3.27) we get

$$-A_{11}x_v+A_{12}x_u+B_{11}t_v-B_{12}t_u=0 \tag{3.32}$$

and

$$-A_{21}x_v+A_{22}x_u+B_{21}t_v-B_{22}t_u=0. \tag{3.33}$$

If $j\neq$ zero or infinity, then every solution of (3.26) and (3.27) gives a solution of (3.32) and (3.33) and vice-versa. We note that the latter system is linear.

*Vanishing of a Jacobian of a tranformation leads to interesting results regarding the mapping from (x, t)-plane to (u, v)-plane (Lighthill, 1953).

This transformation was first used in fluid dynamics, where u and v denote velocity components in a steady two dimensional flow, t being replaced by y. It is called "hodograph transformation". Though the transformed equations are linear and it is much easier to find their solutions, the boundary curves in the (x, y)-plane are transformed into complicated boundaries in (u, v)-plane. Therefore the hodograph transformation has been of very limited use. This transformation has been extended to unsteady flows as the example below indicates.

Example 3.2 Consider the equations of unsteady motion of a polytropic gas

$$\rho_t + q\rho_x + \rho q_x = 0$$

and

$$q_t + q\,q_x + \frac{a^2}{\rho}\rho_x = 0 \tag{3.34}$$

with

$$a^2 = k^2\rho^{\gamma-1}, \quad k = \text{constant}, \quad \gamma = \text{constant}.$$

Here ρ is the mass density and q, the velocity of the gas. Assuming that

$$\rho_x q_t - \rho_t q_x \neq 0 \tag{3.35}$$

we make use of the hodograph transformation. Equation (3.34) then become

$$-x_q + qt_q - \rho t_\rho = 0$$

and

$$-x_\rho + \frac{a^2}{\rho}t_q - qt_\rho = 0 \tag{3.36}$$

which are linear.

A more elegant form of the equations in the hodograph plane would be obtained, if we had started with equations

$$\left(\frac{\partial}{\partial t} + \left(\frac{\gamma+1}{2}r + \frac{\gamma-3}{2}s\right)\frac{\partial}{\partial x}\right) r = 0 \tag{3.37}$$

$$\left(\frac{\partial}{\partial t} + \left(\frac{\gamma-3}{2}r + \frac{\gamma+1}{2}s\right)\frac{\partial}{\partial x}\right) s = 0 \tag{3.38}$$

with dependent variables r and s instead of q and ρ, where r and s are called the Riemann invariants (see §3.3) and are defined by

$$r = \frac{1}{2}q + \frac{1}{2}\int\frac{a\,d\rho}{\rho} = \frac{1}{2}q + \frac{a}{\gamma-1} \tag{3.39}$$

and

$$s = -\frac{1}{2}q + \frac{1}{2}\int\frac{a\,d\rho}{\rho} = -\frac{1}{2}q + \frac{a}{\gamma-1}. \tag{3.40}$$

The hodograph equations now are

$$x_s - (q+a)t_s = 0 \quad \text{or} \quad x_s - \left\{\frac{\gamma+1}{2}r + \frac{\gamma-3}{2}s\right\}t_s = 0 \tag{3.41}$$

and

$$x_r - (q-a)t_r = 0 \quad \text{or} \quad x_r + \left\{\frac{\gamma-3}{2}r + \frac{\gamma+1}{2}s\right\}t_r = 0. \tag{3.42}$$

The equations (3.41) and (3.42) can also be derived from (3.37) and (3.38) without going through the hodograph transformation. Equation (3.37) implies

$$r = \text{constant on } \frac{dx}{dt} = q + a \tag{3.43}$$

and equation (3.38) implies

$$s = \text{constant on } \frac{dx}{dt} = q - a. \tag{3.44}$$

On r = constant, x and t are functions of s only so that (3.43) can be written as

$$x_s - (q + a)t_s = 0 \tag{3.45}$$

which is precisely the equation (3.41). Similarly, (3.44) leads to (3.42).

Eliminating x from (3.41) and (3.42) we get

$$(r+s)t_{rs} + \frac{\gamma+1}{2(\gamma-1)}(t_r + t_s) = 0. \tag{3.46}$$

Method of solution: We present here an interesting method of solution of the equation (3.46), which we write in the form

$$u_{xy} + \frac{M}{x+y}(u_x + u_y) = 0 \tag{3.47}$$

where M is a constant. If u satisfies (3.47), then

$$v = (x+y)^{2M-1}u \tag{3.48a}$$

satisfies

$$v_{xy} + \frac{1-M}{x+y}(v_x + v_y) = 0 \tag{3.48b}$$

and

$$w = \frac{1}{x+y}(u_x + u_y) \tag{3.49a}$$

satisfies

$$w_{xy} + \frac{M+1}{x+y}(w_x + w_y) = 0. \tag{3.49b}$$

The formulae (3.48) and (3.49) can be used to determine the solution of (3.47) for all positive integral values of M. We note that for $M = 0$, the equation reduces to the wave equation

$$u_{xy} = 0$$

whose general solution is

$$u = \varphi(x) + \psi(y)$$

where φ and ψ are arbitrary functions of x and y respectively. From (3.48) it follows that

$$u_1 = \frac{1}{x+y}\{\varphi(x) + \psi(y)\}$$

is a solution of (3.47) with $M=1$. Consider only one part

$$u_1=\frac{\varphi(x)}{x+y}$$

of the solution. From (3.49) it follows that

$$u_2=\frac{1}{x+y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\varphi(x)}{x+y}\right)=\frac{\partial}{\partial x}\left(\frac{\varphi(x)}{(x+y)^2}\right)$$

is a solution of (3.47) for $M=2$.

Using (3.49) successively, we find that for any integer M, $\frac{\partial^{M-1}}{\partial x^{M-1}}\left(\frac{\varphi(x)}{(x+y)^M}\right)$ is a solution of (3.47). Similarly, $\frac{\partial^{M-1}}{\partial y^{M-1}}\left(\frac{\psi(y)}{(x+y)^M}\right)$ is also a solution. Therefore

$$u=\frac{\partial^{M-1}}{\partial x^{M-1}}\left(\frac{\varphi(x)}{(x+y)^M}\right)+\frac{\partial^{M-1}}{\partial y^{M-1}}\left(\frac{\psi(y)}{(x+y)^M}\right),\quad M=\text{positive integer} \tag{3.50}$$

containing two arbitrary functions, gives the general solution of the equation (3.47) for any positive integral value of M.

Using (3.48) and (3.50), we find that the general solution of (3.47) for negative integral values M is

$$u=[x+y]^{1-2M}\left[\frac{\partial^{(-M)}}{\partial x^{(-M)}}\left\{\frac{\varphi(x)}{(x+y)^{1-M}}\right\}+\frac{\partial^{(-M)}}{\partial y^{(-M)}}\left\{\frac{\psi(y)}{(x+y)^{1-M}}\right\}\right]. \tag{3.51}$$

Using the theory of complex variables, we can represent the first term of the solution (3.50) in the form of a contour integral

$$u(x,y)=\frac{(M-1)!}{2\pi i}\int_\Gamma\frac{\varphi(\xi)\,d\xi}{(\xi-x)^M(\xi+y)^M} \tag{3.52}$$

where Γ is a simple closed contour in the complex ξ-plane with the point $\xi=x$ in its interior and point $\xi=-y$ outside Γ. The arbitrary function $\varphi(\xi)$ is necessarily an analytic function of ξ, regular inside and on Γ.

We note that the factor $(\xi-x)^{-M}(\xi+y)^{-M}$ satisfies the equation (3.47) not only for integral values of M but for all real values of M. However, it is no longer a (single valued) function of ξ. There are two branch point singularities one at $\xi=x$ and another at $\xi=-y$. Therefore if Γ is a path in the complex ξ-plane such that $\xi\neq x$ or $-y$ for all real x, y belonging to D, then (3.52) represents a solution of the equation (3.47) for an arbitrary value of M.

§3.3 Simple Wave Solution for a System of Two Equations

We shall discuss now an important class of solutions of a reducible system of two equations. These solutions, called "simple waves", are some of the very few known exact solutions of quasilinear equations and play an extremely important role in understanding and construction of more general solutions. In fact, simple waves are the fundamental or basic solutions and any other genuine solution is formed by the "interaction" of two simple waves.

First we notice that for a reducible system of equations, the compatibility conditions (3.12) and (3.13) not only become homogeneous but also get decoupled from the characteristic equations. The equations (3.12) to (3.15) reduce to

$$h_{11}(u, v)u_\alpha + h_{12}(u, v)v_\alpha = 0 \tag{3.53}$$

$$h_{21}(u, v)u_\beta + h_{22}(u, v)v_\beta = 0 \tag{3.54}$$

$$x_\alpha - c_1(u, v)t_\alpha = 0 \tag{3.55}$$

and

$$x_\beta - c_2(u, v)t_\beta = 0. \tag{3.56}$$

Since x and t do not appear in the compatibility conditions (3.53) and (3.54), these conditions can be integrated independently and we get

$$r(u, v) = \text{constant} = \beta' \text{ (say) along } \frac{dx}{dt} = c_1(u, v) \tag{3.57}$$

and

$$s(u, v) = \text{constant} = \alpha' \text{ (say) along } \frac{dx}{dt} = c_2(u, v). \tag{3.58}$$

We call the functions r and s *Riemann invariants* of the second and first characteristic family respectively.

A nonconstant solution, of a reducible system of two equations, in a domain S of the (x, t)-plane is called a simple wave if one of the two Riemann invariants is constant in S. This domain S is called a *simple wave region.*

The constancy of one of the Riemann invariants in S implies a functional relation, between u and v, of the form $f(u, v) = \text{constant}$, so that the Jacobian $\partial(u, v)/\partial(x, t) = 0$ everywhere in S. Therefore, a hodograph transformation cannot be used to find a simple wave solution.

Theorem 3.1 In a simple wave region if Riemann invariant* $r\,(s)$ is a constant, then u and v are constant along the second (first) family of characteristics which are straight lines.

Proof: Let us assume that the Rieman invariant r is constant in the simple wave region S, then β' is the same constant everywhere in S. Along a member, say C_c, of the second family of characteristics (3.56), there is another relation between u, v given by $s(u, v) = \alpha'$, where 'α'' is a constant. Solving u and v in terms of α' and β' from $r(u, v) = \beta'$ and $s(u, v) = \alpha'$ we get

$$u = u(\alpha', \beta'), \qquad v = v(\alpha', \beta') \tag{3.59}$$

---

*In literature the variables r and s are simply called *Riemann invariants*. Consistent with the general theory of Riemann invariants (see the problem in exercise 4.1) we call r as a Riemann invariant of the second characteristic family and s as that of the first characteristic family. The theory of simple waves for a hyperbolic system of more than two equations can also be presented in terms of the Riemann invariants. However, for more than two equations, we have given an equivalent but simpler account of simple waves in §4. The theory of simple waves in terms Riemann invariants simplifies considerably for two equations, and therefore, we present it here.

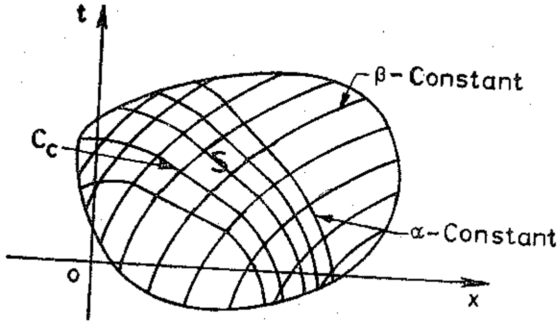

Fig. 3.2 Simple wave region S, where $r(u, v)$=constant

where β' is constant everywhere in S and α' is constant along C_c. Therefore u, v are both constants along C_c. The function $c_2(u, v)$, the second characteristic velocity is also constant along C_c, so its slope $\frac{dt}{dx} = \frac{1}{c_2}$ is constant. Consequently the characteristic C_c and similarly all characteristics of the second family are straight lines in the simple wave region S.

Now we state here a fundamental theorem which identifies simple waves as basic building material for more general solutions.

Theorem 3.2: If a section of a characteristic carries constant values of u and v, then in regions adjacent to this section, the solution is either a constant state or a simple wave.

Proof: Consider a region D, of the (x, t)-plane, in which u and v are continuous functions and which contains a section δ of a characteristic of the C_I family, on which u and v are both constants. Through each point of the region D a member of the C_{II} family of characteristic passes. Consider those members which intersect δ. The region covered by these curves is denoted by S. Since the values of u and v are constant on δ, say $u=u_0$ and $v=v_0$, the C_{II} family of characteristics in S carry the same values of $s=s(u_0, v_0)=s_0$ (say). Hence the Riemann invariant of the first characteristic family is constant in S. Therefore, the region S is either a simple wave region or a constant state region.

An immediate corollary of the above theorem is:

Corollary: The solution in a region adjacent to a region of constant state is a simple wave solution.

To prove the corollary, we note that if C is a curve, in the (x, t)-plane, across which the solution changes continuously from a constant state to a nonconstant state, at least one derivative of the solution must be discontinuous across C so that the curve C is a characteristic curve. Since on one side of C there is a constant state and since the solution is continuous, u and v both must be constant on C. Using theorem 3.2 it follows that the nonconstant solution on the other side of C must be a simple wave.

After this elemantary theory of the simple waves, we take an example from gasdynamics.

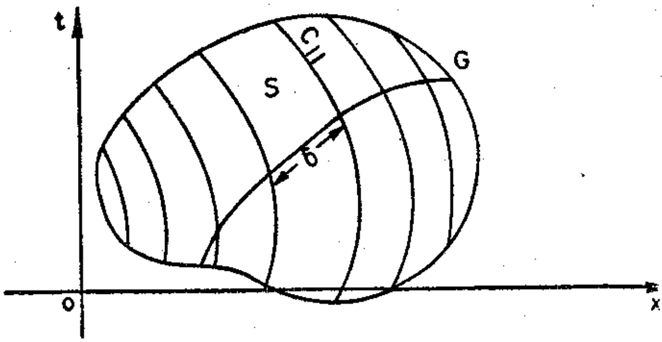

Fig. 3.3 The region adjacent to the section δ is simple wave or constant state region

§3.4 Application of the Theory: Simple Waves in Gasdynamics

Let us consider the motion produced by a moving piston in an initially undisturbed compressible medium at uniform state and contained in a semi-infinite tube bounded on the left by the piston. We assume the medium to be a polytropic gas, equations of one-dimensional motion for which can be written in the matrix form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \rho \\ q \end{bmatrix} + \begin{bmatrix} q & \rho \\ \dfrac{a^2}{\rho} & q \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ q \end{bmatrix} = 0 \tag{3.60}$$

where ρ is the mass density, q the particle velocity and a the sound velocity satisfying

$$a^2 \equiv a^2(\rho) = k^2 \rho^{\gamma-1}, \qquad k^2 = \text{constant}, \qquad \gamma = \text{constant}. \tag{3.61}$$

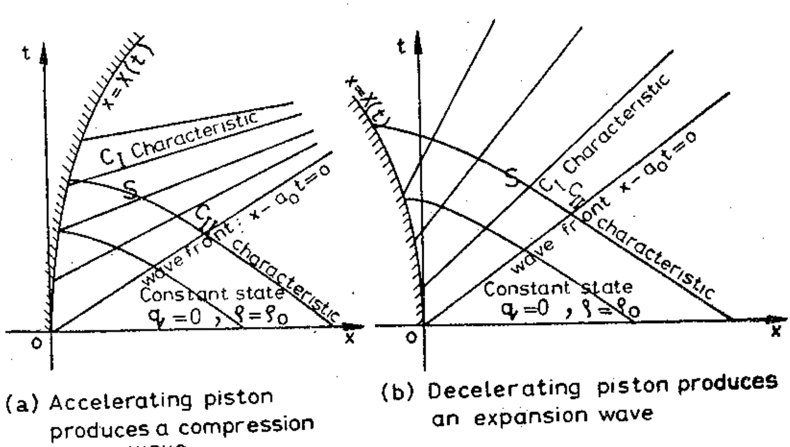

Fig. 3.4 Waves produced by a piston starting with zero velocity

The system (3.60) is reducible with distinct characteristic velocities

$$c_1 = q + a, \qquad c_2 = q - a. \tag{3.62}$$

Let the equation of the piston be given by

$$x = X(t), \qquad X(0) = 0 \tag{3.63}$$

where we assume that initially the piston starts with zero velocity, i.e.

$$X'(0) = 0. \tag{3.64}$$

For an accelerating piston motion we have

$$X''(t) > 0 \tag{3.65}$$

and for a decelerating motion

$$X''(t) < 0. \tag{3.66}$$

The problem is to find the solution of the system of equations (3.60) in the region on the right of the piston, i.e. in the domain D of the (x, t)-plane:

$$D : x \geqslant X(t), \qquad t \geqslant 0. \tag{3.67}$$

Initially the gas in the tube is at rest with constant density, so the solution must satisfy the initial conditions

$$\rho(x, 0) = \text{constant} = \rho_0(\text{say}), \qquad q(x, 0) = 0, \qquad x \geqslant 0. \tag{3.68}$$

At the piston the fluid velocity is equal to the particle velocity, so the solution must satisfy the boundary condition

$$q(X(\eta), \eta) = X'(\eta), \qquad t = \eta, \qquad \eta \geqslant 0. \tag{3.69}$$

On the initial line $t = 0$, the two characteristics velocities are $\pm a_0$, so from each point of it two characteristics enter into the region D. On the boundary $x = X(t)$, the piston velocity is equal to the velocity of the gas, so the characteristics C_{I} lie to the right of the boundary and C_{II} to the left. So from each point of the boundary one characteristic of C_{I} family enters into D. Thus the initial conditions (3.68) and one boundary condition (3.69) are sufficient to give a unique solution.

The characteristic equations and the compatibility conditions are:

$$\text{along } \frac{dx}{dt} = q + a, \qquad dq + dl = 0 \tag{3.70}$$

and

$$\text{along } \frac{dx}{dt} = q - a, \qquad dq - dl = 0 \tag{3.71}$$

where

$$l = \int \frac{a\,d\rho}{\rho} = \frac{2a}{\gamma - 1}. \tag{3.72}$$

Therefore we can choose the Riemann invariants to be

$$r = \frac{q}{2} + \frac{a}{\gamma - 1}, \qquad s = -\frac{q}{2} + \frac{a}{\gamma - 1}. \tag{3.73}$$

Since the initial motion is continuous, the effect of piston motion will be felt in a region behind a wavefront running into the undisturbed state with the velocity of the forward facing wave. Therefore the region in (x, t)-plane, disturbed due to the motion of the piston, will be separated from the undisturbed region by a C_{I} characteristic starting from the origin, i.e. by $x = a_0 t$. Since this characteristic adjoins a constant state, the values of q and ρ are constant on it and hence, using theorem 3.2, the flow produced by the piston is a simple wave in which the Riemann invariant s is constant. Therefore we have

$$\frac{q}{2} = \frac{a}{\gamma-1} - \frac{a_0}{\gamma-1}. \tag{3.74}$$

The C_{I} family of characteristics are straight lines and are given by

$$x = (q+a)t + \text{constant}$$

or

$$x = (a_0 + \frac{\gamma+1}{2} q)t + \text{constant}. \tag{3.75}$$

Evaluating the constant with the help of the boundary condition (3.69) and noting that q at any point (x, t) is same as that at the point of intersection of the piston path and the C_{I} characteristic through (x, t), we get the following equation of the C_{I} family of characteristics:

$$x = X(\eta) + \{a_0 + \frac{\gamma+1}{2} \dot{X}(\eta)\}(t-\eta). \tag{3.76}$$

Different values of η give different members of the C_{I} characteristics starting at time $t = \eta$ from the piston. On a C_{I} characteristic curve, the constant value of q is equal to the piston velocity at that point of the piston where the characteristic curve, intersects the piston path. The constant value of a on a C_{I} characteristic can be easily obtained from (3.74). Thus

$$q(x, t) = \dot{X}(\eta), \qquad a(x, t) = a_0 + \frac{\gamma-1}{2} \dot{X}(\eta). \tag{3.77}$$

Solving η from (3.76) in terms of x and t and substituting it in (3.77) we get q and a as functions of x and t. Thus, the simple wave problem is completely solved.

Though the above process is usually followed to solve a simple wave problem, it is important to note that a simple wave is governed by a single first order quasilinear partial differential equation of a very simple type and hence it can be obtained as its solution. The compatibility condition along the C_{I} characteristics give

$$\left\{\frac{\partial}{\partial t} + (q+a)\frac{\partial}{\partial x}\right\}\left(\frac{q}{2} + \frac{a}{\gamma-1}\right) = 0. \tag{3.78}$$

Eliminating a from (3.74) and (3.78) we have

$$\frac{\partial q}{\partial t} + (a_0 + \frac{\gamma+1}{2} q)\frac{\partial q}{\partial x} = 0 \tag{3.79}$$

which can be completely solved for any type of initial and boundary value problem. Note that the equation (3.79) is valid not only for arbitrary small q but for all finite values of q.

EXERCISE 3.2

1. Obtain the hodograph transformation of the equations

$$v_x - u_y = 0$$

$$(a^2 - u^2)u_x - uv(u_y + v_x) + (a^2 - v^2)v_y = 0$$

where a^2 is a given function of $q^2 = u^2 + v^2$ and satisfies

$$\mu^2 q^2 + (1-\mu^2)a^2 = a_*^2;\ a^*,\ \mu = \text{constant}.$$

Show that the characteristic curves in the hodograph plane are epicycloids generated by the movement of a point fixed on the circumference of a circle which is of diameter $a_* \left(\frac{1}{\mu} - 1\right)$ and which rolls on the sonic circle $u^2 + v^2 = a_*^2$.

2. In the example of the simple waves in gasdynamics (§3.4), show that when the piston is accelerating the characteristics of the first family, carrying different constant values of u and ρ, converge in the compression wave and the solution ceases to be single-valued after some finite time.

3. Consider the general reducible hyperbolic system and a simple wave solution in which the second Riemann invariant $r(u, v)$ has a constant value β'. Then the characteristic velocity $c_2(u, v)$ can be expressed as a function of u only: $c_2(u, v) = C_2(u, \beta')$. Show that the simple wave is described by the quasilinear equation

$$\frac{\partial u}{\partial t} + C_2(u, \beta')\frac{\partial u}{\partial x} = 0.$$

4. The characteristic field C_{II} is defined to be *genuinely nonlinear* if the function $C_2(u, \beta')$ defined in the preceding problem satisfies $\partial C_2/\partial u \neq 0$. Now if the simple wave is weakly nonlinear, i.e. if u and v differ from some constant values u_0 and v_0 respectively by small quantities, show that the simple wave in the genuinely nonlinear characteristic field is governed approximately by

$$\frac{\partial w}{\partial t} + (c_0 + c_w w)\frac{\partial w}{\partial x} = 0$$

where $w = u - u_0$, and c_0, c_w = constant.

5. Given two families of characteristic curves

$$\varphi(x, y) = \beta \text{ and } \psi(x, y) = \alpha$$

and another noncharacteristic curve $C: f(x, y)=0$, show that it is possible to choose another set of characteristic parameters α', β' such that

$$\alpha'=\alpha'(\alpha), \quad \beta'=\beta'(\beta)$$

and the curve C becomes the straight line $\alpha'+\beta'=0$ in the (α', β')-plane.

6. For a hyperbolic pair of two quasiliaear equations with a double characteristic velocity c, show that the genuine solution of a noncharacteristic Cauchy problem (3.22) can be obtained uniquely with the help of an iterative scheme

$$u^{n+1}(\eta, t)=u_0(\eta)+\int_0^{t-t_0(\eta)} \gamma_1(u^n(s, \eta), v^n(s, \eta), x^n(s, \eta), s+t_0(\eta))\, ds$$

$$v^{n+1}(\eta, t)=v_0(\eta)+\int_0^{t-t_0(\eta)} \gamma_2(u^n(s, \eta), v^n(s, \eta), x^n(s, \eta), s+t_0(\eta))\, ds$$

and

$$x^{n+1}(\eta, t)=x_0(\eta)+\int_0^{t-t_0(\eta)} c(u^n(s, \eta), v^n(s, \eta), x^n(s, \eta), s+t_0(\eta))\, ds$$

where γ_1, γ_2 are functions appearing in equations (3.17) and (3.18). While proving the result, state the conditions that are required to be imposed on the functions γ_1, γ_2 and c, determine a value of T such that the genuine solution is valid in $t_0(\eta) < t < t_0(\eta)+T$, and show that the solution is stable.

*§4. GENERAL THEORY OF A SIMPLE WAVE

As we have mentioned in the footnote on page 185 and we shall indicate in the problem at the end of this section, we can extend the theory of simple waves, in terms of Riemann invariants (presented in §3.3 for a reducible system of two equations) to a hyperbolic system of n homogeneous equations in the form

$$A(U)\frac{\partial U}{\partial t}+B(U)\frac{\partial U}{\partial x}=0 \tag{4.1}$$

where the coefficient matrices A and B do not depend on x and t explicitly. Let the set of all U, for which $A(U)$ and $B(U)$ are defined, be denoted by D_U. In this section we shall present briefly a simplified theory which would make use of an extension of the concept of characteristic variables w_k defined in § 2.2 for linear equations.

Our interest is to find a genuine solution of the system in a domain D of the (x, t)-plane such that all components u_i of U can be expressed in terms of a single function $w(x, t) \in C^1(D)$:

$$U(x, t)=\Phi(w(x, t)). \tag{4.2}$$

Substituting it in (4.1) and assuming $w_x \neq 0$, we get

$$[B(\Phi)-(-w_t/w_x)\, A(\Phi)]\frac{d\Phi}{dw}=0. \tag{4.3}$$

For a nontrival solution of $\frac{d\Phi}{dw}$ we require that $-w_t/w_x = c$, i.e.

$$w_t + c(\Phi)w_x = 0 \tag{4.4}$$

where c is a characteristic velocity of (4.1).

Since Φ is a function of w alone and does not explicitly contain x and t, $c(\Phi)$ is a function of w alone. Therefore the equation (4.4) implies that

$$w = \text{constant}, \ \Phi = \text{constant along each of lines } x - c(\Phi)t = \text{constant}. \tag{4.5}$$

The straight lines $x - c(\Phi)t = \text{constant}$, are characteristic curves of the system (4.1). The derivative $d\Phi/dw$ is parallel to the corresponding right eigenvector.

c is one of the n characteristic velocities $c_1, c_2, \cdots, c_n$ which, for simplicity, we assume to be distinct in D_U. For each c_k we can find a solution which remains constant along a family of straight lines which are, in fact, the kth family of characteristics in D. In this case we shall put a subscript k on the variable w appearing above, i.e. $w(x, t) = w_k(x, t)$. The solution $U(x, t)$ is said to be a *simple wave of the kth family*. As given in Problem 1, Exercise 4.1, corresponding to each characteristic velocity $c = c_k$, there exist $n-1$ Riemann invariants which remain constant in the kth simple wave region. This conforms with our earlier definition of a simple wave for a system of two equations given in Section 3.3.

Consider a constant value w_{k0} of w_k along a particular characteristic curve of the kth family in a kth simple wave. In the neighbourhood of this characteristic curve, we have

$$c_k(\Phi) = c_{k0} + \left\{ (\text{grad}_U c_k) \cdot \frac{d\Phi}{dw_k} \right\}_0 (w_k - w_{k0}) + 0\{(w_k - w_{k0})^2\}, \quad \text{no sum over } k.$$

Therefore using the fact that $d\Phi/dw_k$ is parallel to $r^{(k)}$, we find that to the first order in $(w_k - w_{k0})$ the characteristic velocity c_k in (4.4) depends linearly on w_k if $((\text{grad}_U c_k) \cdot r^{(k)})_0 \neq 0$. If,

$$(\text{grad}_U c_k) \cdot r^{(k)}, \neq 0, \text{ no sum over } k \tag{4.6}$$

for all U in the domain D_U of the dependent variables, we say that the kth characteristic field of (4.1) is *genuinely nonlinear*. We note that if the kth characteristic field is not genuinely nonlinear, i.e. $(\text{grad}_U c_k) \cdot r^{(k)} = 0$ for some U, one can get a kth simple wave governed by a linear equation and hence a nonlinear deformation in the shape of a pulse in such a simple wave will not take place in the wave moving with the kth characteristic velocity.

The variable w_k, which remains constant along a kth characteristic in a kth simple wave, is defined to be the kth *characteristic variable* for a reducible system (4.1). In §2.2, where the system of equations considered is linear, the characteristic variable w_k satisfies equation (2.38) which governs its rate of change along the kth characteristic. The equation governing the evolution of the kth characteristic variable in the kth simple wave is a single first order quasilinear equation (4.4), which we write now in the form

$$\frac{\partial w_k}{\partial t} + C_k(w_k)\frac{\partial w_k}{\partial x} = 0, \text{ no sum over } k \tag{4.7}$$

where

$$C_k(w_k) = c_k(\Phi(w_k)). \tag{4.8}$$

Results corresponding to the theorem 3.2 and its corollary in section 3.3 can be proved for a kth simple wave of a reducible hyperbolic system (4.1) of n equations.

We now discuss the types of constant states which can be connected by a simple wave. Consider all the states U_r which can be joined to a given state U_l on the left by a simple wave of the kth family. Then the states U_r form a one parameter family of states (a choice of the continuously varying parameter being $\delta = (w_k)_r - (w_k)_l$):

$$U_r \text{ (through a simple wave of the } k\text{th family)} = \Phi((w_k)_r) \equiv U_{\text{sim}}(\delta) \tag{4.9a}$$

where δ is a parameter such that

$$U_{\text{sim}}(0) = U_l. \tag{4.9b}$$

Treating the state U_l to be fixed, we note that δ is a function of $(w_k)_r$ only. Hence, differentiating (4.9a) with respect to δ and using the fact that $d\Phi/dw_k$ is parallel to $r^{(k)}$, we get

$$\dot{U}_{\text{sim}}(0) = \alpha r^{(k)}(U_l) \tag{4.10}$$

where a dot denotes differentiation with respect to δ and α is a scalar. *Thus we have proved that the derivative $\dot{U}_{\text{sim}}(0)$ in a kth simple wave is parallel to the kth right eigenvector $r^{(k)}$.*

Now we proceed to discuss a special class of simple wave solutions, namely *centered simple waves*, with the help of which we shall solve the fundamental Riemann's problem in §5.6. A *centered simple wave*, more appropriately called *centered* rarefaction wave, is a genuine solution of (4.1) for $t > t_0$, in which U depends only on the ratio $(x - x_0)/(t - t_0)$; (x_0, t_0) being centre of the wave. Choosing $(x - x_0)/(t - t_0)$ to be the variable w in (4.2), it follows that such a solution is a simple wave with one of the n families of characteristics, say kth, being straight lines represented by $(x - x_0)/(t - t_0) = \text{constant}$, all of which pass-through the point (x_0, t_0). Now we state a theorem without proof.

Theorem 4.1 Let a constant state U_l on left* be connected to another constant state U_r on right by a centered simple wave of the kth family, then

$$c_k(U_l) < c_k(U_r). \tag{4.11}$$

Figure 4.1 shows that the inequality (4.11) is a necessary condition for the existence of a centered simple wave for $t_0 > 0$.

The inequality (4.11) is of great importance to us. Let us explain it with a particular choice of the parameter δ:

$$\delta = c_k(U_r) - c_k(U_l). \tag{4.12}$$

In general, for an arbitrarily simple wave (not necessarily centered) the parameter δ could have both positive and negative values (see section 3.4). However, the inequality (4.11) permits only positive values of δ, i.e., "rarefaction simple waves" in a centered wave. We shall come to this remark again in §5.5.

*With respect to the reader (see footnote on page 115).

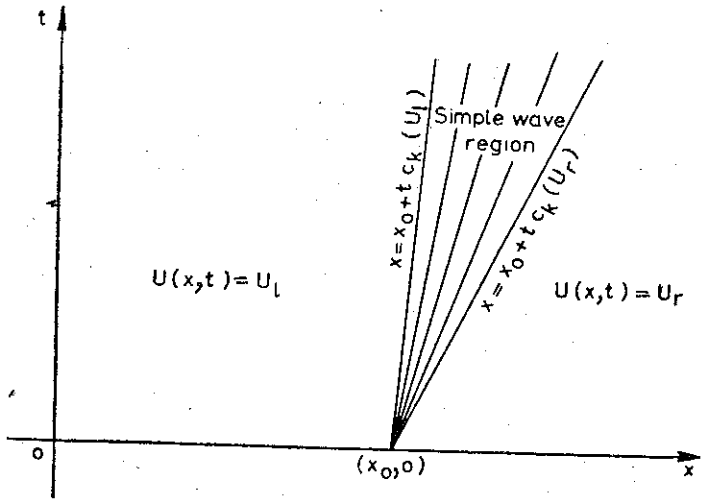

Fig. 4. Centered simple wave of the kth family joining states U_l and U_r (with $t_0=0$)

*EXERCISE 4.1

1. Consider a first order hyperbolic system of n equations

$$A(U)U_t+B(U)U_x=0$$

for which the kth characteristic velocity c_k is assumed to be simple. The set of $n-1$ nonconstant independent functions $w_j^{(k)}(U)$, $j=1, 2, \cdots, k-1, k+1, \cdots, n$ satisfying the first order quasi-linear equation

$$(\text{grad}_U\, w \cdot r^{(k)})=0$$

are defined to be $n-1$ *Riemann invariants* of the kth characteristic field. A function $w^{(k)}(U)$ such that $\partial(w_1^{(k)}, w_2^{(k)}, \cdots, w_n^{(k)})/\partial(u_1, u_2, \cdots, u_n)\neq 0$ is called kth *characteristic variable.* A solution of the hyperbolic system in a domain of the (x, t)-plane, for which all Riemann invariants of the kth field are constants, is called a *kth simple wave.* Prove the theorems 3.1 and 3.2 with this definition of the simple wave (Lax, 1957).

§ 5. WEAK SOLUTION OF QUASILINEAR EQUATIONS

A classical or genuine solution of a first order system of equations (2.1) is defined in a domain D of (x, t)-plane if the functions u_i possess continuous first order partial derivatives at every point of D and satisfy the differential equations in D. However the class of genuine solutions is too restricted to represent all physical phenomena. In fact, discontinuities in the variables specifying the state of a physical system and their derivatives are very common. For such solutions, the differential equations lose their significance. In this section we shall discuss a generalisation of the meaning of the solution, with special emphasis on the nonlinear equations.

§5.1 Conservation Laws

Before we proceed to discuss this, let us consider an initial value problem for a specific quasilinear partial differential equation:

$$\frac{\partial u}{\partial t}+a(u)\frac{\partial u}{\partial x}=0,\ t>0,\ -\infty<x<\infty \tag{5.1}$$

$$u(x,0)=\varphi(x),\ -\infty<x<\infty \tag{5.2}$$

where we assume for simplicity that $a'(u)$, the derivative of the coefficient of $\partial u/\partial x$, satisfies

$$a'(u)>0 \tag{5.3}$$

for all u for which $a(u)$ is defined. Assuming that the coefficient $a(u)$ and the Cauchy data are C^1 functions of their arguments, we conclude from the existence and uniqueness theorem for a single quasilinear equation of first order (see Theorem 2.1, Chapter 1) that the genuine solution exists for all values of $t<T$, where T depends on the functions $a(u)$ and $\varphi(x)$. The solution is given by

$$F(u)\equiv u-\varphi(x-a(u)t)=0 \tag{5.4}$$

which defines u as a continuously differentiable function of x and t as long as $F'(u)\neq 0$ i.e.

$$1+ta'(u)\varphi'(\xi)\neq 0,\ \xi=x-a(u)t. \tag{5.5}$$

The region of the (x, t)-plane of interest to us is that which is covered by the characteristics emanating from the points on the initial line. Considering a point $x=\xi$ on the initial line $t=0$, the condition (5.5) is violated on the characteristic through $(x=\xi, t=0)$ at a time

$$t=-\frac{1}{a'(\varphi(\xi))\varphi'(\xi)} \tag{5.6}$$

which is greater than zero provided $\varphi'(\xi)<0$. Therefore, the solution given by (5.4) ceases to exist for all time if the initial data be such that

$$\varphi'(\xi)<0 \tag{5.7}$$

for some value of ξ. Let T be the time when the solution first develops a singularity for some value of ξ. Then

$$T=-\frac{1}{\min\limits_{-\infty<\xi<\infty}\{a'(\varphi(\xi))\varphi'(\xi)\}}>0. \tag{5.8}$$

The fact that a singularity develops in the solution for $t\geqslant T$, can also be seen by the following consideration: if $\varphi'(\xi)<0$, we can find two points ξ_1

and $\xi_2(\xi_1 < \xi_2)$ on the initial line such that the characteristics through them have different slopes $1/a(u_1)$ and $1/a(u_2)$, where $u_1 = \varphi(\xi_1)$ and $u_2 = \varphi(\xi_2)$ and $a(u_1) > a(u_2)$. These characteristics will intersect at a point in (x, t)-plane with $t > 0$. Since the characteristics carry constant values of u, the solution ceases to be single valued at their point of intersection. If we examine the (u, x)-curve at various times, we find that after time T, the profile folds itself and at any instant there exists an interval on the x-axis where there

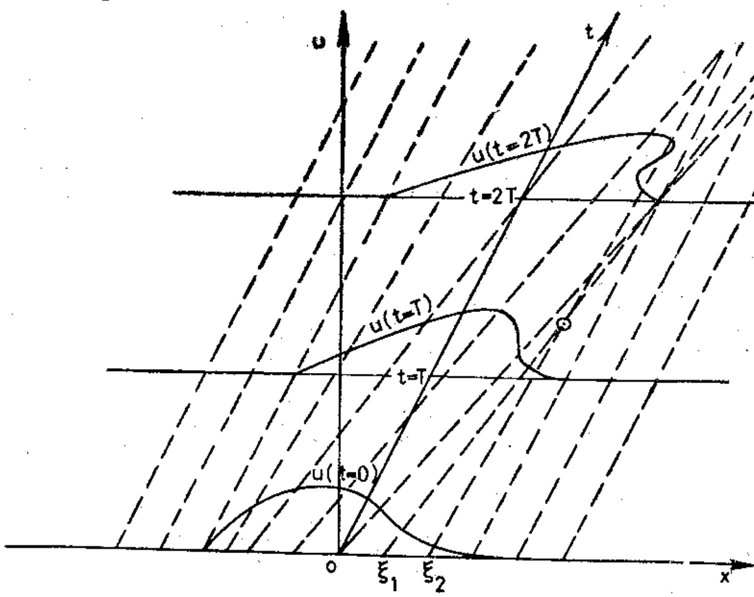

Fig. 5.1 The solution of an initial value problem for the equation $u_t + uu_x = 0$ is shown by continuous line at $t=0$, $t=T$ and $t=2T$. Dotted lines represent characteristic curves. The characteristics from two points ξ_1 and ξ_2 intersect at $t>T$.

are three values of u for every value of x. Consideration of similar situations in physical problems (such as waves in gas dynamics) tells us that a possible way to make the solution single-valued is to introduce a shock wave type of discontinuity in the solution at an appropriate place where the value of u jumps from u_l on one branch of the solution to a point u_r on another branch of the solution. If the condition (5.7) is not satisfied, the solution remains a genuine solution for all times. We note that in contrast to the linear equations (see section 2.3), a continuous Cauchy data can lead to a discontinuous solution for a quasilinear equation.

The above example shows that we must generalise the concept of solution in such a manner that even discontinuous solutions are admissible. We shall consider this generalisation for a restricted class of quasilinear equations which can be put in the form

$$\frac{\partial H}{\partial t} + \frac{\partial F}{\partial x} + K = 0. \tag{5.9}$$

We assume for our discustion in this chapter that the vector functions H, F and K are twice continuously differentiable functions of x, t and U in some domain D_2 of $(x, t, u_1, \cdots, u_n)$-space. Every system of linear first order equations can be put in the form (5.9) but this is not true for a quasilinear system. If H_i, F_i and K_i are the ith components of the column vectors H, F and K respectively, the vector equation (5.9) is the same as

$$\frac{\partial H_i}{\partial t}+\frac{\partial F_i}{\partial x}+K_i=0,\ i=1, 2, \cdots, n \tag{5.10}$$

where

$$H_i=H_i(x, t, U),\ F_i=F_i(x, t, U),\ K_i=K_i(x, t, U) \tag{5.11}$$

are generally nonlinear functions of $u_1, \cdots, u_n$. (5.9) is equivalent to the system

$$A\frac{\partial U}{\partial t}+B\frac{\partial U}{\partial x}+C=0 \tag{5.12}$$

where the matrices A and B and the column vector C are defined by

$$A=\text{grad}_U H,\ B=\text{grad}_U F,\ C=K+\left(\frac{\partial H}{\partial t}\right)_{U=\text{const}}+\left(\frac{\partial F}{\partial x}\right)_{U=\text{const}}. \tag{5.13}$$

Let $x_1(t)$ and $x_2(t)$ be two continuously differentiable functions of t satisfying

$$x_1(t) < x_2(t),\ \text{for all } t. \tag{5.14}$$

Integrating (5.9) with respect to x from $x_1(t)$ to $x_2(t)$ and using

$$\frac{\partial}{\partial t}\int_{x_1(t)}^{x_2(t)} H\,dx=\int_{x_1(t)}^{x_2(t)}\frac{\partial H}{\partial t}dx+\dot{x}_2(t)H(x_2)-\dot{x}_1(t)H(x_1) \tag{5.15}$$

we get

$$\frac{\partial}{\partial t}\int_{x_1(t)}^{x_2(t)} H\,dx=\{-F(x_2)+\dot{x}_2(t)H(x_2)\}-\{-F(x_1)+\dot{x}_1(t)H(x_1)\}$$
$$+\int_{x_1(t)}^{x_2(t)}(-K)\,dx \tag{5.16}$$

where

$$F(x_i)=F(x_i, t, U(x_i, t)),\ H(x_i)=H(x_i, t, U(x_i, t)),\ i=1, 2. \tag{5.17}$$

When the end points are fixed, this equation with $\dot{x}_i(t)=0$, expresses the fact that the time rate of change of the total quantities represented by the vector function H contained in the fixed interval (x_1, x_2) is equal to the difference of the flux $F(x_1)$ and the flux $F(x_2)$ through the end points of the interval and the increase due to the source $-K$ distributed in the interior of the interval.

Many physical laws governing the evolution of the state of a system are initially expressed in the integral form (5.16). In general, in physical problems, the quantities H and F depend on the variables U(describing the state of the system) and also on their derivatives. However, we have assumed here that dissipitive mechanisms such as viscosity, heat conduction, radiation etc. are neglegted. In this case H and F are functions of state variables only. When it is assumed that the state variables U are continuously differentiable, the integral form (5.16) can be differentiated with respect to t and we get the system of equations (5.9). A special sub-class of equations (5.9) or (5.16) has played an extremely important role in the theory of generalised or weak solutions. This special class, called *conservation law*, is obtained when the source function K is zero and the quantities H and F do not depend on x and t explicitly. Therefore, a vector conservation law is a system of equations of the form

$$\frac{\partial H}{\partial t}+\frac{\partial F}{\partial x}=0;\ H=H(U),\ F=F(U). \tag{5.18}$$

According to the definition given earlier, the system of conservation laws (5.18) is hyperbolic if the first order system

$$A\frac{\partial U}{\partial t}+B\frac{\partial U}{\partial x}=0 \tag{5.19}$$

where A and B are given by (5.13), is hyperbolic.

§5.2 An Example

Let us now consider a specific single first order equation of the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \tag{5.20}$$

for a scalar function u in order to introduce the various concepts involved in the theory of weak solution. Equation (5.20) immediately gives a conservation law

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(\tfrac{1}{2}u^2)=0. \tag{5.21}$$

Multiplying (5.20) by nu^{n-1}, where we take, for simplicity, n to be a positive integer, we derive an infinite sequence of conservation laws :

$$\frac{\partial u^n}{\partial t}+\frac{\partial}{\partial x}\left(\frac{n}{n+1}u^{n+1}\right)=0. \tag{5.22}$$

We note that $n=1$ gives (5.21) and different value of n give distinct conservation laws. The integral equation corresponding to the conservation law (5.22) is

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}u^n\,dx=\frac{n}{n+1}\,u^{n+1}(x_1)-\frac{n}{n+1}u^{n+1}(x_2) \tag{5.23}$$

where we have taken x_1 and x_2 to be fixed points. We refer to this as integral form of the conservation law.

Though the equation (5.20) ceases to have any meaning for functions which are not differentiable (much less if they are discontinuous), the terms appearing in the integral form (5.23) of the conservation law are defined for a function which is integrable and which need not be continuous in the interval (x_1, x_2). Therefore, we admit as solutions of (5.23) even discontinuous functions with finite jumps of u, u_t and u_x at the points of the interval (x_1, x_2). Such a function is called *generalised solution* of the corresponding conservation law (5.22). Let us try to derive the jump condition across a point of discontinuity $x = X(t)$ which moves with the velocity $S(t) = dX(t)/dt$ and which is the only point of discontinuity in the open interval (x_1, x_2) at a given time t. Now

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} u^n\,dx = \frac{\partial}{\partial t}\left[\int_{x_1}^{X(t)} u^n\,dx + \int_{X(t)}^{x_2} u^n\,dx\right]$$

$$= \frac{dX(t)}{dt}(u_l^n - u_r^n) + \int_{x_1}^{X(t)} nu^{n-1}u_t\,dx + \int_{X(t)}^{x_2} nu^{n-1}u_t\,dx$$

$$= \frac{dX(t)}{dt}(u_l^n - u_r^n) + \int_{x_1}^{x_2} nu^{n-1}u_t\,dx \tag{5.24}$$

where

$$u_l = \lim_{x \to X(t)-0} u(x, t), \quad u_r = \lim_{x \to X(t)+0} u(x, t) \tag{5.25}$$

are the limits of the function u as we approach the point of discontinuity from the left and right respectively. Therefore, at any given time, from (5.23) we have the equality

$$S(t)(u_l^n - u_r^n) + \int_{x_1}^{x_2} nu^{n-1}u_t\,dx = \frac{n}{n+1}[u^{n+1}(x_1) - u^{n+1}(x_2)]. \tag{5.26}$$

Taking $x_1 \to X(t)-0$ and $x_2 \to X(t)+0$ and noting that the integal appearing in (5.26) tends to zero (since its integrand remains bounded) we get in the limit

$$S(t)\cdot[u^n] = \frac{n}{n+1}[u^{n+1}] \tag{5.27}$$

where $[f]$ represents the jump of the quantity f from left to right across the point of discontinuity i.e. $[f] = f_r - f_l$.

The equation (5.27) is a single relation between three quantities u_l, u_r and $S(t)$. Therefore, it determines only one of the three quantities in terms of the other two. If the states u_l and u_r on the two sides of the shock are known, the velocity $S(t)$ of the discontinuity is uniquely determined from

$$S(t) = \frac{n[u^{n+1}]}{(n+1)[u^n]} \tag{5.28}$$

i.e. $$S(t) = \frac{u_l + u_r}{2} \text{ for } n = 1, \quad S(t) = \frac{2}{3}\,\frac{u_l^2 + u_l u_r + u_r^2}{u_l + u_r} \text{ for } n = 2.$$

The velocity $S(t)$ of the discontinuity depends on the value of n, i.e. on the conservation law whose generalised solution is considered. We note that from a single differential equation (5.20) we obtained an infinite number of conservation laws (5.22) with $n = 1, 2, 3, \ldots$. A genuine solution of the equation (5.20) satisfies each of the conservation laws and also the corresponding integral forms so that for a genuine solution all conservation laws and their integral forms are equivalent to the given equation. However, a discontinuous generalised solution of a conservation law is not necessarily a generalised solution of another conservation law derived from the same equation. Therefore, for the class of generalised solutions, these conservation laws are not equivalent. The importance of this remark will be noted when we solve a particular Cauchy problem.

In the domain of generalised solutions we can consider not only a Cauchy problem with C^1 initial data, but with an initial data which could be discontinuous Let us consider a discontinuous initial data of the form

$$u(x, 0) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ 1 & \text{for } 0 < x. \end{cases} \tag{5.29}$$

Two generalised solutions of the conservations law (5.21) are

$$u(x, t) = \begin{cases} 0 & \text{for } 2x \leqslant t \\ 1 & \text{for } t < 2x \end{cases} \tag{5.30}$$

and

$$u(x, t) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ x/t & \text{for } 0 < x \leqslant t \\ 1 & \text{for } t < x \end{cases} \tag{5.31}$$

which satisfy the initial condition (5.29). The solution (5.30) is discontinuous; it satisfies (5.20) everywhere except at the points of the curve of discontinuity : $2x = t$ where the jump condition (5.28) with $n = 1$ is satisfied. The same holds for (5.31), which however has additional property that the solution is continuous but its first derivatives do not exist along the straight lines $x = 0$ and $x = t$. This solution shows that, unlike in the case of linear equations (see section 2.3), a discontinuous Cauchy data could lead to a continuous solution for a quasilinear equation. These are not the only generalised solutions, there exist an infinity of generalised solutions of which (5.30) and (5.31) are particular cases:

$$u(x, t) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ x/t & \text{for} \quad 0 < x \leqslant \alpha t \\ \alpha & \text{for} \quad \alpha t < x \leqslant \frac{1}{2}(1+\alpha)t \\ 1 & \text{for} \quad \frac{1}{2}(1+\alpha)t < x \end{cases} \tag{5.32}$$

where α is a constant and satisfies

$$0 \leqslant \alpha \leqslant 1 \tag{5.33}$$

and the jump condition (5.28) with $n = 1$ is satisfied along $x = \frac{1}{2}(1+\alpha)t$. For $\alpha = 0$ we get the solution (5.30) and for $\alpha = 1$ we get (5.31). The characteristics and the lines of discontinuity for these solutions have been drawn in the Fig. 5.2.

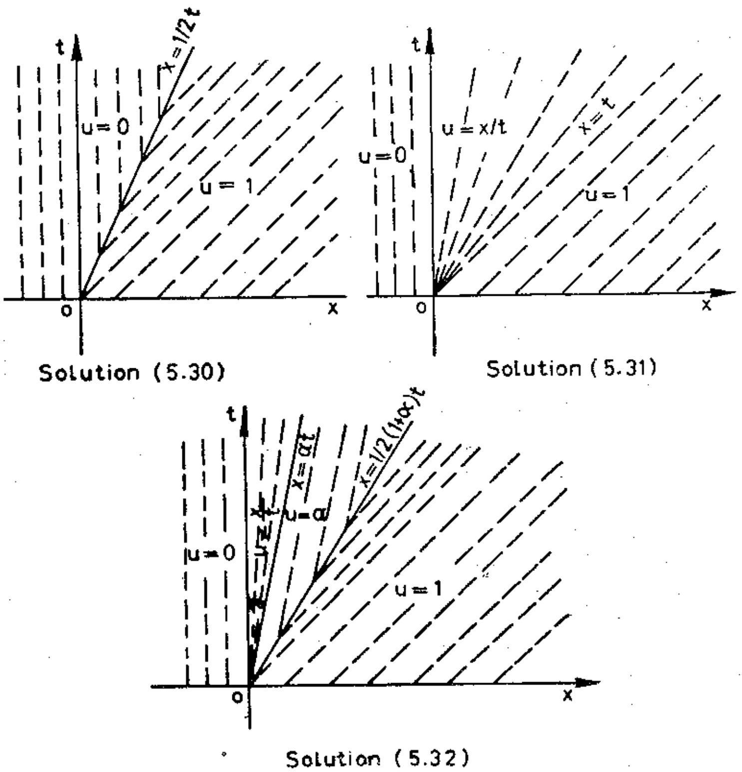

Fig. 5.2 The characteristic curves have been shown by dotted lines

Instead of the Cauchy data (5.29) if we take

$$u(x, 0) = \begin{cases} 1 & \text{for} \quad x \leqslant 0 \\ 0 & \text{for} \quad 0 < x \end{cases} \tag{5.34}$$

we shall show later that with this initial condition, the conservation law (5.21) has a unique generalised solution

$$u(x, t) = \begin{cases} 1 & \text{for} \quad 2x \leqslant t \\ 0 & \text{for} \quad t < 2x. \end{cases} \tag{5.35}$$

The solution (5.35) has been shown in Fig. 5.3.

The above example shows that, in general, the generalised solution of a Cauchy problem for a conservation law is not unique. What is needed now is a mathematical principle characterising a class of *permissible solutions* in which every Cauchy problem has a unique solution. This will be taken up later. However, we can deduce such a principle from the following consideration. We have noted that a discontinuity appears whenever characteristics of the equation (5.20) start intersecting. Therefore, a discontinuity is permissible only if it prevents the intersection of the characteristics coming from the points of the initial line on the two sides of it. If we accept this principle, Figs. 5.2 and 5.3 show that among the solutions given here (5.31) is the only permissible solution with Cauchy data (5.29) and that the unique weak solution (5.35) with the Cauchy data (5.34) is also permissible.

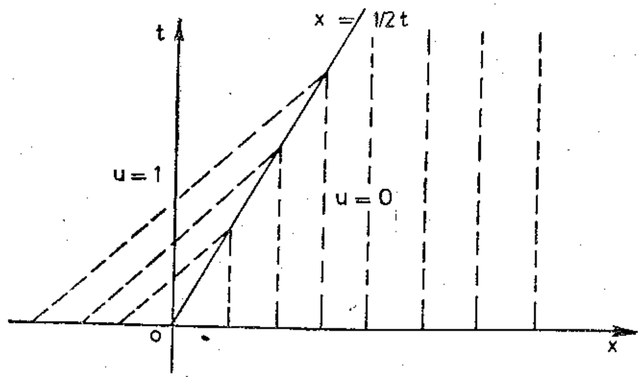

Fig. 5.3 Characteristic curves of the solution (5.35) have been shown by dotted lines.

Let us now examine the remark made earlier that different conservation laws derived from (5.20) are not equivalent so far as the generalised solutions are concerned. The solution of the Cauchy problem for the conservation law (5.22) for $n=2$ and with the initial data (5.34) is

$$u(x, t)=\begin{cases} 1 & \text{for} \quad 3x \leqslant 2t \\ 0 & \text{for} \quad 2t < 3x \end{cases} \tag{5.36}$$

where the point of discontinuity moves with the velocity 2/3 as given by (5.28). The solution (5.36) is evidently different from the solution (5.35) of the conservation law (5.21) with the same Cauchy data.

§5.3 Definition of Weak Solution

The moving discontinuities in the solutions of quasilinear equations were first introduced as shock waves in the theory of waves of finite amplitude in gasdynamics. This was done by exactly the same procedure as discussed in the last section while generalising the meaning of a solution. The use of the integral form (5.16) of the equations in the derivation of the jump condition is related to the basic fact that a physical law is generally stated in the form (5.16) which, therefore, is fundamental and (5.9) can be deduced from it. Our aim in this section is to give a more elegant definition of the generalised solution.

As mentioned earlier, we consider only those quasilinear equations, which can be put in the divergence form (5.10), i.e.

$$\frac{\partial H}{\partial t}+\frac{\partial F}{\partial x}+K=0. \tag{5.37}$$

Let the Cauchy data on $t=0$ be given by

$$U(x, 0)=\varphi(x). \tag{5.38}$$

Let V belong to a class of vector functions of x and t, called *test functions*, such that $V \in C^{\infty}$ and it vanishes outside a closed bounded domain in the (x, t) plane. Consider now a genuine solution $U(x, t)$ of the differential equations (5.37). Pre-multiplying the left hand side of (5.37) by the test function V and then integrating it over the region R $(t \geqslant 0, -\infty < x < \infty)$, which is the upper half of the (x, t)-plane including the x-axis, we get

$$\iint_{R^2_+} V\left(\frac{\partial H}{\partial t}+\frac{\partial F}{\partial x}+K\right) dx\, dt=\iint_{R^2_+}\left(-\frac{\partial V}{\partial t}H-\frac{\partial V}{\partial x}F+VK\right) dx\, dt$$
$$+\iint_{R^2_+}\left\{\frac{\partial}{\partial t}(VH)+\frac{\partial}{\partial x}(VF)\right\} dx\, dt. \tag{5.39}$$

Using the Gauss divergence theorem for the second integral on the right hand side and noting that V is zero on the boundary of R^2_+ (actually it is of compact support) except possibly on the portion where the boundary is the x-axis, we get

$$\iint_{R^2_+} V\left(\frac{\partial H}{\partial t}+\frac{\partial F}{\partial x}+K\right) dx\, dt=\iint_{R^2_+}\left(-\frac{\partial V}{\partial t}H-\frac{\partial V}{\partial x}F+VK\right) dx\, dt$$
$$-\int_{-\infty}^{\infty} V(x, 0)H(x, 0, U(x, 0))\, dx. \tag{5.40}$$

When U is a genuine solution of (5.37) and (5.38), the identity (5.40) gives

$$\iint_{R^2_+}\left\{-\frac{\partial V}{\partial t}H-\frac{\partial V}{\partial x}F+VK\right\} dx\, dt$$
$$=\int_{-\infty}^{\infty} V(x, 0)H(x, 0, \varphi(x))\, dx \tag{5.41}$$

for every admissible test function V. However, since in (5.41) only integrals of functions of U (and not its derivatives) appear, it can be satisfied by a larger class of functions U than the class of genuine solutions of (5.37) and (5.38).

Definition: We define, a *weak solution* of the Cauchy problem (5.37) and (5.38) to be a measurable vector function $U(x, t)$ on R^2_+ which satisfies (4.41) for every admissible test function V.

Let $\bar{V}$ be an admissible test function such that the intersection of the support of $\bar{V}$ with the x-axis is null. Then $\bar{V}(x, 0)=0$ for all x and (5.41) reduces to

$$\iint_{R_+^2} \left(-\frac{\partial \bar{V}}{\partial t}H-\frac{\partial \bar{V}}{\partial x}F+\bar{V}K\right) dx\, dt=0 \tag{5.42}$$

for all $\bar{V}$.

Theorem 5.1 A weak solution $U(x, t)$ of (5.37) and (5.38), which has continuous partial derivatives, is a genuine solution of the problem.

Proof: To prove this theorem we proceed in two steps. The first step consists of showing that $U(x, t)$ satisfies the differential equations (5.37) and for this we shall consider only the test functions of the type $\bar{V}$ so that we use the result (5.42) which is independent of the Cauchy data $\varphi(x)$. The second step is to show that $U(x, t)$ satisfies the initial condition (5.38), and for this we shall consider the general class of test functions and use the relation (5.41) involving the initial condition.

For a weak solution which is continuously differentiable in R_+^2 the identity (5.40) with V replaced by $\bar{V}$ and the equation (5.42) give

$$\iint_{R_+^2} \bar{V}\left(\frac{\partial H}{\partial t}+\frac{\partial F}{\partial x}+K\right) dx\, dt=0. \tag{5.43}$$

Since $\bar{V}$ is arbitrary with compact support in $t>0$, $-\infty<x<\infty$, from the fundamental Lemma of the calculus of variations we find that U satisfies the differential equations (5.37).

Next we consider general test functions for which $V(x, 0)$ need not be identically zero. U as a generalised solution satisfies (5.41). Further, we have just shown that if $U \in C^1(R_+^2)$ it also satisfies (5.37). Substracting (5.41) from (5.40) and using (5.37) we get

$$\int_{-\infty}^{\infty} V(x, 0)\{H(x, 0, U(x, 0))-H(x, 0, \varphi(x))\}\, dx=0$$

where $V(x, 0)$ can be arbitrarily chosen. This shows that $H(x, 0, U(x, 0)) = H(x, 0, \varphi(x))$ which under the condition that $A=\text{grad}_U H$ is nonsingular (see equation (2.8)), implies that $U(x, 0)=\varphi(x)$. Thus a continuously differentiable weak solution also satisfies the initial condition. The theorem is now proved.

In this and the subsequent sections, we shall discuss only those weak solutions $U(x, t)$, which along with their first derivatives are piecewise continuous, i.e. for which U, U_t, U_x possess jump discontinuities along piecewise smooth curves. Let $U(x, t)$ be such a discontinuous solution which is regular in all sub-domains of R_+^2 not containing a curve $C: x=X(t)$. We shall derive the jump-conditions across the curve of discontinuity C. Let C divide the domain D in the upper half of the (x, t)-plane into two subdomains D_1 and D_2 and, as before, let $\bar{V}$ be a test function with compact support entirely within D. Since in D_α ($\alpha = 1, 2$), the function $U(x, t)$ is a genuine solution, it satisfies (5.37). Therefore, from the identity

$$\iint_{D_\alpha} \bar{V}\left(\frac{\partial H}{\partial t} + \frac{\partial F}{\partial x} + K\right) dx\, dt = \iint_{D_\alpha} \left(-\frac{\partial \bar{V}}{\partial t} H - \frac{\partial \bar{V}}{\partial x} F + \bar{V} K\right) dx\, dt$$

$$+ \iint_{D_\alpha} \left\{\frac{\partial}{\partial t}(\bar{V} H) + \frac{\partial}{\partial x}(\bar{V} F)\right\} dx\, dt,\ \alpha = 1, 2 \qquad (5.44)$$

we get

$$-\iint_{D_\alpha} \left(-\frac{\partial \bar{V}}{\partial t} H - \frac{\partial \bar{V}}{\partial x} F + \bar{V}\right) dx\, dt = \iint_{D_\alpha} \left\{\frac{\partial}{\partial t}(\bar{V} H) + \frac{\partial}{\partial x}(\bar{V} F)\right\} dx\, dt,\ \alpha = 1, 2.$$

$$(5.45)$$

Converting the integral on the right hand side by the Gauss divergence theorem into a curvilinear integral on the boundary ∂D_α and noting that $\bar{V} = 0$ on ∂D_α except at most on a part of it formed by C, we obtain

$$-\iint_{D_1} \left(-\frac{\partial \bar{V}}{\partial x} H - \frac{\partial \bar{V}}{\partial x} F + \bar{V} K\right) dx\, dt = \int_C \bar{V}\{n_t(H)_l + n_x(F)_l\}\, ds \qquad (5.46)$$

and

$$-\iint_{D_2} \left(-\frac{\partial \bar{V}}{\partial t} H - \frac{\partial \bar{V}}{\partial x} F + \bar{V} K\right) dx\, dt = -\int_C \bar{V}\{n_t(H)_r + n_x(H)_r\}\, ds \quad (5.47)$$

where (n_x, n_t) is the unit normal drawn from side 1 to 2 and the suffixes l and r represent the limiting values of the quantities as we approach the

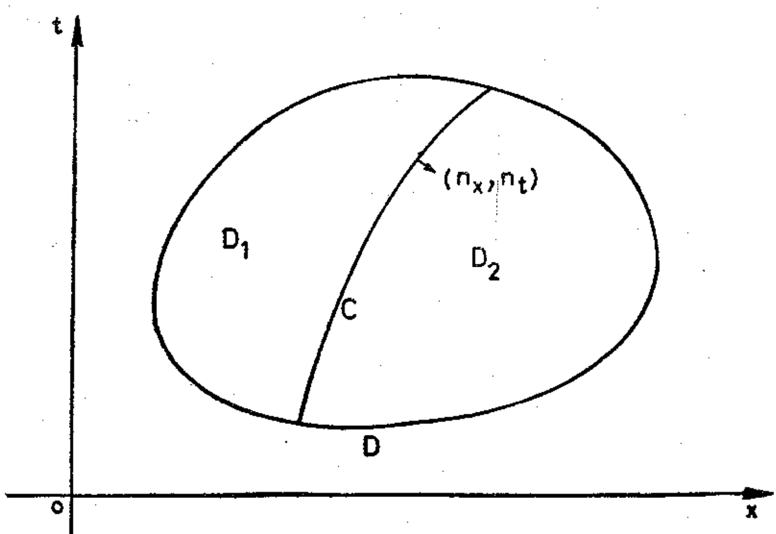

Fig. 5.4 Domain $D = D_1 \cup D_2 \cdot C$ is the curve across which the weak solution suffers jump

curve C from regions D_1 and D_2 respectively. Adding (5.46) and (5.47), we get

$$-\iint_D \left(-\frac{\partial \bar{V}}{\partial x}H - \frac{\partial \bar{V}}{\partial x}F + \bar{V}K\right) dx\, dt = -\int_C \bar{V}\{n_t[H] + n_x[F]\}\, ds \quad (5.48)$$

where $[f]$ is the jump of a quantity f from left to right* (see, for example, the definition (2.42)). The left hand side is zero, since U is a weak solution and the support of $\bar{V}$ is contained in D. Therefore, the line integral on right hand side of (5.48) identically vanishes for all such test functions $\bar{V}$. From the fundamental theorem of the calculus of variations, we deduce that

$$n_t[H] + n_x[F] = 0. \quad (5.49)$$

For a physical interpretation of (5.49), we note that $-n_t/n_x$ represents the rate of change of x with respect to t along the curve of discontinuity and hence represents the velocity of propagation of the discontinuity in x-direction so that

$$-\frac{n_t}{n_x} = S(t) = \frac{dX(t)}{dt}. \quad (5.50)$$

Then (5.49) gives the jump condition in the form

$$[H]S(t) - [F] = 0. \quad (5.51)$$

Definition: The condition (5.49) or (5.51) which joins the state u_l on the left of a discontinuity to the state u_r on its right is called a *Rankine-Hugoniot* condition.

We note that the *Rankine-Hugoniot* condition (5.51) represents n relations between $2n+1$ quantities:

n components u_{1l}, u_{2l}, $\ldots$, u_{nl} of u_l

n components u_{1r}, u_{2r}, $\ldots$, u_{nr} of u_r

and

S the velocity of the discontinuity.

Therefore, if U_l is kept fixed, the states U_r form a one-parameter family of states given by

$$U_r = U_{sh}(\epsilon), \qquad U_{sh}(0) = U_l \quad (5.52)$$

where U_{sh} is a vector function of a single variable and ϵ is a measure of the amplitude of the discontinuity. The velocity of the point of discontinuity is also a function of the parameter ϵ, i.e.

$$S = S(\epsilon). \quad (5.53)$$

An immediate consequence of (5.49) is the following theorem.

*With respect to the reader (see footnote on page 115).

Theorem 5.2: For a linear system of differential equations the curve of discontinuity C is a characteristic curve.
Proof: In this case, A and B given by (5.13) are functions of x and t only. Consequently H and F are of the form

$$H = AU, \ F = BU. \tag{5.54}$$

From (5.51) and (5.54) we have

$$(B - SA)[U] = 0. \tag{5.55}$$

For a nontrivial solution for $[U]$, S satisfies the characteristic equation $\det(B - SA) = 0$, showing that S equals a characteristic velocity c. The discontinuity curve, is therefore, a characteristic curve.

For a system of quasilinear equations, the jump velocity $S(t)$ is, in general, different from a characteristic velocity. However, for sufficiently weak discontinuities, i.e. for discontinuous functions for which the parameter ϵ (introduced in (5.52)) tends to zero, we shall show that the shock velocity $S(\epsilon)$ tends to a characteristic velocity.

Using the notation introduced in (5.52) we write the jump condition (5.51) in the form

$$S(\epsilon)(H_{sh}(\epsilon) - H(0)) = F_{sh}(\epsilon) - F(0) \tag{5.56}$$

where

$$H_{sh}(\epsilon) \equiv H(U_{sh}(\epsilon)), \ F_{sh}(\epsilon) \equiv F(U_{sh}(\epsilon)).$$

Differentiating (5.56) with respect to ϵ, using $\text{grad}_U H = A$ and $\text{grad}_U F = B$, and letting $\epsilon \to 0$ we get

$$S(0)A_{sh}(0)\dot{U}_{sh}(0) = B_{sh}(0)\ \dot{U}_{sh}(0)$$

where a dot denotes differentiation with respect to ϵ. We note that $A_{sh}(0) = A(U_l)$, $B_{sh}(0) = B(U_l)$ and write the last result as

$$\{B(U_l) - S(0)A(U_l)\}\ \dot{U}_{sh}(0) = 0. \tag{5.57}$$

We can always choose the parameter ϵ to be such that $\dot{U}_{sh}(0)$ is not a null vector. Therefore, it follows from (5.57) that $S(0)$ must be equal to a characteristic velocity, i.e.

$$S(0) = c(U_l) \tag{5.57(a)}$$

which proves the assertion that as $\epsilon \to 0$, $S(\epsilon)$ tends to a characteristic velocity. We also note that the derivative $\dot{U}_{sh}(0)$ must be a scalar multiple of the corsesponding right eigenvector, i.e.

$$\dot{U}_{sh}(0) = \alpha r(U_l) \tag{5.57(b)}$$

where α is a scalar constant.

*§5.4 The Entropy Condition and a Shock

From the example in section 5.2 it is clear that a weak solution of a Cauchy problem with admissible Cauchy data is generally not unique. Instead of going into the question of uniqueness of a general Cauchy problem, we note that the reason for the nonuniqueness of the solution in the example discussed in section 5.2 is the existence of discontinuous solutions along with a continuous solution with the same Cauchy data. Therefore, it appears that criterion for selecting a physically relevant weak solution is closely associated with the search for a criterion for an admissible discontinuous solution. A mathematical criterion for admissible discontinuities can be derived from the following stability consideration.

A discontinuity is permissible if when small amplitude waves are incident upon the discontinuity, the resulting perturbations in the velocity of the discontinuity and in the resulting waves moving away from the discontinuity are uniquely determined and remain small.

An admissible discontinuity satisfying the stability condition is called a *shock*. From the stability criterion, we can deduce the following simple and easily verifiable condition for a discontinuity to be a shock (Gelfand, 1962).

Let us consider a piecewise smooth weak solution $U(x, t)$ of hyperbolic conservation law (5.18) with distinct eigenvalues $c_1, c_2, \cdots, c_n$ satisfying the condition

$$c_1(U) < c_2(U) < \ldots < c_n(U) \tag{5.58}$$

for all values U for which the characteristic velocities are defined. Let P be a point on a line of discountinuity and let as before, U_l and U_r be the limiting values of U as we approach P from left and right. Let us first assume that the discontinuity curve itself is not a characteristic curve. Let us draw those characteristic curves which start from the point P and go to the left side of the discontinuity curve as t decreases. Similarly, we draw those characteristic curves which start from the point P and go to the right side of the discontinuity curve as t decreases. The characteristics of this type that are on the left are those for which

$$c_i(U_l) > S(t) \tag{5.59}$$

and that are on the right are those for which

$$c_j(U_r) < S(t). \tag{5.60}$$

The stability criterion for the shock leads to the following result which we state without proof (see Gelfand (1962)).

Theorem: A discontinuity is a shock if the total number of characteristic curves converging at P from both sides is $n+1$.

We briefly mention here that the $n+1$ characteristics which reach P from both sides carry $n+1$ pieces of information from the state of the system at a smaller value of t and these pieces of information together with the n Rankine-Hugoniot relations (5.51) uniquely determine the $2n+1$ quantities: the n components of U_l, the n components of U_r and $S(t)$.

Analytically, we can express the requirement of the above theorem for a discontinuity to be a shock, as follows:

There exists an index k, $1 < k < n$, for which the inequalities

$$c_{k-1}(U_l) < S < c_k(U_l) \tag{5.61a}$$

$$c_k(U_r) < S < c_{k+1}(U_r) \tag{5.61b}$$

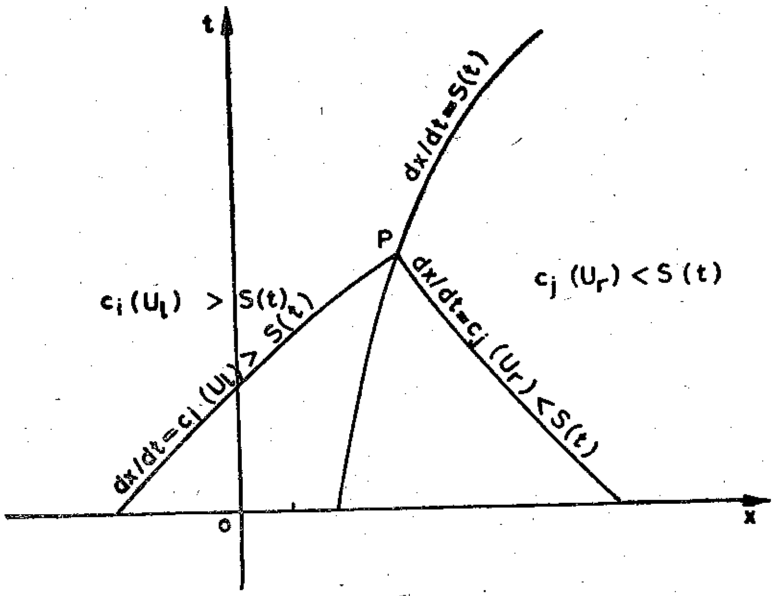

Fig. 5.5 The characteristics which converge at P from both sides

should be satisfied. We note that for $k=1$ the inequality (5.61a) and for $k=n$, (5.61b) become meaningless. In these cases (5.61) are replaced by

$$S < c_1(U_l),\ c_1(U_r) < S < c_2(U_r) \text{ for } k=1 \tag{5.62}$$

and

$$c_{n-1}(U_l) < S < c_n(U_l),\ c_n(U_r) < S \text{ for } k=n. \tag{5.63}$$

Definition: A criterion which ensures that a discontinuity is a shock is called an *entropy condition.*

The term entropy condition comes from the fact that this condition, in the form of the second law of thermodynamics, was first encountered in gasdynamics where an ambiguity arose. The Rankine-Hugoniot conditions allow both compression and expansion discontinuities. Rayleigh first pointed out in 1910 that the physical principle which says that the entropy per unit mass of the gas must increase as the fluid particles cross the discontinuity surface, allow only a compression discontinuity called shock.

For each value of k, the entropy condition mentioned above gives a shock called a *shock of the kth characteristic family* or simply *kth shock*. Thus we see that there are n different kinds of shocks corresponding to the different values of k from 1 to n. For the kth shock the inequalities in (5.61)-(5.63) imply that $c_k(U_r) < S < c_k(U_l)$. Therefore the kth shock, when it appears, prevents the intersection of the characteristics of the kth family coming from an initial data. On the contrary, if for any k, the characteristic curves on the right and those on the left diverge from the curve of discontinuity, then the discontinuity is not admissible.

It may happen that the curve of discontinuity may coincide with a characteristic curve on one or both sides (but belonging to the same family, say kth) of the discontinuity curve. Such discontinuities are also admissible and are called *contact discontinuities.* We mention here that a contact discontinuity appears in a characteristic field which is not genuinely nonlinear (Lax, 1957).

*§5.5 Application of the Theory to Gasdynamics

Consider the one dimensional inviscid, nonconducting motion of a polytropic gas (see §3.4 but here the motion is not assumed to be isentropic). The differential equations of motion are the three equations, namely equations of continuity, momentum and energy and they are derived from the integral form of the three conservation laws representing the conservation of mass, momentum and energy. However, from the above three partial differential equations, we can determine another partial differential equation which can be put in conservation form

$$\frac{\partial}{\partial t}(\rho\sigma)+\frac{\partial}{\partial x}(\rho q\sigma)=0 \tag{5.64}$$

where ρ is the mass density, σ the specific entropy and q the velocity of the gas. Equation (5.64) represents the conservation of entropy. However, we have seen in section 5.2 that different conservation laws are not equivalent for weak solutions. From continuum mechanics we know that the conservation of mass, momentum and energy are fundamental laws and the entropy may be permitted to increase according to the second law of thermodynamics. The law of conservation of entropy holds for continuous flows without dissipation but does not apply to discontinuous flows.

Therefore we start with the following form of the equations of gas dynamics

$$\frac{\partial H}{\partial t}+\frac{\partial F}{\partial x}=0 \tag{5.65}$$

where

$$H=\begin{bmatrix} \rho \\ \rho q \\ \rho\left(e+\frac{1}{2}q^2\right) \end{bmatrix}, \quad F=\begin{bmatrix} \rho q \\ \rho q^2+p \\ \rho q\left(e+\frac{p}{\rho}+\frac{1}{2}q^2\right) \end{bmatrix}. \tag{5.66}$$

Here p is the pressure and e is the internal energy per unit mass, i.e. the specific internal energy given by

$$e=p/\{(\gamma-1)\rho\}.$$

For a polytropic gas ρ, p and σ are related by

$$p=A(\sigma)\rho^{\gamma} \tag{5.67}$$

where A is a function of σ only and γ is a constant. Further, the local velocity of sound, a, is given by

$$a^2 = \frac{\gamma p}{\rho}. \tag{5.68}$$

The characteristic velocities are

$$c_1 = q - a,\ c_2 = q,\ c_3 = q + a. \tag{5.69}$$

The Rankine-Hugoniot conditions (5.51) give

$$\rho_l(q_l - S) = \rho_r(q_r - S) \tag{5.70}$$

$$q_l(q_l - S) + p_l = \rho_r q_r(q_r - S) + p_r \tag{5.71}$$

and

$$\rho_l(e_l + \tfrac{1}{2}q_l^2)\,(q_l - S) + q_l p_l = \rho_r(e_r + \tfrac{1}{2}q_r^2)\,(q_r - S) + q_r p_r. \tag{5.72}$$

A possible solution of the Rankine-Hugoniot conditions is the degenerate solution

$$q_l = S = q_r,\ p_l = p_r,\ \rho_l = \text{arbitrary},\ \rho_r = \text{arbitrary}. \tag{5.73}$$

In this case we find that the relations

$$c_1(U_l) < S = c_2(U_l),\ c_2(U_r) = S < c_3\,(U_r) \tag{5.74}$$

are satisfied. Therefore, we have a contact discontinuity for which the characteristic curves of the second family on both the sides of the shock merge with the curve of discontinuity. The contact surface moves with the gas particles and separates two zones of different density. The pressure and the fluid velocity are continuous across the contact discontinuity. We note that the second family of characteristics is not genuinely nonlinear.

Assuming that $S \neq q_l$ which is equivalent to $S \neq q_r$ since $\rho \neq 0$, we can show that (for details, see Courant and Friedrichs (1948))

$$(q_l - S)\,(q_r - S) = a^{*2} \tag{5.75}$$

where

$$a^{*2} = \frac{\gamma - 1}{\gamma + 1}(q_l - S)^2 + \frac{2}{\gamma + 1}a_l^2 = \frac{\gamma - 1}{\gamma + 1}\,(q_r - S)^2 + \frac{2}{\gamma + 1}a_r^2. \tag{5.76}$$

a^* represents the common critical (or sonic) velocity on both sides of the shock. We can further prove that

$$\left.\begin{aligned} &|\ q_i - S\ | > a^* \text{ implies } a_i < a^* \\ \text{and}\quad &|\ q_i - S\ | < a^* \text{ implies } a_i > a^* \end{aligned}\right\} \tag{5.77}$$

where the suffix i stands for both l and r. Now we get the following types of discontinuities:

(i) $S - q_l < -a^*$ (5.78)

From (5.75) we get $-a^* < S - q_r < 0$. Inequality (5.77) implies

$$S - q_l < -a_l \text{ and } -a_r < S - q_r < 0. \tag{5.79}$$

The entropy condition is satisfied in the form

$$S < c_1(U_l),\ c_1(U_r) < S < c_2(U_r). \tag{5.80}$$

Since $S < q_l, S < q_r$, i.e. the discontinuity is a backward facing shock which is crossed by the fluid particles from left to right. The relative position of the first family of characteristics and the shock path are shown in Fig. 5.6. We can show that in this case specific entropy (entropy per unit mass) σ and pressure p increase when the fluid particles cross the shock (see Courant and Friedrichs, 1948)).

(ii) $-a^* < S - q_l < 0.$ (5.81)

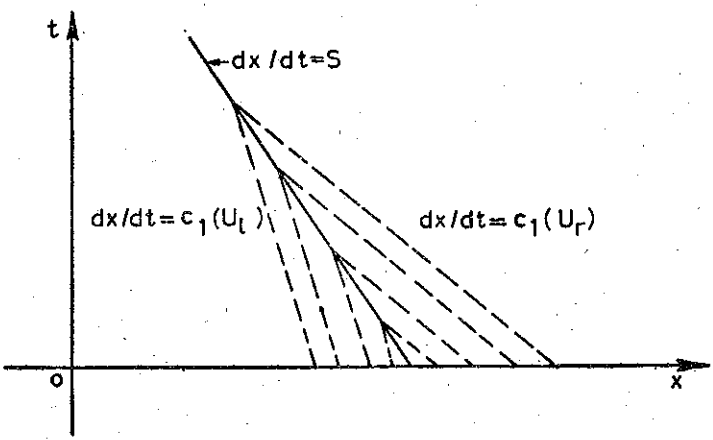

Fig. 5.6 Relative position of the characteristics of the first family with backward facing shock. The characteristic curves and shock have been shown by dotted lines and continuous line respectively

In this case, we have the following results

$$S - q_r < -a^*, -a_l < S - q_l < 0, S - q_r < -a_r. \tag{5.82}$$

Since $c_1(U_l) < S < c_1(U_r)$, the entropy condition necessary for the stability of the discontinuity is no longer satisfied and consequently it is not an admissible discontinuity. Here the pressure and entropy decrease as the fluid particles cross it from left to right.

(iii) $a^* < S - q_l$ (5.83)

which implies

$$0 < S - q_r < a^*, a_l < S - q_l, 0 < S - q_r < a_r. \tag{5.84}$$

Again the entropy condition (5.61)-(5.63) is not satisfied for any value of k. The discontinuity surface, which is now a forward facing wave and is crossed by the fluid particles from right to left, is not an admissible discontinuity. We can show that both the pressure and specific entropy of a fluid element decrease after passing through the discontinuity. As in the case of Fig. 5.7, the two characteristics (of the third family) starting from an arbitrary point of the discontinuity curve diverge away on the two sides showing that discontinuity is not admissible.

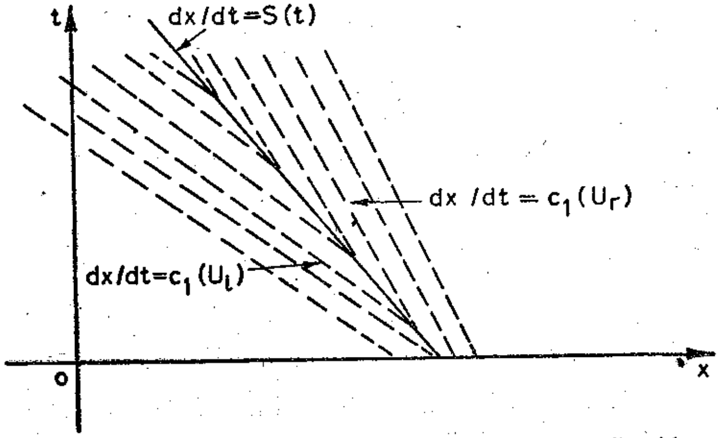

Fig. 5.7 Relative position of the characteristics of the first family with respect to the curve of discontinuity in case (ii)

(iv) $0 < S - q_l < a^*$ (5.85)

In this case we get

$$a^* < S - q_r,\ 0 < S - q_l < a_l,\ a_r < S - q_r. \tag{5.86}$$

The entropy condition

$$c_2(U_l) < S < c_3(U_l),\ c_3(U_r) < S \tag{5.87}$$

is satisfied so that the discontinuity surface is a shock of the third characteristic family. This is a forward facing shock which is crossed by the fluid particles from right to left. We can show that the pressure and the specific entropy increase in the medium after the shock transition. If we draw the characteristics of the third family on two sides, we note that the shock prevents intersection of the curves (of this family) coming from the two sides as in the Fig. 5.6.

Thus out of the four cases, only two cases, namely (i) and (iv) represent shocks. In both these cases, we note that the flow in front of the shock is supersonic relative to the shock and the flow behind it is subsonic relative to it. Finally, we surmise that the only admissible discontinuities in one dimensional gas dynamics are

Backward facing shock: $S < c_1(U_l),\ c_1(U_r) < S < c_2(U_r)$

Contact discontinuity : $c_1(U_l) < c_2(U_l) = S = c_2(U_r) < c_3(U_r)$

and

Forward facing shock : $c_2(U_l) < S < c_3(U_l),\ c_3(U_r) < S.$

EXERCISE 5.1

*1. Find the solution of the following initial-boundary value problem for the equations (5.)

$$q(x, 0) = 0,\ \rho(x, 0) = \rho_0,\ p(x, 0) = p_0,\ x > 0$$

and $\qquad q(Ut, t) = \text{constant} = U,\ t > 0$

at the points in the domain $x > Ut,\ t > 0$ of the (x, t)-plane.

*2. Prove the following statement for a forward facing shock discussed in the section 5.5:

The jumps $[p] = p_r - p_l$ and $[\rho] = \rho_r - \rho_l$ in pressure and density respectively are greater than zero.

*3. Show that the shock velocity $S(\epsilon)$, as a function of the parameter ϵ, satisfies $\dot{S}(0) = \frac{1}{2}\dot{c}_{sh}(0)$, where the dot denotes differentiation, and $c_{sh}(\epsilon) = c(U_{sh}(\epsilon))$. Using (5.57a), show that

$$S(\epsilon) = \tfrac{1}{2}\{c(U_l) + c(U_r)\} + 0(\epsilon^2)$$

i.e. for a weak shock of the kth family, the shock speed $S(\epsilon)$ is the arithmetic mean of the kth characteristic velocities in the regions on the two sides of the shock.

*4. Verify the assertion of problem number 3 5.5 for the gasdynamics shocks.

5. Find the solution of the following initial value problems

$$u_t + (u^2/2)_x = 0,\ t > 0,\ -\infty < x < \infty$$

$$\text{(i)}\quad u(x, 0) = \begin{cases} 0 & \text{for} \quad x < -1 \\ 1 & \text{for} \quad -1 < x < 1 \\ 0 & \text{for} \quad x > 1 \end{cases}$$

$$\text{(ii)}\quad u(x, 0) = \begin{cases} 2 & \text{for} \quad x < -1 \\ 1 & \text{for} \quad -1 < x < 1 \\ 0 & \text{for} \quad x > 1. \end{cases}$$

6. In the case of *a* linear system of equations

$$LU \equiv A(x, t)\frac{\partial U}{\partial t} + B(x, t)\frac{\partial U}{\partial x} + D(x, t)U = 0$$

show that the weak solution satisfies the equation

$$\iint_{R_+^2} (L^*V)\, dx\, dt = \int_{-\infty}^{\infty} V(x, 0)A(x, 0)\varphi(x)\, dx$$

where L^* is the the adjoint operator and other quantities have the same meaning as those in section 5.3.

*7. For a hyperbolic conservation law, define Riemann invariants of the kth characteristic field as in the problem of exercise 4.1. Show that for a weak shock, the change in a Riemann invariant of the kth field across a kth shock is at most of the third order in ϵ [Lax (1957)].

*8. Consider a system of conservation laws and two one-parameter family of states $U_{sim}(\delta)$ and $U_{sh}(\epsilon)$ which can be joined to a state U_l on the left respectively by simple waves and shock waves of the kth characteristic family (see (4.9) and (5.52)). Prove that for small values of δ and ϵ, the transition through the shock agrees with those through the simple wave up to the second power in the shock strength.

*§5.6 Riemann Initial Value Problem: Solution with the Help of Shocks and Simple Waves

We shall discuss this problem very briefly assuming that all the characteristic fields are genuinely nonlinear.

Any constant state U_l on the left can be joined to a one-parameter family of states $U_{sim}(\delta)$ (see (4.9)) on the right by a kth *centered* simple wave, where δ, when defined by (4.12), can take only positive values due to the inequality (4.11). The states $U_{sim}(\delta)$ depend not only on the amplitude δ of the simple wave but also on the state U_l. In fact, in this section we shall denote these states by the symbol $U_{sim}(U_l, \delta)$. Similarly, the constant state U_l can be joined by a kth shock to another one-parameter family of states $U_{sh}(\epsilon)$ (see (5.52)) on the right, where the parameter ϵ, also defined by $\epsilon = c_k(U_r) - c_k(U_l)$, can take only negative values due to the entropy condition (5.57)-(5.58). These states also depend on U_l and in this section we shall denote them by the symbol $U_{sh}(U_l, \epsilon)$. The two families $U_{sim}(U_l, \delta)$ and $U_{sh}(U_l, \epsilon)$ are non-overlapping, i.e. the states $U_{sim}(U_l, \delta)$ cannot be joined to the state U_l on the left by a shock and the states $U_{sh}(U_l, \epsilon)$ cannot be joined to U_l by a *centered* simple wave. Since there is a lot of freedom in the choice of the parameters δ and ϵ (i.e. we can replace δ by any monotomically increasing function of itself), we can suitably choose these parameters so that the following theorem is true.

Theorem 2.3 Given a constant state U_l on the left, it can be connected to a one-parameter family of states $U_r = \bar{U}(U_l, \epsilon)$, $-\epsilon_0 < \epsilon < \epsilon_0$ $(\epsilon_0 > 0)$ either through a shock or a centered simple wave of the kth characteristic field. We can choose the parameter ϵ such that a value of ϵ in the interval $(-\epsilon_0, 0)$ corresponds to a transition through a shock and that in $(0, \epsilon_0)$ to a transition through a centered simple wave. The function $\bar{U}(U_l, \epsilon)$ is twice continuously able with respect to ϵ.

For the proof of the theorem, refer to Lax (1957).

Now, we are in a position to solve one of the very basic problems of nonlinear hyperbolic equations, namely the *Riemann problem*. This consists of finding a solution of the system of conservation laws (5.18) satisfying the different initial condition

$$U(x, 0) = \begin{cases} U_0 \text{ for } x < 0 \\ U_n \text{ for } x > 0 \end{cases} \tag{5.88}$$

where U_0 and U_n are constant vectors.

If $U(x, t,)$ is a weak solution with initial value (5.88), then $U(\alpha x, \alpha t)$, where α is any constant > 0, is also a weak solution of the same problem. Assuming that the solution is unique, we find $U(\alpha x, \alpha t) = U(x, t)$ for all $\alpha > 0$. This is possible if and only if $U(x, t)$ is a function of x/t.

This solution consists of $n+1$ constant states $U_0, U_1, \ldots, U_{n-1}, U_n$; the state U_{k-1} on the left is joined to the state U_k on the right by a kth shock or a kth centered simple wave. The two end states U_0 and U_n are given and the $n-1$ intermediate constant states are found as follows.

According to the therem 5.3, U_1 is to be selected from one-parameter family of states $\bar{U}(U_0, \epsilon_1) \equiv U^{(1)}(U_0, \epsilon_1)$ say, U_2 from two-parameters family of states $\bar{U}(U^{(1)}(U_0, \epsilon_1), \epsilon_2) \equiv U^{(2)}(U_0, \epsilon_1, \epsilon_2)$, ..., and U_n from the n-parameter family of states

$$\bar{U}(U^{(n-1)}(U_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}), \epsilon_n) \equiv U^{(n)}(U_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_n) \text{ say.}$$

Therefore for some values of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ we have

$$U_n = U^{(n)}(U_0, \epsilon_1, \ldots, \epsilon_n). \tag{5.89}$$

Further

$$U_0 = U^{(n)}(U_0, 0, \ldots, 0). \tag{5.90}$$

The derivative $\left(\dfrac{\partial U^{(n)}}{\partial \epsilon_k}\right)$ at $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = (0, 0, \ldots 0)$ is parallel to the kth right eigenvector (for shock it follows from (5.57b) and for simple wave it follows from (4.10)). Therefore, the Jacobian of the transformation (5.89) from $(\epsilon_1, \epsilon_2, \ldots \epsilon_n)$ to U_n is not zero at origin in $(\epsilon_1, \epsilon_2,, \ldots, \epsilon_n)$-space. By the implicit function theorem, a sufficiently small cube in ϵ-space about origin is mapped in one to one way onto a neighbourhood of U_0 in $(u_1, u_2, \ldots, u_n)$-space. Therefore, given a state U_n sufficiently close to U_0 in $(u_1, u_2, \ldots, u_n)$-space, we can uniquely find a set of values of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that the relation (5.89) is satisfied. This can be summarised in

Theorem 5.4 Every state U_0, has a neighbourhood in $(u_1, u_2, \ldots, u_n)$-space such that if U_n belongs to this neighbourhood, the Riemann initial value problem defined by (5.18) and (5.88) has a solution. There is exactly one solution consisting of centered waves (shocks and simple waves) provided the intermediate states are restricted to lie in that neighbourhood of U_0.

Interpreted in the language of wave propagation, the solution of the Riemann problem shows that the initial discontinuity in U breaks into n waves belonging to n characteristic fields of $c_1, c_2 \ldots, c_n$. These waves are separated by $n-1$ intermediate constant states $U_1, U_2, \ldots, U_{n-1}$.

The existence of a weak solution of an arbitrary initial value problem for a system of nonlinear conservation laws can be proved with the help of the solution of the Riemann problem.

Part B: EQUATIONS IN MORE THAN TWO INDEPENDENT VARIABLES

Let us recollect our summation convention. A repeated suffix from the set $\{i, j, k, \alpha, \beta, \gamma\}$ will represent summation over the range of the suffix. The range of i, j, k is $1, 2, \ldots, n$ and that of α, β, γ is $1, 2, \ldots, m$. Let us also note that we shall treat linear and quasilinear equations together. However, in the second case our results are true for a given solution.

*§6 THE CAUCHY PROBLEM AND A CHARACTERISTIC MANIFOLD

Consider a single nth order equation for a single unknown function u of $m+1$ independent variables $x_1, x_2, \cdots x_m, t$:

$$F\left(x_1, \ldots, x_m, t; u; \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}, \frac{\partial u}{\partial t}; \ldots; \frac{\partial^n u}{\partial x_1^n}, \ldots, \frac{\partial^n u}{\partial x_m \partial t^{n-1}}, \frac{\partial^n u}{\partial t^n}\right)=0 \tag{6.1}$$

and an m dimensional manifold (or surface)

$$\gamma: \varphi(x_\alpha, t)=0 \text{ or } x_\alpha = x_{\alpha 0}(\eta_\beta),\ t=t_0(\eta_\beta). \tag{6.2}$$

In §5.1 of Chapter 1 we have shown that it is possible to write the parametric representation $x_\alpha = x_{\alpha 0}(\eta_\beta)$, $t=t_0(\eta_\beta)$ of an m dimensional manifold when its equation is given in the form $\varphi(x_\alpha, t)=0$. Let us suppose that the values of u and the $n-1$ exterior derivatives $\partial^i u/\partial \varphi^i$, $i=1, 2,, \ldots, n-1$, are prescribed by

$$(u)_\gamma = u(x_{\alpha 0}(\eta_\beta), t_0(\eta_\beta)) \equiv u_0(\eta_\beta) \tag{6.3}$$

and

$$\left(\frac{\partial^i u}{\partial \varphi^i}\right)_\gamma = u_0^{(i)}(\eta_\beta),\ i=1, 2, \ldots, n-1 \tag{6.4}$$

where $u_0, u_0^{(1)}, \ldots, u_0^{(n-1)}$ are known functions of $\eta_1, \eta_2, \ldots, \eta_m$.

The Cauchy problem for the nth order equation (6.1) is to find a solution which satisfies the n conditions (6.3) and (6.4) on the manifold γ.

In our attempt to solve a Cauchy problem for a single first order equation in more than two independent variables (§ 5, Chapt. 1) we notice that a unique solution can be found only if the mainfold γ is not tangential to certain "exceptional" manifolds called *characteristic manifolds* (or characteristic surfaces). We also notice that if the surface γ is coincident with a characteristic surface and if the solution of the Cauchy problem exists, the Cauchy data cannot be arbitrarily prescribed but must satisfy a *compatibility condition*. In this section we shall discuss these results in connection with a higher order equation (6.1) (or a first order system) in more than two independent variables.

Consider first a single linear or quansilinear equation of nth order:

$$\sum_{i_1+\ldots+i_{m+1}=n} A_{i_1 \ldots i_m i_{m+1}} \frac{\partial^n u}{\partial x_1^{i_1} \ldots \partial x_m^{i_m}\, \partial t^{i_{m+1}}} + B = 0 \tag{6.5}$$

where the coefficients $A_{i_1 \ldots i_m i_{m+1}}$ and B depend on x_α, t, u and the partial derivatives of u up to order $n-1$. The values of the coefficients are known on the datum manifold γ. The condition that $\gamma: \varphi(x_\alpha, t)=0$ is a characteristic manifold C is

$$Q(x_\alpha, t; \varphi_{x_\alpha}, \varphi_t) \equiv \sum_{i_1+\dots+i_{m+1}=n} A_{i_1 \dots i_m i_{m+1}} \varphi_{x_1}^{i_1} \dots \varphi_{x_m}^{i_m} \varphi_t^{i_{m+1}} = 0, \text{ on } \gamma. \quad (6.6)$$

Similarly, let us consider a single nonlinear equation of the second order

$$F(x_\alpha, t; u; u_{x_\alpha}, u_t; u_{x_\alpha x_\beta}, u_{x_\alpha t}, u_{tt}) = 0. \quad (6.7)$$

It can be shown that the condition that the surface γ: $\varphi(x_\alpha, t)=0$ is a characteristic surface is

$$Q(x_\alpha, t; \varphi_{x_\alpha}, \varphi_t) \equiv \varphi_t^2 F_{u_{tt}} + \varphi_t \varphi_{x_\alpha} F_{u_{tx_\alpha}} + \varphi_{x_\alpha} \varphi_{x_\beta} F_{u_{x_\alpha x_\beta}} = 0 \text{ on } \gamma. \quad (6.8)$$

Next we pass on to a *system of n first order quasilinear equations* in n dependent variables $u_1, u_2, \dots, u_n$. Consider n equations of the form

$$A_{ij}\frac{\partial u_j}{\partial t} + B_{ij}^{(\alpha)} \frac{\partial u_j}{\partial x_\alpha} + C_i = 0; \; i = 1, 2, \dots, n \quad (6.9)$$

where $u_1, \dots, u_n$ are the n dependent variables, $x_1, \dots, x_m, t$ are the $m+1$ independent variables and the coefficients $A_{ij}, B_{ij}, C_i, (i, j = 1, 2, \dots, n)$ are functions of x_α, t, u_i over a domain D_2 in (x_α, t, u_i)-space. Let A and $B^{(\alpha)}$ be $n \times n$ matrices whose elements in the ith row and jth column are A_{ij} and $B_{ij}^{(\alpha)}$ respectively and let U and C be the column vectors with components $u_1, u_2, \dots, u_n$ and $C_1, C_2, \dots, C_n$ respectively. The system of n equations (6.9) can be written in a more compact form as

$$A\frac{\partial U}{\partial t} + B^{(\alpha)}\frac{\partial U}{\partial x_\alpha} + C = 0. \quad (6.10)$$

Consider an m-dimensional manifold

$$\gamma : \varphi(x_\alpha, t) = 0 \text{ or } x_\alpha = x_{\alpha 0}(\eta_\beta), \; t = t_0(\eta_\beta) \quad (6.11)$$

in (x_α, t)-space and n functions $u_{i0}(\eta_\beta)$, such that $(x_{\alpha 0}, t_0, u_{i0}) \in D_2$ for $(\eta_\beta) \in I$, where I is a domain in an m-dimensional space of variables $\eta_1, \eta_2, \dots, \eta_m$. *The Cauchy problem for the system (6.9) or (6.10) is to find a solution, $U(x_\alpha, t)$ in some domain D in (x_α, t)-space, of the equations and satisfying*

$$U(x_{\alpha 0}(\eta_\beta), t_0(\eta_\beta)) = U_0(\eta_\beta), \quad (n_\alpha) \in I \quad (6.12)$$

where $u_{i0}(\eta_\beta)$ are the components of the column matrix U_0. As in § 5.1 of Chapt. 1 we introduce $m+1$ independent functions $\eta_\alpha(x_\beta, t)$, $\varphi(x_\beta, t)$ as new independent variables and transform the system of equations (6.10) and the initial conditions (6.12) respectively to

$$(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha})\frac{\partial U}{\partial \varphi} + \left(A \frac{\partial \eta_\beta}{\partial t} + B^{(\alpha)}\frac{\partial \eta_\beta}{\partial x_\alpha}\right)\frac{\partial U}{\partial \eta_\beta} + C = 0 \quad (6.13)$$

and

$$U(\eta_\alpha, \varphi = 0) = U_0(\eta_\alpha). \quad (6.14)$$

From the Cauchy data (6.14), all interior derivatives $\partial U/\partial \eta_\beta$ can be determined at $\gamma : \varphi = 0$. The equation (6.13), in which all quantities except $\partial U/\partial \varphi$ are known on γ, shows that the exterior derivative $\partial U/\partial \varphi$ can be uniquely determined if

$$Q(x_\alpha, t; \varphi_{x_\alpha}, \varphi_t) \equiv \det (A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}) \neq 0 \text{ on } \varphi = 0 \tag{6.15}$$

where det H means determinant of the square matrix H. We can multiply the equation (6.13) by the inverse of the matrix $(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha})$ which is non-singular in a neighbourhood of the initial manifold $(x_{\alpha 0}, t_0, u_{\lambda 0})$ in D_2, and get

$$\frac{\partial U}{\partial \varphi} = -B'^{(\beta)} \frac{\partial U}{\partial \eta_\beta} - C' \tag{6.16}$$

where the right hand side is known on γ. When (6.15) is satisfied, we say that the datum manifold is *free* or *noncharacteristic*.

When $Q = 0$ on γ, the datum manifold is said to be a characteristic manifold, the matrix $(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha})_{\varphi=0}$ is singular and has at least one left null vector l_0 (a row vector with components $l_{01}, l_{02}, \cdots, l_{0n}$) satisfying

$$l_0(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha})_{\varphi=0} = 0. \tag{6.17}$$

Premultiplying (6.13) by l_0, we get a restriction on the Cauchy data:

$$l_0\left[\left(A\frac{\partial \eta_\beta}{\partial t} + B^{(\alpha)}\frac{\partial \eta_\beta}{\partial x_\alpha}\right)\frac{\partial U}{\partial \eta_\beta} + C\right]_{\varphi=0} = 0 \tag{6.18}$$

corresponding to a choice of the null vector l_0. The number of restrictions is equal to the number of linearly independent left null vectors of the matrix $A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}$.

As usual, for the quasilinear system (6.10), we can determine whether an initial manifold is a characteristic manifold or not only when the dependent variable U is known on γ. For a linear equation, the characteristic manifolds are determined without any reference to the values of U on the manifold.

We make here two important remarks. The first is about the characteristic equation $Q = 0$ in all cases when Q is given either by (6.6) or (6.8) or (6.15). The remark applies equally to linear, quasilinear and nonlinear equations with an assumption that for quasilinear and nonlinear equations, it applies only when a known solution of the partial differential equation has been substituted in Q. The characteristic condition $Q = 0$ is required to be satisfied on the manifold γ: $\varphi = 0$, i.e. the function φ need not satisfy the equation $Q = 0$ identically in $m+1$ variables x_α, t but only after using the condition $\varphi = 0$. However, if $Q = 0$ is satisfied identically, we get a one-parameter family of characteristic manifolds $\varphi(x_\alpha, t) = c$ in which the manifold $\varphi = 0$ is embedded. A simple example will make the point clear.

Example 6.1

For the wave equation

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} \tag{6.19}$$

in the four dimensional space of variables x, y, z, t, the characteristic equation is

$$\varphi_t^2 - \varphi_x^2 - \varphi_y^2 - \varphi_z^2 = 0. \tag{6.20}$$

Let us choose a special characteristic surface C_3 (a sphere of expanding radius t in (x, y, z)-space, for which $t > 0$) given by

$$\varphi \equiv x^2 + y^2 + z^2 - t^2 = 0. \tag{6.21}$$

It is a characteristic surface, since

$$\varphi_t^2 - \varphi_x^2 - \varphi_y^2 - \varphi_z^2 = 4(t^2 - x^2 - y^2 - z^2)$$

vanishes on C_3 due to the relation (6.21). However, it is simple to see that the function φ defined by (6.21) is not a solution of the partial differential equation (6.20). The manifold C_3 can also be represented by

$$\psi \equiv t - \sqrt{x^2 + y^2 + z^2} = 0 \tag{6.22}$$

where the function ψ satisfies the characteristic equation $Q = 0$ identically. Therefore we get a family of characteristic surfaces:

$$\sqrt{x^2 + y^2 + z^2} = t - c \tag{6.23}$$

where c is the parameter. The surface C_3 is a member of the family (6.23) when $c = 0$.

Similarly, it is simple to see that the family of surfaces $\chi \equiv t + \sqrt{x^2 + y^2 + z^2} = d$ is also a family of characteristic manifolds. A particular member of this family is D_3 obtained by taking $d = 0$. We note that the equation (6.21) is a combined equation for the two surfaces C_3 and D_3.

In this example we have shown that it is possible to embed the characteristic surface C_3 in a one-parameter family of characteristic surfaces. This is possible for every characteristic surface of any partial differential equation (see problem 2 in exercise 6.1). Thus all characteristic surfaces of an equation can be obtained by solving the characteristic partial differential equation

$$Q(x_\alpha, t, \varphi_{x_\alpha}, \varphi_t) = 0. \tag{6.24}$$

The second remark is regarding the invariance of characteristic manifolds. We state this result in the form of a theorem.

Theorem 6.1 A characteristic surface of a partial differential equation is invariant under a transformation of dependent and independent variables.
Proof We prove this theorem for a system of n first order equations (6.9). We first change only the dependent variables from U to V. Let us suppose that the components of U and V are related by a nonsingular transformation

$$u_j = \tilde{u}_j(x_\alpha, t, v_k). \tag{6.25}$$

The determinant of the matrix

$$T = [T_{jk}] = \left[\frac{\partial \tilde{u}_j}{\partial v_k}\right] \tag{6.26}$$

is neither zero nor infinity in a domain of $(x_\alpha,\ t,\ u_i)$-space. Equation (6.9) now becomes

$$AT\frac{\partial V}{\partial t}+B^{(\alpha)}T\,\frac{\partial V}{\partial x_\alpha}+\left(C+A\frac{\partial \tilde{U}}{\partial t}+B^{(\alpha)}\frac{\partial \tilde{U}}{\partial x_\alpha}\right)=0. \tag{6.27}$$

A characteristic manifold $\Phi(x_\alpha,\ t)=0$ of (6.27) satisfies

$$\det\,(AT\Phi_t+B^{(\alpha)}T\Phi_{x_\alpha})=0 \tag{6.28}$$

or

$$\det\,(A\Phi_t+B^{(\alpha)}\Phi_{x_\alpha})\cdot\det\,(T)=0. \tag{6.29}$$

Since $\det\,(T)\neq 0$, it follows that Φ satisfies the equation

$$\det\,(A\Phi_t+B^{(\alpha)}\Phi_{x_\alpha})=0 \tag{6.30}$$

which is the characteristic equation of the original system (6.9). This proves the invariance of the characteristic manifolds under an arbitrary transformation of dependent variables.

To prove the invariance under a change of independent variables we consider a new set of independent variables ξ_β, τ defined by

$$\xi_\beta=\xi_\beta(x_\alpha,\ t),\ \tau=\tau(x_\alpha,\ t). \tag{6.31}$$

Then

$$\frac{\partial}{\partial t}=\frac{\partial\tau}{\partial t}\,\frac{\partial}{\partial\tau}+\frac{\partial\xi_\beta}{\partial t}\,\frac{\partial}{\partial\xi_\beta},\quad \frac{\partial}{\partial x_\alpha}=\frac{\partial\tau}{\partial x_\alpha}\,\frac{\partial}{\partial\tau}+\frac{\partial\xi_\beta}{\partial x_\alpha}\,\frac{\partial}{\partial\xi_\beta} \tag{6.32}$$

and the equation (6.9) transforms to

$$\left(A\frac{\partial\tau}{\partial t}+B^{(\alpha)}\frac{\partial\tau}{\partial x_\alpha}\right)\frac{\partial u_j}{\partial\tau}+\left(A\frac{\partial\xi_\beta}{\partial t}+B^{(\alpha)}\frac{\partial\xi_\beta}{\partial x_\alpha}\right)\frac{\partial u_j}{\partial\xi_\beta}=0. \tag{6.33}$$

The characteristic equation of (6.33) is

$$\det\left[\left(A\,\frac{\partial\tau}{\partial t}+B^{(\alpha)}\frac{\partial\tau}{\partial x_\alpha}\right)\varphi_\tau+\left(A\,\frac{\partial\xi_\beta}{\partial t}+B^{(\alpha)}\frac{\partial\xi_\beta}{\partial x_\alpha}\right)\varphi_{\xi_\beta}\right]=0. \tag{6.34}$$

Collecting the coefficients of A and $B^{(\alpha)}$ and using (6.32) we find that this equation reduces to

$$\det\,(A\varphi_t+B^{(\alpha)}\varphi_{x_\alpha})=0. \tag{6.35}$$

This is the same equation which is satisfied by a characteristic manifold of (6.9), i.e. the original system. This completes the proof of the invariance of characteristic surfaces.

These invariant properties give a special significance to the characteristic surfaces. Let $C:\varphi(x_\alpha,\ t)=0$ be a characteristic surface where φ satisfies the equation $Q(\varphi_{x_\alpha},\ \varphi_t,\ x_\alpha,\ t)=0$. Let us denote by $S(t)$ the surface in $(x_1,\ x_2,\ \ldots,\ x_m)$-space represented by $\varphi(x_\alpha,\ t)=0$ for a given value of t. Then the moving surface $S(t)$ is a surface across which discontinuities or singularities of the solution $U(x_\alpha,\ t)$ propagate and, therefore, $S(t)$ can be interpreted as a wave front. If (n_α) be the unit normal to $S(t)$ and c the wave front velocity, then by considering two successive positions of the wavefront at time t and $t+\delta t$ we can show that

$$n_\alpha = \frac{\varphi_{x\alpha}}{(\varphi_{x_1}^2+\varphi_{x_2}^2+\dots+\varphi_{x_m}^2)^{1/2}} \tag{6.36}$$

and

$$c = -\frac{\varphi_t}{(\varphi_{x_1}^2+\varphi_{x_2}^2+\dots+\varphi_{x_m}^2)^{1/2}}. \tag{6.37}$$

*EXERCISE 6.1

*1. Show that a solution of the partial differential equation

$$u_{xyt}+u_{ytt}=0$$

with Cauchy data

$$u(x, y, 0)=f(x, y), \qquad u_t(x, y, 0)=g(x, y), \qquad u_{tt}(x, y, 0)=h(x, y)$$

exists if the data satisfies the condition

$$g_{xy}(x, y)+h_y(x, y)=0.$$

Show also that when this condition is satisfied, the solution is non-unique and we can add any function $\chi(t)$ to a solution provided $\chi(0)=\chi'(0)=\chi''(0)=0$.

*2. Prove that any characteristic surface $\varphi=0$ can be embedded in a one-parameter family of characteristic manifolds $\Phi(x_\alpha, t)=c$ satisfying the partial differential equation

$$Q(\Phi_{x\alpha}, \Phi_t; x_\alpha, t)=0$$

*§7 THE WAVE EQUATION

In §4 of Chapter 2 we discussed well-posedness of the various boundary value problems for the wave equation and derived explicit expressions for the solution of the initial value problem (a Cauchy problem) in the case of one, two and three space variables. In this section, we shall view the wave equation as a particular case of general hyperbolic equation and gradually bring out those properties of this equation which can be generalised to other hyperbolic equations. The wave equation in m-space dimensions for the function $u(x_\alpha, t)$ is

$$u_{tt}-c^2(u_{x_1x_1}+u_{x_2x_2}+\dots+u_{x_mx_m})=0. \tag{7.1}$$

The characteristic partial differential equation of the wave equation is

$$Q(\varphi_{x\alpha}, t)\equiv\varphi_t^2-c^2\varphi_{x\alpha}\varphi_{x\alpha}=0. \tag{7.2*}$$

If we interpret $\varphi(x_\alpha, t)=0$ as the locus of a moving surface $S(t)$ in $(x_1, x_2, \dots, x_m)$-space, it follows from (6.36) and (6.37) that it represents a wave front with wavefront velocity equal to $\pm c$, which is constant and independent of the direction of the wavefront normal (n_α).

*Since x_α, t do not appear explicitly in the characteristic polynomial $Q(x_\alpha, t, \varphi_{x_\alpha}, \varphi_t)$ we denote it by $Q(\varphi_{x_\alpha}, \varphi_t)$.

A closed form explicit solution of the initial value problem of the wave equation for $m \geqslant 4$ has also been obtained (see Courant and Hilbert (1962), §11, Chapter VI). The solution of the wave equation shows that the disturbances which are initially of finite extent are always bounded by a sharply defined leading front and in addition if the number m of spatial variables is odd and > 1, they are bounded also by a sharply defined trailing front. These solutions help us also in determining the domains of dependence and influence as shown below.

The most important solution of the characteristic equation (7.2) represents a *characteristic conoid* in (x_α, t)-space with its vertex at an arbitrary point $P_0(x_\alpha^{(0)}, t^{(0)})$:

$$\varphi \equiv (t-t^{(0)}) \pm \frac{1}{c}\{(x_\alpha - x_\alpha^{(0)})(x_\alpha - x_\alpha^{(0)})\}^{1/2} = 0. \qquad (7.3)$$

(7.3) with plus (+) and minus (−) signs represent respectively the lower and upper branches of the right circular conoid in space-time with its vertex at P_0:

$$\psi(x_\alpha, t) \equiv (t-t^{(0)})^2 - \frac{1}{c^2}(x_\alpha - x_\alpha^{(0)})(x_\alpha - x_\alpha^{(0)}) = 0. \qquad (7.4)$$

Intersection of the conoid (7.4) by the hyperplane $t=0$ is a sphere (a circle when $m=2$, a pair of points when $m=1$)

$$S_0\colon (x_1 - x_1^{(0)})^2 + (x_2 - x_2^{(0)})^2 + \ldots + (x_m - x_m^{(0)})^2 = c^2(t^{(0)})^2 \qquad (7.5)$$

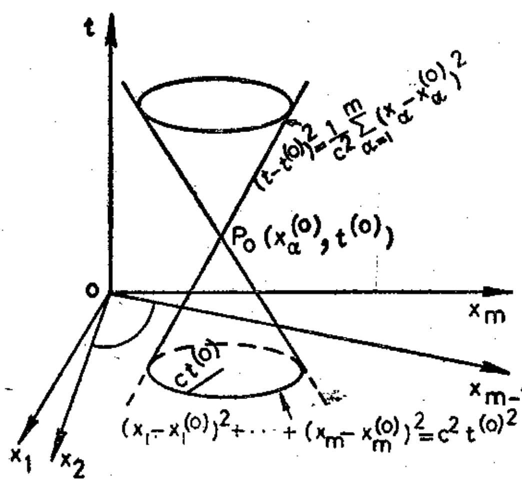

Fig. 7.1 The lower and upper portions of the characteristic conoid through P_0 in space-time.

in (x_α)-space. The explicit solutions of the wave equation show that if the initial data is changed only outside the sphere S_0, the solution remains unchanged in the closed domain bounded by the lower part of the conoid (7.4) and the sphere S_0. More accurately, the *domain of dependence* on the plane $t=0$ of the point P_0 is the set of points on the sphere S_0 (when m is odd except $m=1$) or the set of all points on and inside the sphere S_0 when m is even. We have now met a phenomenon in which the domain of dependence of a point P_0 need not consist of all interior points of a surface S_0.

Consider now the role of a disturbance at a point P_0 in space-time. The values of u at P_0 influences the solution for $t > t^{(0)}$ at all points on the upper part of the characteristic conoid (7.4) when m is odd (except $m=1$) and for all other value of m (including $m=1$) it also influences the solution everywhere on the conoid and its interior. These point sets constitute the *domain of influence* of P_0.

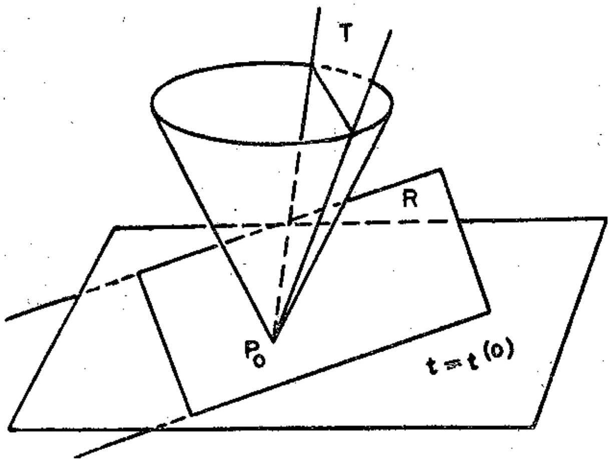

Fig. 7.2 R is a space-like plane.

§7.1 Space-like Manifold and Time-like Direction

Consider a hyperbolic partial differential equation in u and an m-dimensional manifold R in (x_α, t)-space such that the value of u at any point P on R does not influence the solution u at any other point of R. Then the manifold R is said to be a *space-like manifold* for the given equation.

An example of a space like manifold for the wave equation (7.1) is a hyperplane $t=$ constant. Any other plane

$$v(t-t^{(0)})-n_\alpha(x_\alpha-x_\alpha^{(0)})=0, \qquad (v,\ n_\alpha=\text{constant}) \tag{7.6}$$

through the point $P_0(x_\alpha^{(0)}, t^{(0)})$ such that it intersects the characteristic conoid (7.4) through P_0 only at P_0, is also an example of a space-like manifold. We now derive a condition on the coefficients v and n_α in (7.6) for it to be spacelike.

At the points of intersection of the conoid (7.4) and the plane (7.6), we have

$$\frac{v^2}{c^2}\{(x_1-x_1^{(0)})^2+(x_2-x_2^{(0)})^2+\ldots+(x_m-x_m^{(0)})^2\}$$
$$=\{n_1(x_1-x_1^{(0)})+n_2(x_2-x_2^{(0)})+\ldots+n_m(x_m-x_m^{(0)})\}^2. \tag{7.7}$$

Using Schwartz's inequality for the expression on the right hand side, we get

$$\frac{v^2}{c^2}\{(x_1-x_1^{(0)})^2+(x_2-x_2^{(0)})^2+\ldots+(x_m-x_m^{(0)})^2\}$$
$$\leqslant (n_1^2+n_2^2+\ldots+n_m^2)\{(x_1-x_1^{(0)})^2+(x_2-x_2^{(0)})^2+\ldots+(x_m-x_m^{(0)})^2\}$$

which gives

$$\frac{v^2}{c^2}\leqslant n_\alpha n_\alpha. \tag{7.8}$$

Therefore, if the inequality

$$\frac{v^2}{c^2} > n_\alpha n_\alpha \qquad \text{or} \qquad \frac{v^2}{n_\alpha n_\alpha} > c^2 \tag{7.9}$$

is satisfied, (7.7) can be satisfied only at the point P_0, i.e. the only point of intersection of (7.4) and (7.6) is P_0. Thus the condition that the plane (7.6) is a space-like manifold, is (7.9). We note that (7.6) represents the locus in (x_α, t)-space of a moving plane with normal in the direction (n_α) and moving with the velocity $\frac{v}{\sqrt{n_\alpha n_\alpha}}$ which must be greater than the speed c in order that (7.6) is space-like for the wave equation.

Consider now a space-like manifold R

$$f(x_\alpha, t)=0$$

of the wave equation. Then its tangent planes should also be space-like. Therefore, the inequality (7.9) immediately gives the condition

$$(f_t)^2-c^2 f_{x_\alpha} f_{x_\alpha} > 0, \qquad \text{for all points of } R \tag{7.10}$$

Space-like direction: A direction which lies in a space-like plane, is called a *space-like direction.*

Time-like direction and curve: Consider a straight line in space-time passing through a point $P_0(x_\alpha^{(0)}, t^{(0)})$. If the straight line lies in the interior of the characteristic conoid through the point P_0, then the direction of the straight line is said to be a time-like direction for the wave equation. A curve

$$x_\alpha = x_\alpha(\sigma), \qquad t=t(\sigma) \tag{7.11}$$

in the space time is said to be a *time-like curve* if its tangent direction is always a time-like direction. This implies

$$\left(\frac{dt}{d\sigma}\right)^2-\frac{1}{c^2}\left(\frac{dx_\alpha}{d\sigma}\,\frac{dx_\alpha}{d\sigma}\right) > 0. \tag{7.12}$$

A generator of the characteristic conoid is neither space-like nor time-like.

*§7.2 Algebraic Criterion for Hyperbolicity

The term 'hyperbolic equations' includes all those equations or systems for which the Cauchy problem is well-posed with respect to suitably chosen initial manifolds. However, from the point of view of applications, it is necessary to have an algebraic criterion for hyperbolicity, which can be easily verified. In the case of two independent variables the algebraic criterion for hyperbolicity could be given (see section 2) in terms of the real roots for $\lambda = -\frac{\varphi_t}{\varphi_x}$ in the characteristic equation (2.4). As problem 1 in Exercise 7.1 shows, such a criterion is not appropriate when the number of independent variables is more than two. In this section, we shall discuss a criterion, for the wave equation, which can be easily generalised to include a very large class of hyperbolic equations.

The criterion for hyperbolicity is best expressed in terms of the 'normal cone'. Denote φ_{x_α} by k_α and φ_t by k_{m+1}; then the vector $(k_1, k_2, \cdots, k_m, k_{m+1})$ is in the direction of the normal to the characteristic manifold $\varphi(x_\alpha, t) = 0$. Writing the characteristic equation (7.2) in the form

$$k_{m+1}^2 = c^2(k_1^2 + k_2^2 + \ldots + k_m^2) \tag{7.13}$$

we note that the normals to the characteristic manifolds through a point P generate a conoid called normal conoid in $(k_1, k_2, \cdots, k_{m+1})$-space. If we superimpose this space on (x_α, t)-space, the normal conoid is orthogonal to the characteristic conoid at the origin. Note that the semi-vertical angle of the normal conoid is $\pi/2 - \beta$, where

$$c^2 = \tan^2 \beta. \tag{7.14}$$

Take any vector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \cdots, \zeta_{m+1})$ which points in the interior of the normal conoid. For any fixed nonzero vector $\boldsymbol{\theta}$ not parrallel to $\boldsymbol{\zeta}$ and for any arbitrary scalar λ, the vector $\lambda\boldsymbol{\zeta} + \boldsymbol{\theta}$ represents a straight line L parallel to the vector $\boldsymbol{\zeta}$ through the point $\boldsymbol{\theta}$. This straight line intersects the normal conoid at two points, say A and B (in this case the normal cone consists of two separate sheets, the lower sheet being the mirror image of the upper one in the $k_{m+1} = 0$ plane). This implies that the equation

$$Q(\lambda\boldsymbol{\zeta} + \boldsymbol{\theta}) = 0^* \tag{7.15}$$

where

$$Q(\mathbf{k}) \equiv k_{m+1}^2 - c^2 (k_1^2 + \ldots + k_m^2) \tag{7.16}$$

has two distinct roots for λ (i.e. equal to the order of the partial differential equation). *The hyperbolicity of the wave equation is related to the existence of vectors of the type* $\boldsymbol{\zeta}$ *such that the equation* (7.15) *has two distinct roots* λ_1 *and* λ_2 *for an arbitrary nonzero vector* $\boldsymbol{\theta}$ *not parallel to* $\boldsymbol{\zeta}$. This would ensure that if $t = 0$ is space-like, then the characteristic equation gives two real and distinct characteristic velocities (see section 8.1, the second definition of hyperbolicity). We also note that any plane perpendicular to the vector $\boldsymbol{\zeta}$

*In (7.16) the arguments $k_1, k_2, \ldots, k_{m+1}$ of the characteristic polynominal have been denoted by a vector $\mathbf{k} = (k_1, k_2, \ldots, k_{m+1})$.

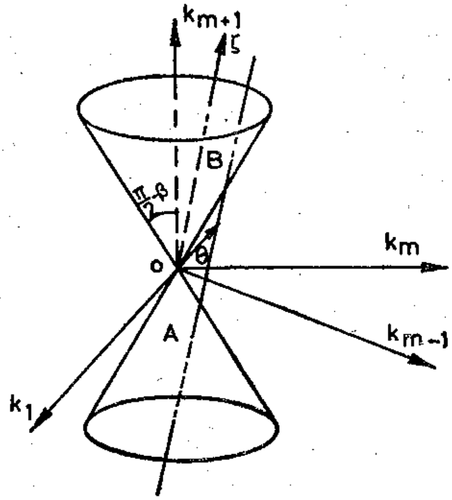

Fig. 7.3 A straight line L parallel to ζ intersects normal conoid in two points

(when ζ points into the normal conoid) and passing through the vertex intersects the characteristic conoid only at the vertex and hence is space-like.

The use of the equation (7.15) in the definition of hyperbolicity has a great advantage in that the definition does not depend on the choice of a particular set of independent variables. This definition of hyperbolicity will be used in §8.1.

EXERCISE 7.1

1. When a new coordinate system (ξ_1, ξ_2, τ) is obtained by rotating the (x_1, x_2, t)-system about x_1-axis by an angle α, i.e. when

$$\xi_1 = x_1, \ \xi_2 = t \sin \alpha + x_2 \cos \alpha, \ \tau = t \cos \alpha - x_2 \sin \alpha, \tag{7.17}$$

show that the characteristic equation of the wave equation:

$$\varphi_t^2 - \varphi_x^2 - \varphi_y^2 - \varphi_z^2 = 0 \tag{7.18}$$

becomes

$$(\cos^2 \alpha - c^2 \sin^2 \alpha)\varphi_\tau^2 + 2 \cos \alpha \sin \alpha \, (1 + c^2)\varphi_\tau \varphi_{\xi_2} + (\sin^2 \alpha - c^2 \cos^2 \alpha) \, \varphi_{\xi_2}^2 - c^2 \varphi_{\xi_1}^2 = 0. \tag{7.19}$$

Show also that the characteristic equation (7.19) gives two real and distinct values of φ_τ for arbitrary real values of φ_{ξ_1} and φ_{ξ_2} if and only if the plane $\tau = 0$ is space-like (note that the $\tau =$ axis need not be time-like).

2. The two axioms of the *special theory of relativity* can be stated together as "the propagation of light in an inertial frame is governed by

the wave equation $u_{tt}-c^2u_{x_\alpha x_\alpha}=0$, where c is the same constant for all inertial frames". Every non-singular linear transformation (transformation from one inertial frame to another inertial frame) of the variables $t, x_1, x_2, \cdots, x_m$, with real coefficients, under which the wave equation remains invariant, is a combination of a Lorentz tranformation, a translation of the origin and a similarity transformation (see Petrovsky, 1954). Show that Lorentz transformation maps the t-axis into a time-like line and $x_1, x_2, \cdots, x_m$ axes into space-like lines.

*§ 7.3 Energy Density and the Law of Conservation of Energy

The wave equation can be put in the conservation form:

$$\frac{1}{2}\left[\frac{\partial}{\partial t}(u_t^2+c^2u_{x_\alpha}u_{x_\alpha})-2c^2\frac{\partial}{\partial x_\alpha}(u_tu_{x_\alpha})\right]=0. \tag{7.20}$$

Let D be a simply connected closed domain (in space-time) bounded by a surface S and let (n_α, λ) be the components of the exterior unit normal of S. Integrating (7.20) over D and using the Gauss divergence theorem to convert a 'volume' integral to a 'surface' integral we get

$$\int_S E(S)\,dS=0 \tag{7.21}$$

where

$$E(S) = \tfrac{1}{2}\{\lambda(u_t^2+c^2u_{x_\alpha}u_{x_\alpha})-2c^2u_tn_\alpha u_{x_\alpha}\}. \tag{7.22}$$

The quantity $E(S)$ is defined to the 'energy density' of the m-dimensional manifold S with unit normal (n_α, λ). For a plane $t=$constant with unit normal pointing in 'future', we have $n_\alpha=0, \lambda=1$. Then

$$E(t=\text{const})=\tfrac{1}{2}u_t^2+\tfrac{1}{2}c^2u_{x_\alpha}u_{x_\alpha}. \tag{7.23}$$

We know that for a system (such as a vibrating membrane) governed by the wave equation, $\frac{1}{2}u_t^2$ is the kinetic energy density and $\frac{1}{2}c^2u_{x_\alpha}u_{x_\alpha}$ is the potential energy density. Therefore the energy density defined by (7.22) does coincide with its classical definition in mechanics.

The expression for $E(S)$ can also be written in the form

$$E(S)=\frac{1}{2\lambda}[(\lambda^2-c^2n_\alpha n_\alpha)u_t^2+c^2(\lambda u_{x_\alpha}-n_\alpha u_t)(\lambda u_{x_\alpha}-n_\alpha u_t)]. \tag{7.24}$$

The energy density in the manifold S is positive definite if (note that we can always choose $\lambda>0$)

$$\lambda^2-c^2n_\alpha n_\alpha>0 \tag{7.25}$$

i.e. only if the manifold S is space-like (compare with (7.10)).

If we choose the surface S to be the characteristic conoid, $\lambda^2-c^2n_\alpha n_\alpha=0$, then $E(S)=\dfrac{1}{2\lambda}c^2(\lambda u_{x_\alpha}-n_\alpha u_t)^2$ which is non-negative.

To deduce the law of conservation of energy, we take two non-intersecting infinite space-like manifolds S_1 and S_2 and assume that values of $u(x_\alpha, t)$ and their derivatives on S_1 and S_2 and also in the infinite domain bounded by them tend to zero at infinity sufficiently rapidly. Using (7.21) for the surface S_1+S_2, we get

$$\iint_{S_1} E(S)\, dS = \iint_{S_2} E(S)\, dS \tag{7.26}$$

where we have chosen the unit normal such that $\lambda > 0$ on both surfaces S_1 and S_2.

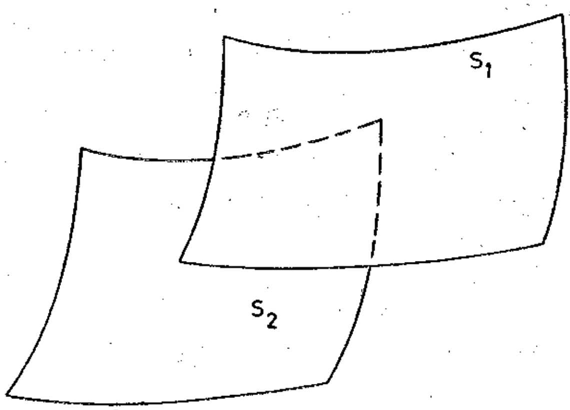

Fig. 7.4

Equation (7.26) represents the law of conservation of energy: energy contained in all space-like manifolds is the same.

*§7.4 The Cauchy Problem is Not Well-posed if the Data is Prescribed on a Manifold Which is Not Space-like

Consider a pair of distinct planes

$$\xi \equiv \lambda_\alpha x_\alpha - \lambda_0 t = 0, \quad \eta \equiv \mu_\alpha x_\alpha - \mu_0 t = 0 \tag{7.27}$$

in (x_α, t)-space. Let us examine the conditions under which the wave equation has a solution of the form

$$u = e^{\xi + i\eta} = \exp\{(\lambda_\alpha x_\alpha - \lambda_0 t) + i(\mu_\alpha x_\alpha - \mu_0 t)\}. \tag{7.28}$$

When (7.28) is a solution, for any constant r, $e^{\pm r(\xi+i\eta)}$ and, therefore, $1/(2r^2)(e^{r(\xi+i\eta)}+e^{-r(\xi+i\eta)})$, also a solution of the wave equation. This shows that the imaginary part, of the last expression, namely

$$v = \frac{1}{r^2} \sinh r\xi \sin r\eta \tag{7.29}$$

is a solution of the wave equation. This solution satisfies the following Cauchy data on the manifold $\xi=0$:

$$v(\xi=0)=0$$

and (7.30)

$$v_\xi(\xi=0)=\frac{1}{r}\sin r\eta.$$

As $r\to\infty$, the Cauchy data $v(\xi=0)$ and $v_\xi(\xi=0)$ both uniformly tend to zero; however, the solution v itself tends to infinity as $r\to\infty$ for any value of $\xi\neq 0$. Noting that the only solution of the Cauchy problem with $u(\xi=0)=0$ and $u_\xi(\xi=0)=0$ is the trivial solution $u=0$, we conclude that the solution of the Cauchy problem with u and u_ξ prescribed on $\xi=0$ does not depend continuously on the Cauchy data. Therefore if the wave equation prossesses a solution of the form (7.28), the Cauchy problem with the data on $\xi=0$ is not well-posed.

Substituting (7.28) in the wave equation we get

$$(\lambda_0+i\mu_0)^2-c^2(\lambda_\alpha+i\mu_\alpha)(\lambda_\alpha+i\mu_\alpha)=0. \tag{7.31}$$

Equating the real and imaginary parts we obtain

$$\mu_0^2-c^2\mu_\alpha\mu_\alpha=\lambda_0^2-c^2\lambda_\alpha\lambda_\alpha \tag{7.32}$$

and

$$\lambda_0\mu_0=c^2\lambda_\alpha\mu_\alpha. \tag{7.33}$$

Eliminating μ_0 from the last two equations we get

$$\frac{c^4}{\lambda_0^2}(\lambda_\alpha\mu_\alpha)^2-c^2\mu_\alpha\mu_\alpha=\lambda_0^2-c^2\lambda_\alpha\lambda_\alpha.$$

Using the Schwartz inequality in the form

$$(\lambda_\alpha\lambda_\alpha)(\mu_\beta\mu_\beta)\geqslant(\lambda_\alpha\mu_\alpha)^2$$

we get

$$\frac{c^4}{\lambda_0^2}(\lambda_\alpha\lambda_\alpha)(\mu_\beta\mu_\beta)-c^2\mu_\alpha\mu_\alpha\geqslant\lambda_0^2-c^2\lambda_\alpha\lambda_\alpha,$$

or

$$\left(\frac{c^2}{\lambda_0^2}\mu_\beta\mu_\beta+1\right)(c^2\lambda_\alpha\lambda_\alpha-\lambda_0^2)\geqslant 0. \tag{7.34}$$

This can be satisfied only if $\lambda_0^2-c^2\lambda_\alpha\lambda_\alpha\leqslant 0$ which ensures that $\xi=0$ is not a space-like plane. $\lambda_0^2-c^2\lambda_\alpha\lambda_\alpha\leqslant 0$ is simultaneously a necessary condition for the wave equation to have a solution of the form (7.28). Therefore, if the data is prescribed on a plane which is not space-like, the Cauchy problem is not well-posed.

*§7.5 Uniqueness Theorem for a Cauchy Problem

Consider the lower part of the characteristic conoid (7.4) through a point $P(x_\alpha^{(0)}, t^{(0)})$, $t^{(0)}>0$ in space-time. Let S_h be the solid sphere of intersection of this characteristic conoid with the plane $t=h$. Let D be the domain bounded by S_0, S_h $(0 < h < t^{(0)})$ and the portion M_h, of the conoid, contained between the planes $t=0$ and $t=h$ as shown in figure 7.5.

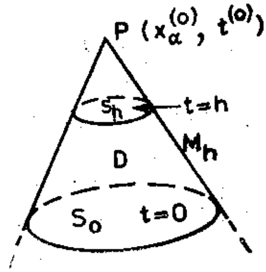

Fig. 7.5

Using (7.21) for the surface of the domain D and noting that the outward normals on S_h and S_0 are in opposite directions, we get

$$\int_{S_h} E(S)\, dS + \int_{M_h} E(S)\, dS = \int_{S_0} E(S)\, dS. \tag{7.35}$$

Since $E(S)$ is non-negative on M_h (see section 7.3), (7.35) gives

$$\int_{S_h} E(S)\, dS \leqslant \int_{S_0} E(S)\, dS. \tag{7.36}$$

Consider a Cauchy problem with values of u and u_t prescribed on the surface $t=0$. Assume that $u=0$ and $u_t=0$ at every point of the solid sphere S_0 (i.e. inside and on the boundary of S_0). Then u_{x_α} also vanishes on $t=0$. Therefore, $E(S)\equiv 0$ on S_0. From (7.36) we get

$$\int_{S_h} E(S)\, dS \equiv \int_{S_h} \tfrac{1}{2}\left\{u_t^2 + c^2 u_{x_\alpha} u_{x_\alpha}\right\} dS \leqslant 0$$

Therefore $\qquad E(S_h)=0$ in S_h. $\qquad$ (7.37)

However, h was chosen arbitrarily which implies that $E(S)=0$ everywhere in the characteristic conoid. Since S_h is a part of plane $t=$ constant, $E(S_h)$ is positive definite. Hence we must have $u_t \equiv 0$ and $u_{x_\alpha} \equiv 0$ in the domain. Since u is continuous and $u=0$ on S_0, it now follows that $u \equiv 0$ everywhere in the characteristic conoid.

This leads to the following uniqueness theorem for a genuine solution of a Cauchy problem:

Theorem: If $u_0(x_\alpha)$ and $u_1(x_\alpha)$ are prescribed as sufficiently smooth functions on S_0, the solution of the Cauchy problem for the wave equation with the Cauchy data

$$u(x_\alpha, 0) = u_0(x_\alpha),\ u_t(x_\alpha, 0) = u_1(x_\alpha)$$

is uniquely determined everywhere in the characteristic conoid.

*§7.6 Bicharacteristics and Rays

In the case of a hyperbolic equation in two independent variables, we could derive compatibility conditions (see equation (2.6)) along the characteristic curves. These conditions contain the entire information regarding propagation of prescribed initial values. Here we shall examine whether these results can be extended to the wave equation in more than two independent variables.

Consider a one-parameter family of characteristic manifolds: $\varphi(x_\alpha, t) = \text{constant}$. Then φ satisfies the first order non-linear partial differential equation (7.2). The characteristic curves of (7.2) are defined to be the *bicharacteristic curves* of the wave equation. These are curves in space-time and are given by a coupled system of ordinary differential equations for t, x_α, φ_t and φ_{x_α}

$$\frac{dt}{d\sigma} = \frac{1}{2} Q_{\varphi_t} = \varphi_t, \quad \frac{dx_\alpha}{d\sigma} = \frac{1}{2} Q_{\varphi_{x_\alpha}} = -c^2 \varphi_{x_\alpha} \tag{7.38}$$

and

$$\frac{d\varphi_t}{d\sigma} = -\frac{1}{2} Q_t = 0, \quad \frac{d\varphi_{x_\alpha}}{d\sigma} = -\frac{1}{2} Q_{x_\alpha} = 0. \tag{7.39}$$

With this choice of σ, σ is increasing in positive t-direction.

Note that these are nothing but Charpit equations of (7.2). A characteristic surface of the wave equation is an integral surface of the equation (7.2) and hence is generated by a family of bicharacteristic curves of the wave equation.

The equations (7.39) show that along the bicharacteristic curves φ_t and φ_{x_α} are constants. Then equations (7.38) show that bicharacteristic curves of the wave equation are straight lines in space-time. Spatial projections of these curves are called rays. These are curves in (x_α)-space. The rays of the wave equation are also straight lines. Their equations in terms of the parameter t are given by

$$\frac{dx_\alpha}{dt} = -\frac{c^2 \varphi_{x_\alpha}}{\varphi_t} = c^2 \frac{\varphi_{x_\alpha} / |\operatorname{grad} \varphi|}{-\varphi_t / |\operatorname{grad} \varphi|} = n_\alpha c \tag{7.40}$$

where the unit normal (n_α) of a wavefront (moving in (x_α)-space) is also constant along a ray. Therefore the rays of the wave equation starting from an arbitrary point $(x_{\alpha 0})$ at time $t = t_0$ are given by

$$x_\alpha - x_{\alpha 0} = n_\alpha c(t - t_0), \quad \alpha = 1, 2, \ldots, m. \tag{7.41}$$

Compatibility conditions on a characteristic manifold

Let us make a transformation of independent variables from (x_α, t) to (x'_α, φ), where

$$x'_\alpha = x_\alpha, \quad \varphi = \varphi(x_\alpha, t). \tag{7.42}$$

Then the wave equation transforms to

$$(\varphi_t^2 - c^2 \varphi_{x_\alpha} \varphi_{x_\alpha}) \frac{\partial^2 u}{\partial \varphi^2} + (\varphi_{tt} - c^2 \varphi_{x_\alpha x_\alpha}) \frac{\partial u}{\partial \varphi} - 2c^2 \varphi_{x_\alpha} \frac{\partial}{\partial x'_\alpha} \left(\frac{\partial u}{\partial \varphi} \right) - c^2 \frac{\partial^2 u}{\partial x'_\alpha \partial x'_\alpha} = 0 \tag{7.43}$$

The operator $d/d\sigma$ for the directional derivative along a bicharacteristic curve becomes

Note $-\frac{\varphi_t}{|\nabla\varphi|} = c$

$$\frac{d}{d\sigma}=\frac{dt}{d\sigma}\frac{\partial}{\partial t}+\frac{dx_\alpha}{d\sigma}\frac{\partial}{\partial x_\alpha}=\varphi_t\frac{\partial}{\partial t}-c^2\varphi_{x_\alpha}\frac{\partial}{\partial x_\alpha} = \varphi_t\left(\frac{\partial}{\partial t}+cn_\alpha\frac{\partial}{\partial x_\alpha}\right)$$

$$=(\varphi_t^2-c^2\varphi_{x_\alpha}\varphi_{x_\alpha})\frac{\partial}{\partial\varphi}-c^2\varphi_{x_\alpha}\frac{\partial}{\partial x'_\alpha}=-c^2\varphi_{x_\alpha}\frac{\partial}{\partial x'_\alpha} \tag{7.44}$$

since φ satisfies (7.2). Using the results of the Section 5.1 of Chapter 1, we verify that the operator $d/d\sigma$ gives an interior derivative on the characteristic manifold. As we are interested in a condition involving only interior derivatives on a characteristic surface $\varphi=$ constant, we set

$$\frac{\partial u}{\partial\varphi}=v \tag{7.45}$$

and noting that the coefficient of $\partial^2u/\partial\varphi^2$ in (7.43) is also zero, we get the following form of the wave equation

$$2\frac{dv}{d\sigma}-c^2\frac{\partial^2u}{\partial x'_\alpha\partial x'_\alpha}+(\varphi_{tt}-c^2\varphi_{x_\alpha x_\alpha})v=0. \tag{7.46}$$

All the derivatives of u and v appearing in (7.46) are interior derivatives on a characteristic manifold $\varphi=$ constant and thus (7.46) represents a compatibility condition along it. This compatibility condition involves two quantities u and v, both of which are prescribed in a Cauchy problem for a manifold $\varphi=$ constant. The first term represents the rate of the change of v along a bicharacteristic curve. We also note that the coefficient of v in (7.46) is the wave operator itself operating on the function φ.

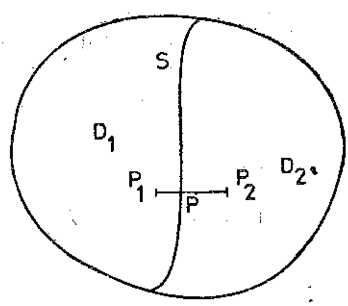

Fig. 7.6 $D=D_1\cup D_2$.

*§7.7 Propagation of Discontinuities Along Rays

Let us consider a solution $u(x_\alpha, t)$ of the wave equation which is C^2 in a domain D of the space-time except for jump discontinuities in the second derivatives of u across an m dimensional manifold S: $\varphi(x_\alpha, t)=0$ which divides D into two subdomain D_1 and D_2. In terms of a new set of independent variables (x'_α, φ), introduced by (7.42), the wave equation reduces to the equation (7.43) which is valid separately in D_1 and D_2. Since u and its first derivatives are continuous across S, the second order interior derivatives $\dfrac{\partial^2 u}{\partial x'_\alpha \partial x'_\beta}$, and interior derivatives of the first order exterior derivatives, namely $\dfrac{\partial}{\partial x'_\alpha}\left(\dfrac{\partial u}{\partial \varphi}\right)$ are continuous across S. So $\partial^2 u/\partial\varphi^2$ must be discontinuous across S. Writing (7.43) at P_1 in D_1, and at P_2 in D_2, taking the limit as P_1 and P_2 both tend to a point P on S, and subtracting the resultant equations we get

$$(\varphi_t^2 - c^2\varphi_{x_\alpha}\varphi_{x_\alpha})\left[\frac{\partial^2 u}{\partial\varphi^2}\right] = 0 \tag{7.47}$$

where $[\partial^2 u/\partial\varphi^2]$ represents the jump of the quantity $\partial^2 u/\partial\varphi^2$ across S.

Since $[\partial^2 u/\partial\varphi^2] \neq 0$, it follows that $\varphi_t^2 - c^2\varphi_{x_\alpha}\varphi_{x_\alpha} = 0$ on $\varphi = 0$ showing that the surface of discontinuity S must be a characteristic manifold. We take the equation of S to be such that S is embedded in a family of characteristic surfaces $\varphi = \text{constant}$, so that φ satisfies $\varphi_t^2 - c^2\varphi_{x_\alpha}\varphi_{x_\alpha} = 0$. The equation (7.43) now becomes

$$(\varphi_{tt} - c^2\varphi_{x_\alpha x_\alpha})\frac{\partial u}{\partial\varphi} - 2c^2\varphi_{x_\alpha}\frac{\partial}{\partial x'_\alpha}\left(\frac{\partial u}{\partial\varphi}\right) - c^2\frac{\partial^2 u}{\partial x'_\alpha \partial x'_\alpha} = 0. \tag{7.48}$$

The quantities $\varphi_{tt} - c^2\varphi_{x_\alpha x_\alpha}$ and φ_{x_α} appearing in the coefficients of (7.48) can be expressed as functions of φ and x'_α. We write

$$\varphi_{tt} - c^2\varphi_{x_\alpha x_\alpha} = A(x'_\alpha, \varphi),\ \varphi_{x_\alpha} = B(x'_\beta, \varphi) \tag{7.49}$$

where, since φ is given, A and B are known functions of x'_α and φ. Differentiating (7.48) with respect to φ and denoting $\partial^2 u/\partial\varphi^2$ by w, we get

$$(\varphi_{tt} - c^2\varphi_{x_\alpha x_\alpha})w + A_\varphi\frac{\partial u}{\partial\varphi} - 2c^2\varphi_{x_\alpha}\frac{\partial}{\partial x'_\alpha}(w) - 2c^2 B_\varphi\frac{\partial^2 u}{\partial x'_\alpha\partial\varphi} - c^2\frac{\partial^3 u}{\partial x'_\alpha\partial x'_\alpha\partial\varphi} = 0. \tag{7.50}$$

We note that the first order derivative $\partial u/\partial\varphi$, and hence also its interior derivatives $\dfrac{\partial^2 u}{\partial x'_\alpha\partial\varphi}$, $\dfrac{\partial^3 u}{\partial x'_\alpha\partial x'_\beta\partial\varphi}$ are continuous across $\varphi = 0$. Also, all the coefficients in (7.50) are continuous across it. Writing equation (7.50) on two sides of the manifold $\varphi = 0$ and taking the difference we get

$$(\varphi_{tt} - c^2\varphi_{x_\alpha x_\alpha})[w] - 2c^2\varphi_{x_\alpha}\frac{\partial}{\partial x'_\alpha}[w] = 0. \tag{7.51}$$

Using $d/d\sigma$ for the bicharacteristic derivative as in (7.44) we finally write (7.51) as

$$2\frac{d}{d\sigma}[w] + (\varphi_{tt} - c^2\varphi_{x_\alpha x_\alpha})[w] = 0. \tag{7.52}$$

Equation (7.52) is the law governing the propagation of discontinuities in second order derivatives along bicharacteristics (or rays in (x_α)-space) in the characteristic manifold $\varphi=0$. Since $\varphi_{tt}-c^2\varphi_{x_\alpha x_\alpha}$ is a known function along a bicharacteristic, (7.52) is a first order linear homogeneous ordinary differential equation along a bicharacteristic. Therefore the discontinuity $[w]$ cannot vanish at any point of a bicharacteristic on which it is somewhere different from zero.

*§ 8 HYPERBOLIC SYSTEM OF FIRST ORDER EQUATIONS

The results of the last section on the wave equation will be generalised in these sections to a hyperbolic system. The aim is mainly to introduce the concepts involved so that the reader will find it easy to go through a detailed theory of hyperbolic equations in more advanced books such as Courant and Hilbert (1962) Chapter VI.

*§ 8.1 Normal Conoid, Characteristic Conoid and Definition of a Hyperbolic System

To start with we shall not distinguish between the variables x_α and t. Our aim is to suitably indentify a time-like variable for a hyperbolic system and then use symbol t for it. Consider a system of n first order partial differential equations in the form

$$\sum_{p=1}^{m+1} B^{(p)} \frac{\partial U}{\partial x_p}+C=0 \tag{8.1}$$

for n dependent variables u_i forming the components of the columm vector U. Note that the range of the suffix p is 1, 2, ..., m, $m+1$. The system may be linear, semilinear or quasilinear. In the last case, we shall first take a known solution $U_0(x_p)$ and substitute it for the function U in the matrices $B^{(p)}$. However, we shall have to remember that our results are true only for the particular solution under consideration. The characteristic equation of (8.1) is a nonlinear first order partial differential equation

$$Q(x_p,\varphi_{x_p})\equiv\det\left[\sum_{p=1}^{m+1} B^{(p)}\varphi_{x_p}\right]=0 \tag{8.2}$$

where $\varphi=$ constant is a one-parameter family of characteristic manifolds. We set

$$\varphi_{x_p}=k_p,\ p=1,\ 2,\ \dots,\ m+1. \tag{8.3}$$

We shall discuss the algebraic property of the characteristic polynomial as a function of φ_{x_p} at a fixed point of (x_p)-space. Therefore we shall denote the characteristic polynomial $Q(x_p,\varphi_{x_p})$ by $Q(\varphi_{x_p})$. Using (8.3) we see that the characteristic equation

$$Q(k_p)\equiv\det\left[\sum_{p=1}^{m+1} B^{(p)}k_p\right]=0 \tag{8.4}$$

is a homogeneous algebraic equation of degree n in k_p and represents the equation of a conoid (in $(k_1, k_2, \ldots, k_m, k_{m+1})$-space) which is called a *normal conoid* at the point (k_p) (see equation (7.17) for the wave equation). The characteristic conoid is obtained as the envelope of the planes which are perpendicular to the generators of the normal conoid, i.e. the planes

$$k_1x_1+k_2x_2+\ldots+k_mx_m+k_{m+1}x_{m+1}=0 \tag{8.5}$$

where k_p satisfy the relation $Q(k_p)=0$.

Before proceeding further, we first discuss an example (Courant and Hilbert, 1962). Since we are interested only in the algebraic properties of the normal and characteristic conoids, we consider a single equation of higher order rather than the corresponding first order system.

Example 8.1: Consider the partial differential equation

$$\left[\left(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_2}\right)^2-\frac{\partial^2}{\partial x_1^2}\left(2\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_2}\right)\right]u=0. \tag{8.6}$$

The normal conoid of (8.6) is given by

$$Q(k_1, k_2, k_3) \equiv (k_3-k_2)(k_3+k_2)^2-k_1^2\,(2k_3+k_2)=0. \tag{8.7}$$

To trace this surface, we first note that the equation (8.7) is homogeneous in k_1, k_2, k_3 and hence represents a surface generated by the movement of a straight line passing through the origin. It is symmetric about the plane $k_1=0$ and the equation remains unchanged under the transformation $(k_1, k_2, k_3) \Rightarrow (-k_1, -k_2, -k_3)$. Therefore it is sufficient if we discuss the nature of the surface for $k_3<0$. We also note that the only generator of the surface in the plane $k_3=0$ is the k_1-axis (i.e. $k_3=0$, $k_2=0$).

To study the nature of the curve of intersection of (8.7) by the plane $k_3=\text{constant}<0$, we write (8.7) as

$$k_1^2=-\frac{(-k_3)+k_2}{(-2k_3)-k_2}(k_3+k_2)^2,\ k_3=\text{constant}<0 \tag{8.8}$$

and study it in a (k_1, k_2)-plane treating k_3 as a parameter. For $k_2>-2k_3$ and $k_2<k_3$, the right hand side of (8.8) is negative and hence there is no real value of k_1. The curve of intersection lies only in $k_3<k_2<-2k_3$. Let us trace the curve of intersection. As $k_0 \to -2k_3-0$, $k_1 \to \pm\infty$; so the line $k_2=-2k_3$ is an asymptote which is approached at each of its ends by two branches of the curve from below. For $k_2=\pm k_3$, the two values of k_1 coincide with zero. For all values of k_2 satisfying $k_3 \leqslant k_2 < -2k_3$, we get a pair of finite values of k_1. The point $(0, -k_3)$ is a node and the curve has no other singular point. The graph of the curve is shown in Fig. 8.1.

The normal conoid for $k_3<0$ is generated by a straight line whose one end is fixed at the origin and which is constrained to move on the curve shown in figure 8.1 in the plane $k_3=\text{constant}<0$. To get the full normal conoid, we have to take the generating straight line to be infinite on the other side of the origin also. The normal conoid of (8.6) is not closed.

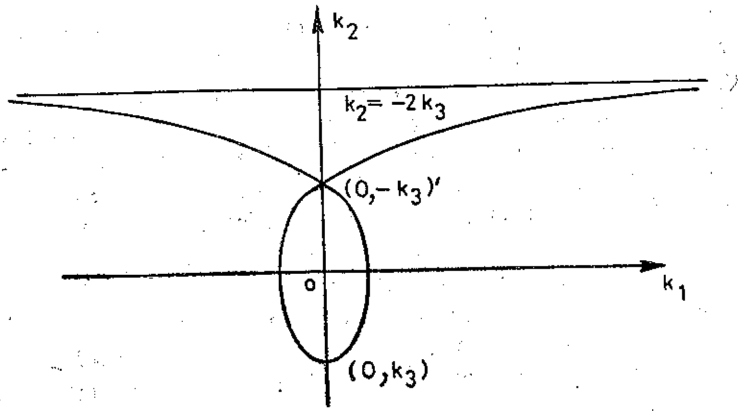

Fig. 8.1 The curve of intersection of the normal conoid (8.7) and the plane k_3=constant < 0

We also note that any straight line (except the k_1 axis itself) parallel to the k_1-axis intersects the conoid at most in two distinct points, a straight line parallel to k_2-axis intersects it either in one or two or three distinct points depending on the position of the straight line but a straight line parallel to the k_3-axis always intersects it in three distinct points (except that the straight lines lying in the plane $k_1 = 0$, they intersect at singular points). The straight lines parallel to k_3-axis and lying in the half space $k_2 > 0$ intersect the conoid at two points in the domain $k_3 < 0$ and at one point in the domain $k_3 > 0$. Existence of three distinct* points of intersection with the conoid by lines parallel to k_3-axis is equivalent to the statement that given arbitrary values of k_1 and k_2, the equation (8.7) always gives three distinct values of k_3. Thus we notice that the direction of k_3-axis plays some significant role for the equation (8.6).

We consider a particular section of the normal conoid by the plane $k_3 = -1$; this is a curve given by the equation

$$(-1-k_2)(-1+k_2)^2 - k_1^2(-2+k_2) = 0. \tag{8.9}$$

This curve is commonly known in literature as the *normal curve*. Similarly, we may consider the curve of intersection of the characteristic conoid by the plane $x_3 = 1$; this curve is called *ray curve*. From (8.5) it follows that the ray curve is obtained by taking the envelope of the straight lines

$$k_1 x_1 + k_2 x_2 = 1 \tag{8.10}$$

in (x_1, x_2)-space with k_1, k_2 related by the equation (8.9). Knowledge of the ray curve is sufficient for constructing the characteristic conoid.

*Except for those lying in the plane $k_1 = 0$. The equation (8.6) is hyperbolic in a sense more general than the definitions, given in the subsequent pages, which in strict sense refer to strongly hyperbolic equations.

The relation between the normal curve and the ray curve is reciprocal. As the point (k_1, k_2) moves on the normal curve, the straight line (8.10) envelopes the ray curve and conversely as the point (x_1, x_2) moves on the ray curve, the line (8.10) envelopes the normal curve. We further note that the equation (8.10), with k_1, k_2 as variables and with fixed x_1, x_2, represents the polar of the point (x_1, x_2) with respect to the unit circle about origin. Similarly, the same equation, with x_1, x_2 as variables and with fixed k_1, k_2 represents the polar* of the point (k_1, k_2) with respect to the unit circle. These results imply the following: the ray curve is the locus of the poles with respect to the unit circle of the tangents of the normal curve (8.9). The normal curve is as shown in figure 8.1. The ray curve for this problem has been shown in figure 8.2 as the curve *ABCDEFA*—and has been traced with the

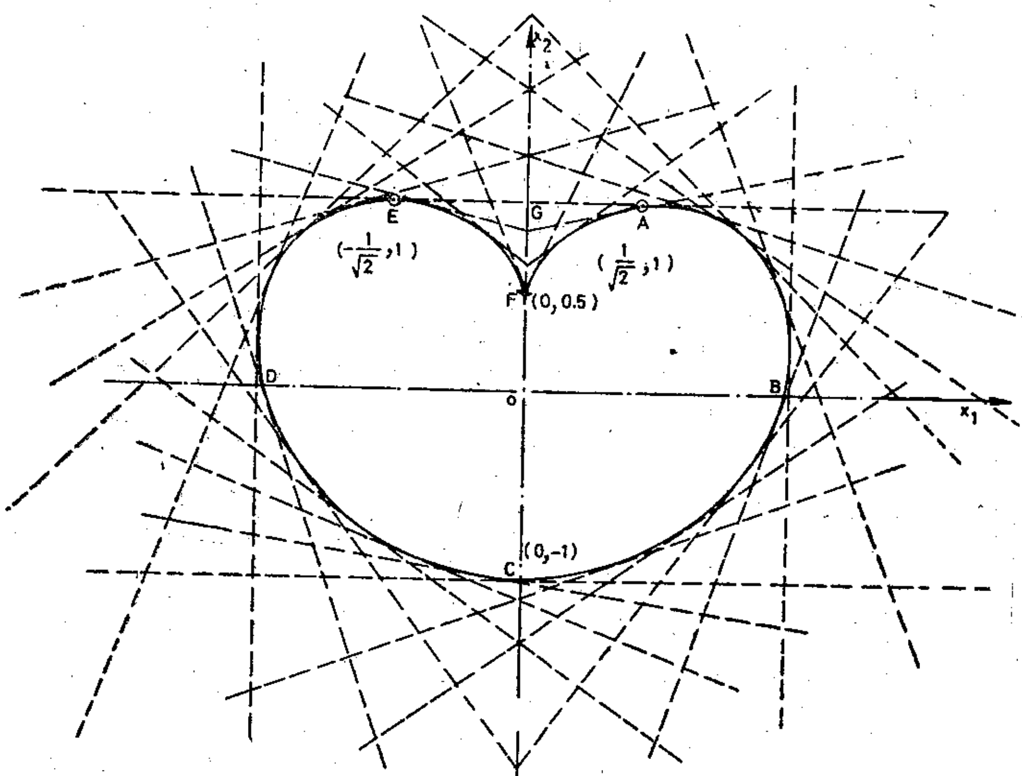

Fig. 8.2 Ray curve *ABCDEFA* of Eq. (8.6). The dotted lines are represented by (8.10) for different values of k_1 and k_2.

help of the straight lines (8.10) when k_1, k_2 satisfy (8.9). The ray curve is not convex; the closed loop of the normal curve lying below the point (0, 1) (note $k_3 = -1$) is mapped onto the convex portion *ABCDE* between *A* and *B*, and both the points *A* and *E* correspond to the node of the normal curve. The two branches of the normal curve above (0, 1) are mapped onto

*See G.T. Bell 'Coordinate Geometry of Three Dimensions'.

the concave portion EFA. The branches, extending towards points at infinity, of the normal curve give rise to a cusp at F. Further, if the normal curve has a double point (such as the node here), then the ray curve, or the characteristic conoid may not be convex and a lid such as EGA (a portion of the straight line) must be added to the ray curve to form its convex hull, where the convex hull is the envelope of the supporting lines (8.10). This gives rise to the convex hull Γ of the characteristic conoid. It is found that it is the convex hull Γ of the characteristic conoid which forms the boundary of the domain of influence of its vertex. In this particular example the domain of influence is larger than the domain bounded by the characteristic conoid.

The above example shows a few complicated geometrical features which may arise in the normal conoid and the characteristic conoid. As in the case of the above example, the role of the normal curve and ray curve can be recognised in the study of the normal conoid and characteristic conoid for an arbitrary system of equations and the concepts can be generalised in higher dimensions wherein they are called *normal surface* and *ray surface*. In what follows, we shall not go into the intricate geometrical features of these surfaces but concentrate only on other concepts which arise when the existence of these surfaces is assumed. We shall start with two definitions of hyperbolicity, based on algebraic criteria, which are easily verifiable. The motivation for these definitions has already been explained in § 7.2.

First definition of hyperbolicity: At a point $P(x_p)$, the first order system (8.1) of n equations is said to be hyperbolic if there exist directions $\boldsymbol{\zeta}=(\zeta_1, \zeta_2, \ldots, \zeta_m, \zeta_{m+1})$ such that all the straight lines (except the one passing through the vertex) parallel to the vectors $\boldsymbol{\zeta}$ intersect the normal conoid in exactly n distinct* points.

Algebraically this statement is the equivalent to the following one and if $\boldsymbol{\theta}=(\theta_1, \theta_2, \ldots, \theta_m, \theta_{m+1})$ is an arbitrary nonzero vector not parallel to $\boldsymbol{\zeta}$, then the equation

$$Q(\lambda\boldsymbol{\zeta}+\boldsymbol{\theta})=0 \tag{8.11}$$

in λ must have n real and distinct roots.

Space-like Surface: If a vector $\boldsymbol{\zeta}$ satisfying the above condition exists, the plane element at P orthogonal to ζ is called a space-like element. A surface in $(m+1)$-dimensional space (x_p) is defined to be space-like if its surface elements are space-like.

Second definition of hyperbolicity: An m-dimensional manifold (or an element of it), which we may, by suitable coordinate transformation, write as $x_{m+1}=0$, is called *space-like* if at every point of the manifold the equation

$$Q(k_1, k_2, \ldots, k_m, k_{m+1})=0 \tag{8.12}$$

*The equations (8.1) for which the normal conoid has n distinct sheets or equivalently for which the equation (8.11) has n real and distinct roots are called *strictly hyperbolic* equations. A discussion of general hyperbolic equations, where the roots of (8.11) are not necessarily distinct, is beyond the scope of this book (see Garding condition in Hörmander's book (1979)).

in k_{m+1} has n real distinct roots for arbitrary real values of $k_1, k_1, \cdots, k_m$. The first order system is called hyperbolic at a point P if space-like surface elements through P exist.

Equivalence of the two definitions: First let us assume that the system is hyperbolic according to the second definition. Choose $\boldsymbol{\zeta}=(0, 0, \cdots, 0, 1)$ and take an arbitrary vector, $\boldsymbol{\theta}=(\theta_1, \theta_2, \cdots, \theta_m, \theta_{m+1})$, not parallel to ζ. The equation (8.11) becomes $Q(\theta_1, \theta_2, \cdots, \theta_m, \lambda+\theta_{m+1})=0$ which according to (8.12) has n real and distinct roots λ. Hence a vector $\boldsymbol{\zeta}$ such that (8.11) has n real and distinct roots, exists. Therefore the system of equations, (8.1) is hyperbolic according to the first difinition and the plane $x_{m+1}=0$ is a space-like surface. Now we assume that a vector $\boldsymbol{\zeta}$ satisfying the first definition exists. We decompose an arbitrary non-zero vector $\bar{\boldsymbol{\theta}}$ into the sum of two vectors one parallel and another orthogonal to $\boldsymbol{\zeta}$ and combine the former with $\boldsymbol{\zeta}$ so that $\lambda\boldsymbol{\zeta}+\bar{\boldsymbol{\theta}}=\lambda\boldsymbol{\zeta}+\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is perpendicular to $\boldsymbol{\zeta}$. Next we choose the coordinate system such that $\boldsymbol{\zeta}$ has the components $(0, 0, \cdots, 0, 1)$; hence $\boldsymbol{\theta}$ has its last component 0. The equation (8.11) now becomes $Q(\theta_1, \theta_2, \cdots, \theta_m, \lambda)=0$ which has n real and distinct roots for λ. Therefore the system is hyperbolic according to the second definition.

The vectors of the type $\boldsymbol{\zeta}$, which are orthogonal to space-like elements, form the inner 'core' of the normal conoid bounded by the 'inner sheet' of the conoid. It can be proved that this inner core of the normal conoid is convex. Geometrically then, the normal conoid may be visualized as one consisting of the closed inner sheet bounding the inner core into which the normals to space-like surface elements point, and of further sheets which form subsequent shells around the core (see Duff, 1960). The outer sheets may be closed or may extend up to infinity. We can also prove that the boundary of the cone *supported* by the planes orthogonal to the generators of the convex inner sheet of the normal conoid is the convex hull Γ of the local characteristic conoid; more specifically, it is the hull of the 'outer shell' of the characteristic conoid.

Time-like direction and curve: Every direction from a point P into the convex hull Γ of the outer shell of the characteristic conoid at P is called time-like. A curve in the $(m+1)$-dimensional space is called time-like if its direction is everywhere time-like.

Now we notice that if a system (8.1) is hyperbolic at a point P, then there exist an m-dimensional space-like surface element at P and time-like directions through P. We can choose a local coordinate system at P such that the direction of the x_{m+1}-axis is time-like and the m-dimensional sub-space spanned by the unit vecrors along x_α coordinate axes $(\alpha=1, 2, \cdots, m)$ is space-like at P. We now move to a global discussion. We assume that we have a first order system of equations which is hyperbolic in a domain D_1 of the $m+1$ dimensional $(x_1, x_2, \cdots, x_{m+1})$-space and it is possible to introduce a coordinate system such that the x_{m+1} coordinate axis is time-like and the other axes lie in a space-like manifold at every point of D_1. We now designate the time-like coordinate x_{m+1} by t, the matrix $B^{(m+1)}$ by A and write the system (8.1) in the form

$$A\frac{\partial U}{\partial t}+B^{(\alpha)}\frac{\partial U}{\partial x_\alpha}+C=0, \quad \alpha=1,\,2,\,\dots,\,m. \tag{8.13}$$

The characteristic equation (8.12) with $k_{m+1}=-\lambda$ and $k_i=n_i$ becomes

$$\det\,[n_\alpha B^{(\alpha)}-\lambda A]=0 \tag{8.14}$$

which when solved for λ, according to our assumption, has n real and distinct roots (at every point of D_1 for arbitrary set of values of m constants n_α). Since the equation (8.14) is homogeneous of degree n in n_α and λ, it is sufficient if we choose

$$n_\alpha n_\alpha=1. \tag{8.15}$$

In this case, we denote the n values of λ by

$$c_1,\,c_2,\,\cdots,\,c_n \tag{8.16}$$

which are characteristic roots or velocities. Our assumption of hyperbolicity implies that the real roots $c_i(i=1,\,2,\,\dots,\,n)$ are finite in D_1. The necessary and sufficient condition for finiteness is that the matrix A is nonsingular in D_1, i.e.

$$\det A\neq 0 \quad \text{in} \quad D_1. \tag{8.17}$$

In all physical systems which evolve with time and which are governed by the hyperbolic equations, the time variable t is always time-like and the physical space containing the spatial coordinates x_α is always a space-like manifold in space-time. However, there are examples of time-independent physical systems which are governed by hyperbolic equations and where the time-like directions and space-like manifolds are not immediately clear. Example of such a system is the three-dimensional steady supersonic flow of a compressible gas where the Mach cone at a point plays the role of the characteristic cone. In this case the direction of the axis of the Mach cone is time-like.

Let us now go back to the hyperbolic system (8.13). For the simple root c_i of the characteristic equation (8.14), the matrix $n_\alpha B^{(\alpha)}-cA$ has rank $n-1$ and there exist unique (except for a scalar multiplier) left and right null vectors $l^{(i)}$ and $r^{(i)}$, respectively satisfying

$$l^{(M)}n_\alpha B^{(\alpha)}=c_M A, \qquad n_\alpha B^{(\alpha)}r^{(M)}=c_M Ar^{(M)}. \tag{8.18}$$

Unlike in the case of two independent variables, the characteristic velocity c_i, the left null vector $l^{(i)}$ and the right null vector $r^{(i)}$ not only depend on the position $(x_\alpha,\,t)$ in space-time but also on m arbitrary numbers $n_1,\,n_2,\,\cdots,\,n_m$ satisfying (8.15).

Definition of hyperbolicity can be extended for characteristics of uniformly constant multiplicity (see section 2.1).

The theory of the normal curve and the ray curve, presented in this section, is extremely important for basic understanding of waves in all physical systems governed by hyperbolic equations. For application of the theory to crystal optics and magnetohydrodynamics, reference may be made to Courant and Hilbert (1962), pages 599-617. For application to elasticity, reference may be made to Duff (1960).

*§8.2 Bicharacteristic Curves and Rays

Consider a system of first order equations (8.13). Its characteristic equation (6.15) is a first order nonlinear partial differential equation for the function φ. The characteristic curves of (6.15) are called *bicharacteristic curves* of (8.13). These are curves in space-time whose parametric representation is obtained after solving the ordinary differential equations

$$\frac{dt}{d\sigma}=\tfrac{1}{2}Q_q, \qquad \frac{dx_\alpha}{d\sigma}=\tfrac{1}{2}Q_{p_\alpha} \tag{8.19}$$

and

$$\frac{dq}{d\sigma}=-\tfrac{1}{2}Q_t, \qquad \frac{dp_\alpha}{d\sigma}=-\tfrac{1}{2}Q_{x_\alpha} \tag{8.20}$$

where

$$q=\varphi_t, \qquad p_\alpha=\varphi_{x_\alpha}\,. \tag{8.21}$$

The functions $p_\alpha(\sigma)$, $q(\sigma)$, $x_\alpha(\sigma)$, $t(\sigma)$ along a bicharacteristic curve, must satisfy the relation $Q(p_\alpha, q, x_\alpha, t)=0$. We note that a bicharacteristic curve lies in a characteristic manifold. Further a characteristic manifold $\varphi=0$ is generated by a $m-1$ parameter family of bicharacteristic curves.

Rays are the projections of the bicharacteristic curves on the hyperplane $t=0$.

If the coefficient matrices A and $B^{(\alpha)}$ are constant matrices, then $Q_t=0$ and $Q_{x_\alpha}=0$. This implies that p_α and q are constant along the bicharacteristic curves or rays. The equations (8.19) imply that the bicharacteristics (or rays) are straight lines in space-time (or in (x_α)-space).

Lemma on bicharacteristic directions: Consider a characteristic manifold, give by $\phi(x_\alpha, t)=$ constant, of the first order system (8.13) corresponding to a characteristic velocity c. We assume that c is simple. This implies that $n_\alpha B^{(\alpha)}-cA$, where $n_\alpha=\varphi_{x_\alpha}/|\text{ grade }\varphi\,|$, has a unique left null vector l and a unique right null vector r. Now we state the following important lemma:

With a suitable choice of the parameter σ the variation of x_α and t along a bicharacteristic curve lying on a characteristic of the above family, is given by

$$\frac{dx_\alpha}{d\sigma}=lB^{(\alpha)}r, \quad \frac{dt}{d\sigma}=lAr. \tag{8.22}$$

Proof: Let us denote the characteristic matrix by B:

$$B=\varphi_t A+\varphi_{x_\alpha}B^{(\alpha)} \tag{8.23}$$

so that the characteristic partial differential equation is $Q \equiv \det B = 0$. To obtain the derivatives of Q with respect to φ_t and φ_{x_α} we use the result that a derivative of a determinant Q is equal to the sum of n determinants Q_i, $i = 1, 2, \cdots, n$ where Q_i is obtained from Q by replacing its i-th row (or column) by the derivative of its i-th row (or column). Then, from the equations (8.19), expanding in terms of ith row (or column), we get

$$\frac{dx_\alpha}{d\sigma} = \sum_{i,j=1}^{n} B_{ij}^{(\alpha)} \beta_{ij} \text{ and } \frac{dt}{d\sigma} = \sum_{i,j=1}^{n} A_{ij}\beta_{ij} \tag{8.24}$$

where β_{ij} is the cofactor of the element B_{ij} in the matrix B. Since $Q = \det B = 0$, it follows that

$$\sum_{j=1}^{n} B_{ij}\beta_{ij} = 0 \qquad \text{for } i = 1, 2, \cdots, n \tag{8.25}$$

and

$$\sum_{i=1}^{n} B_{ij}\beta_{ij} = 0 \qquad \text{for } j = 1, 2, \cdots, n. \tag{8.26}$$

(8.25) shows that for a fixed i, the vector β_{ij} is a scalar multiple of the right eigenvector r and (8.26) shows that for a fixed j, the vector β_{ij} is a scalar multiple of l. Therefore (8.25) and (8.26) together imply that

$$\beta_{ij} = k l_i r_j \tag{8.27}$$

where k is a nonzero scalar.

With a suitable choice of the parameter σ, the equations (8.24) give

$$\frac{dx_\alpha}{d\sigma} = \sum_{i,j=1}^{n} l_i B_{ij}^{(\alpha)} r_j \quad \text{and} \quad \frac{dt}{d\sigma} = \sum_{i,j=1}^{n} l_i A_{ij} r_j \tag{8.28}$$

which are the same as the equations (8.22).

*§8.3 Compatibility Condition Along a Characteristic Manifold

Multiplying the system (8.13) by the left eigenvector and using (8.18) with $l^{(M)}$ and c_M replaced by l and c respectively, we get

$$lB^{(\alpha)}\left(n_\alpha \frac{\partial}{\partial t} + c\frac{\partial}{\partial x_\alpha}\right)U + clC = 0. \tag{8.29}$$

The unit normal (n_α) of the wave front and its speed c are related to the function $\varphi(x_\alpha, t)$ by (6.36) and (6.37) respectively. Therefore equation (8.29) becomes

$$lB^{(\alpha)}\left(\varphi_{x_\alpha}\frac{\partial}{\partial t} - \varphi_t\frac{\partial}{\partial x_\alpha}\right)U + c \mid \text{grad}_X\varphi \mid lC = 0. \tag{8.30}$$

For a given value of α, the expression $\varphi_{x_\alpha}\frac{\partial}{\partial t} - \varphi_t\frac{\partial}{\partial x_\alpha}$ represents an interior differentiation in the characteristic surface $\varphi(x_1, x_2, \cdots, x_m, t) = \text{constant}$. Hence in (8.30) (and therefore in (8.29) also) only the interior derivatives with respect to a characteristic manifold appear. Therefore equation (8.29) or (8.30) represents a compatibility condition on a characteristic surface.

A canonical form of the compatibility condition: Through any point on a characteristic manifold, there exists a bicharacteristic direction tangential to the manifold. Therefore the interior derivatives appearing in the compatibility condition can be written as linear combinations of a derivative along a bicharacteristic curve and other $m-1$ independent interior derivatives. We shall derive such a form of the compatibility condition assuming that the system is hyperbolic.

Let $l^{(i)}$ and $r^{(i)}$ be the left and right eigenvectors corresponding to the ith characteristic velocity $c_i (i=1, 2, \ldots, n)$. Let L and R be the $n \times n$ matrices with $l^{(i)}$ as the ith row of L and $r^{(i)}$ as the ith column of R. Let $S=[s_{ij}]$ be the inverse of R. Since $RS=I$, the identity matrix, we can write the compatibility condition

$$l^{(M)}A\frac{\partial U}{\partial t}+l^{(M)}B^{(\alpha)}\frac{\partial U}{\partial x_\alpha}+l^{(M)}C=0, \qquad M=1, 2, \ldots, n \tag{8.31}$$

in the form

$$l^{(M)}ARS\frac{\partial U}{\partial t}+l^{(M)}B^{(\alpha)}RS\frac{\partial U}{\partial x_\alpha}+l^{(M)}C=0. \tag{8.32}$$

Using the summation convention, we rewrite (8.32) as

$$l^{(M)}Ar^{(i)}s_{ij}\frac{\partial u_j}{\partial t}+l^{(M)}B^{(\alpha)}r^{(i)}s_{ij}\frac{\partial u_j}{\partial x_\alpha}+l^{(M)}C=0, \quad M=1, 2, \ldots, n. \tag{8.33}$$

Let us recollect now the result (2.36) regarding the biorthogonality of the left and right eigenvectors with respect to the matrix A:

$$l^{(i)}Ar^{(j)}\begin{cases}=0 & \text{if } i \neq j\\ \neq 0 & \text{if } i=j.\end{cases} \tag{8.34}$$

When we use this we note that the only nonzero term in the sum $l^{(M)}Ar^{(i)}s_{ij}$ is $l^{(M)}Ar^{(M)}s_{Mj}$. The equation (8.33) now becomes

$$s_{Mj}\left[(l^{(M)}Ar^{(M)})\frac{\partial u_j}{\partial t}+(l^{(M)}B^{(\alpha)}r^{(M)})\frac{\partial u_j}{\partial x_\alpha}\right]+\sum_{i\neq M}l^{(M)}B^{(\alpha)}r^{(i)}s_{ij}\frac{\partial u_j}{\partial x_\alpha}+l^{(M)}C=0,$$
$$\text{no sum over } M. \tag{8.35}$$

Let the equation of the bicharacteristic curve on a characteristic manifold of the Mth family (i.e. corresponding to c_M) in (x_α, t)-space be given by $x_\alpha=x_\alpha(\sigma_M)$, $t=t(\sigma_M)$. Then from the lemma on bicharacteristic directions, we can suitably choose σ_M such that

$$\frac{dx_\alpha}{d\sigma_M}=(l^{(M)}B^{(\alpha)}r^{(M)})/(l^{(M)}Ar^{(M)}), \quad \frac{dt}{d\sigma_M}=1. \tag{8.36}$$

Now, the equation (8.35) becomes

$$s_{Mj}(l^{(M)}Ar^{(M)})\frac{du_j}{d\sigma_M}+\sum_{i\neq M} l^{(M)}B^{(\alpha)}r^{(i)}s_{ij}\frac{\partial u_j}{\partial x_\alpha}+l^{(M)}C=0,$$

no sum over M. (8.37)

Since $dt/d\sigma_M=1$, the derivative operator $d/d\sigma_M$ gives the time rate of change of a quantity as we move along the bicharacteristic curves of the Mth characteristic family. The form of the compatibility condition (8.37) is worth noting. The derivative along the bicharacteristic direction has a very special status in that it is the only one which contains the time derivative $\partial/\partial t$; the other interior derivatives $l^{(M)}B^{(\alpha)}r^{(i)}s_{ij}\,\partial/\partial x_\alpha$ on the characteristic manifold contain only spatial derivatives. Therefore from the knowledge of the data at any time t, all the quantities including $\partial u_j/\partial x_\alpha$ can be found out and the derivative operator $d/d\sigma_M$ can be used to find the values of the quantities at the next higher time step $t+\Delta t$ on a particular bicharacteristic. More precisely, we can integrate the equations (8.37) and construct an iterative scheme for solving a Cauchy problem as in the case of two independent variables. The canonical form (8.37) of the compatibility condition was derived by Prasad and Ravindran in 1978.

*§8.4 Propagation of Discontinuities of First Order Derivatives Along Rays

Consider a linear hyperbolic system

$$\mathcal{L}U\equiv A(x_\beta,t)\frac{\partial U}{\partial t}+B^{(\alpha)}(x_\beta,t)\frac{\partial U}{\partial x_\alpha}+F(x_\alpha,t)U=f(x_\alpha,t) \qquad (8.38)$$

where F is an $n\times n$ matrix and f is a column vector. As in the case of the section 2.3, we could use the canonical form (8.37) to discuss the propagation of discontinuities in the function U. However, we shall concentrate here on the discontinuities in the first order derivatives.

Let us take a solution $U(x_\alpha, t)$, of the above system, which is C^1 in a domain D of the (x_α, t)-space except for the jump discontinuities in first order derivatives of U across a surface $S:\varphi(x_\alpha,t)=0$ which divides D into two sub-domains D_1 and D_2 (see Fig. 7.6). We need not repeat the arguments of the sections 2.3 and 7.7 but briefly mention that if we use a new coordinate system (x'_α,φ) given by (7.42), then all the interior derivatives $\partial U/\partial x'_\alpha$ are continuous across S, but the exterior derivative $\partial U/\partial\varphi$ must be discontinuous.

Writing the equation (8.38) in terms of the coordinates (x'_α,φ) (see (7.42)) we get

$$(A\varphi_t+B^{(\alpha)}\varphi_{x_\alpha})\frac{\partial U}{\partial\varphi}+B^{(\alpha)}\frac{\partial U}{x'_\alpha}+FU=f. \qquad (8.39)$$

As described in the section 7.7, we take the jump of this equation across S and get

$$(A\varphi_t+B^{(\alpha)}\varphi_{x_\alpha})[U_\varphi]=0. \qquad (8.40)$$

This implies that $S : \varphi(x_\alpha, t) = 0$ is a characteristic surface and the jump in the exterior derivative is given by

$$[U_\varphi] = rw \tag{8.41}$$

where r is a right eigen vector corresponding to a characteristic velocity $c(\equiv -\varphi_t/|\text{ grad } \varphi |)$ and w is a scalar function defined on S. Multiplying (8.39) by the corresponding left eigen-vector l, using $l(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}) = 0$ and differentiating the result with respect to φ we get

$$(lB^{(\alpha)})_\varphi \frac{\partial U}{\partial x'_\alpha} + (lB^{(\alpha)}) \frac{\partial}{\partial x'_\alpha}\left(\frac{\partial U}{\partial \varphi}\right) + (lF)_\varphi U + (lF)\frac{\partial U}{\partial \varphi} = (lf)_\varphi \tag{8.42}$$

In the equation (8.42) all quantities except $\partial U/\partial\varphi$ are continuous across S. Hence taking the jump across S, we get

$$(lB^{(\alpha)}) \frac{\partial}{\partial x'_\alpha}[U_\varphi] + (lF)[U_\varphi] = 0. \tag{8.43}$$

Substituting $[U_\varphi]$ from (8.41) we get

$$(lB^{(\alpha)}r) \frac{\partial}{\partial x'_\alpha} w + \left(lB^{(\alpha)} \frac{\partial r}{\partial x'_\alpha} + lFr \right) w = 0. \tag{8.44}$$

Using the lemma on bicharacteristic directions, i.e. the result (8.22) and changing from (x_α, t) coordinates to (x'_α, φ) we find that the operator giving the rate of change along the bicharacteristic curve is

$$\frac{d}{d\sigma} = (lAr)\frac{\partial}{\partial t} + (lB^{(\alpha)}r)\frac{\partial}{\partial x_\alpha} = lB^{(\alpha)}r \frac{\partial}{\partial x'_\alpha}. \tag{8.45}$$

We also note that

$$l\left(A\frac{\partial}{\partial t} + B^{(\alpha)} \frac{\partial}{\partial x_\alpha} \right) = lB^{(\alpha)} \frac{\partial}{\partial x'_\alpha}. \tag{8.46}$$

Therefore the equation (8.44) can be written as

$$\frac{dw}{d\sigma} + l(\mathcal{L}r)w = 0 \tag{8.47}$$

where $\mathcal{L}$ represents the linear differential operator appearing on the left hand side (8.38).

Along a bicharacteristic curve the function $\mathcal{L}r$ can be expressed as a function of σ. Therefore equation (8.47) is a linear homogeneous ordinary differential equation for an amplitude w of the discontinuity and gives the rate of change of w along the bicharacteristic curves on the characteristic surface. It follows immediately that if there exists a discontinuity in the normal derivative of U at some point of a characteristic surface, it persists (i.e. it remains nonzero) at all points on the bicharacteristic curve through that point. Interpreted in the language of wave propagation, *discontinuities propagate along rays.*

We have seen in this chapter that the propagation of discontinuities is a remarkable feature of hyperbolic equations. The analysis of propagation of discontinuities gives rise to the concept of generalised solutions which are the physically meaningful solutions. The structure of solutions of a hyperbolic equation is dominated by characteristic surfaces and rays. The main features of the solution can be analysed by using the essential character of the differential operator along the characteristic manifolds.

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Index