Moments of Supreme Happiness and Satisfaction in My Research

Phoolan Prasad
Department of Mathematics, IISc, Bangalore

November 20, 2020

Contents

Chapter 1

Happiness and Satisfaction in Research

0 Introduction - Different Levels of Happiness and Satisfaction in Research

I describe some different types of happiness below.

1. When a student in a school or in a college or university solves a difficult problem in his text book or examination, he derives great happiness. However, these are problems solved also by many other students.

2. There are also researchers whose research involves just a trivial extension or tempering of some existing results. I think these researchers do not derive much happiness, even though they may use some sophisticated mathematical tools.

3. There are some researchers, who have a good grasp of their areas of research, know the current research, create their research problems, which they think are new and worth solving. I do not like to make distinction in applied and pure mathematics (see my definition of "applied mathematics" in (3c) below) but I do it because I have introduced an engineering type of research in (3b) below.

   - (3a) They create new mathematical problems out of their own mind or make new mathematical formulations or models from areas of application. When these problems are solved they lead to a deeper understanding of some areas of mathematics and, if created from areas of application, a deeper understanding of those areas of application.

   - (3b) In applied mathematics, a researcher perceives that a physical (or social or economic) phenomenon looks mysterious and needs a mathematical explanation. The mathematics used by him is not new and may even be elementary. There is great joy when the mysterious phenomenon is explained. I have called it an engineering type of research. I shall give an example of this type of my own research in Section 13.

- (3c) Again in applied mathematics, an attempt to understand a mysterious phenomenon leads either to a new development of mathematics or brings out a new mathematical structure in the governing equations of the phenomenon. I call such developments to be real "applied mathematics" by this I mean "applied mathematics is a creativity in mathematics for a deeper understanding of a phenomenon".

When the research is completed of the type (3a), (3b) or (3c), one derives supreme happiness and satisfaction, though not all results will be of the same value.

4. There is an old problem, which many mathematicians have failed to solve. They either solve it or contribute significantly to show a path to solve it.

5. Some new theory is being developed by a large number of scientists but though many contribute partially, one scientist perceives a deeper understanding and everything becomes transparent. There are also mathematicians, who develop new ideas which give a strong foundation of some topics. I find it very difficult to write this part and hence I quote M. J. Lighthill's dedication in his book [32] on generalised functions "To Paul Dirac, who saw that it must be true, Laurent Schwartz, who proved it, and George Temple, who saw how simple it can be made". Note that Schwartz was awarded the most respected award in mathematics, namely the Fields Medal in 1950. His theory of distributions clarified the (then) mysteries of the Dirac delta function and Heaviside step function. It helped to extend the theory of Fourier transform and is now of great importance in the theory of partial differential equations.

Researchers in items 4 and 5 derive supreme happiness and satisfaction and they become instantaneously famous. There are many outstanding researchers who have made such contributions to mathematics and science several times in their life.

Note 0.1. *In this article I shall also describe some incidents, the circumstances in which my students and collaborators have worked, and my encounters with some outstanding researchers but they do not form parts of "Moments of My Supreme Happiness and Satisfaction in My Research". I shall put them under footnotes.*

1 My Graduate Level Preparation for Research

In order to understand the choice of my research problems, it will be better if the reader is aware of my graduate level training, which I describe first.

1.1 I did had a good formal training in ordinary differential equations (ODE)

It started with classical treatment from D. M. Murray's 1897 book at Presidency College, Calcutta in 1960 and continued later extensively in M.Sc. at Science College, Calcutta University, both for $Z \in \mathbb{C}$ as function of a complex variable z and a functional analytic approach for a system of ODE for $Y \in \mathbb{R}^n$ as a function of a real variable x. I continued learning ODE and teaching it almost till 1995, the last one was a course based on V. I. Arnold, ODE.

1.2 I did not have a formal training in partial differential equations (PDE)

But I got a very good feeling of the subject through 3 intensive courses in fluid dynamics over a period of 2 years 3 months (equivalent to 8 to 9 semester courses today) in M.Sc. in Applied Mathematics, University of Calcutta. In the first basic course dealing with incompressible inviscid fluid mechanics, I learnt properties of Laplace equation. I leant more about the properties of the Laplace equation in a course on "Potential Theory". In an intensive course on viscous flows, I learnt derivation and properties of Navier-Stokes equations and also about diffusion and turbulence. Finally in a course on compressible flows, I learnt about the wave equation, quasilinear hyperbolic PDE and shocks. My training in PDE was by self reading of Sneddon's book - PDE during 1965 and many research papers on nonlinear wave propagation, unfortunately that was all till I started writing my thesis on nonlinear waves and radiation gas-dynamics in May, 1967.

1.3 Graduate level training was in fluid mechanics

I was admitted to Indian Institute of Science on 9th April 1965 as a research student and financial support came from a Research Fellowship from the Ministry of Defence, Govt. of India. My research supervisor Prof. P. L. Bhatnagar[1] (PLB) gave me a list of 10 books suitable for further training in fluid dynamics and topic of my research "blast waves". Following the list, from April to December 1965, I read those books, which included Courant and Friedrichs [16] (I completed study of this beautiful book in 40 days, which I still remember), Schlichting [73], two volumes of Howarth [26], water waves by J. J. Stoker [77] and a book on thermodynamics. Since intense explosions needed knowledge of interaction of radiation with matter, during 1966, I studied S. Chandrasekhar's Stellar Structure [15] (full book) and his Radiative Transfer [14] (only the first five chapters).

[1] PLBs joint research work "BGK model" is one of the most celebrated works in Plasma Physics. It is a model to solve the formidable Boltzmann equation with a very sound mathematical basis (research of the type (3c) in section 0). It is now a text book material and it continues to make waves even today due to its many ramifications; see *http : //math.iisc.ernet.in/ prasad/prasad/plbhatnagar.pdf*. PLB is also well known for his work in astrophysics (inspired by the famous M. N. Saha) and many areas of fluid dynamics.

My research supervisor PLB was very happy that I chose and created my research problems on my own after some discussion with him. Generally, I used to meet him once or twice in a month. Except for a paper containing a minor research problem (also chosen by me), I have only two joint papers with PLB both in Proceedings of Royal Society (PRS) in 1970 and 1971, but for these two papers he got so much interested in a (completely new for us) beautiful and very deep theory that we used to meet almost every day for more than an hour at Bangalore and Jaipur.

2 Self Study and Research Without Much Official Duty 1965-1975

When I was admitted in IISc (9th April, 1965), I was allotted a desk in the corridor behind the present Faculty Hall of IISc. Within a few months, I was shifted to a shared room in a temporary building of the Applied Mathematics Department, where PLB had a very elegant two room office (in which he interviewed me). Sitting quietly in this shared room, I completed a detailed study of the 10 books PLB gave me. After that, my place of study was a table in the very rich library of IISc, where I continued reading relevant research papers. But I spent evenings reading books on topics not directly related to my research.

In the first year I made two good friends: Renuka Rajagopalan (Renuka Ravindran after her marriage) and R. Jayakaran Isaac. With Jayakar I shared a room in the hostel for two years and a flat in Vyalikaval for another two years - for about 6 months with our wives. Renu also moved to a house in Vyalikaval close to our house with her very big family. Friendship with Renu and Jayakar was family friendships and as if we were three siblings: two brothers and a sister - sharing all happy and difficult times. Writing about these friendships will take many pages. Our children also enjoyed the close relationships with Renu's and Jayakar's children.

Renu was full of life, very hard working in research and she also completed her PhD thesis in just two years of joining IISc. PLB liked her dedication very much. With Renu and Jayakar I used to organise evening lectures in 1967, in a big room behind the library, to which some other students also used to join. When a senior faculty in the next room complained to PLB that the seminars were disturbing, PLB told him "they are doing good work, learning from each other". Another senior lady faculty, afraid that Renu will develop more confidence with PLB, complained to him that Renu was criticising a research work of PLB. He was furious and called me to his room and asked "Phoolan, tell me correctly about it". I, frankly and without any fear, told PLB that the complaint was not true. PLB trusted me and the senior faculty lost all the faith PLB had in her.

In December, 1972, there was an Indo-US binational conference in mathematics, the venue was Hotel Ashok in Bangalore. The leader and deputy leader of the Indian team were PLB and R. P. Bambah respectively and the leader of the American team was Marshall

Harvey Stone. Reporters of applied and pure mathematics sections were me and M. S. Raghunathan. The conference had a good effect in teaching mathematical analysis in India but not in differential equations.

This was the period of intensive learning and lots of research. In 1975, I joined PLB at Mehta Research Institute (MRI), Allahabad as a Visiting Professor, IISc granted me leave on duty.
I describe in the following subsections the main points of my work during 1965-75.

2.1 My First Research

After I had completed study of 10 books, PLB gave my colleague P. L. Sachdev (PLS) and me a classical paper by von Neumann and Richtmyer (vN&R) [80], in which the equations of gasdynamics were modified by the inclusion of additional artificial viscosity terms for stepwise numerical solution of the equations in problems involving shocks. After a week, we went to him and I asked "should we try to develop a numerical scheme with only artificial heat conduction term, which modifies only the energy equation leaving the momentum equation undisturbed?". PLB answer was quick "go ahead and try it". We took less than a week to see that all necessary conditions of vN&R are satisfied. We did some initial computation on the hand driven Facit calculator and then we used the Elliot computer at NAL, where we could even recognise the digits in changing numbers on display (very slow computer). PLB read and meticulously modified almost every paragraph of the manuscript we prepared and the paper [72] was published in 1966 in J. Phys. Soc., Japan, without further modification.

I did not get the thrill of deriving a new numerical scheme, because we followed various steps of vN&R's paper. However, experience with programming and actually doing computation was very useful in my research later.

2.2 PLB Asked Me to Work Separately

My collaborator PLS was deeply involved in blast waves problems in astrophysical context and following previous researchers, he was quite convinced that radiation flux should be discontinuous across every shock. I thought that it was not correct and required deeper investigations. Following my idea we published a joint paper in September, 1968 in Publs. Astron. Soc. Japan. Thus even as late as in 1965, the role of radiation on a shock wave was not well understood.

While discussing with PLS and me, PLB saw the difference in our approach to research, which, instead of increasing our productivity, was actually retarding. Then PLB asked both of us to work separately since the middle of 1966. PLS was very hard working and quite productive in research but relied more on following ideas of former investigators and I questioned them and chose my own path, deriving results from a few basic axioms relevant to the physical phenomena - type of applied mathematics mentioned in (3c) in Section 0.

2.3 Dreaming of Radiation Crossing a Shock

Intensive arguments with PLS led me thinking deeply on the above point for months. In a light sleep at about 4 am, in a dream I saw radiation flux continuously passing through a shock. I got up immediately and surprisingly the reasons for the radiation flux to be continuous became very clear. This was a moment of supreme happiness, satisfaction and a great event for my research. Let me describe the outcome of dream-cum-realisation briefly in the next heading.

2.4 Formulation of a New Set of Equations of Radiation Gas Dynamics (RGD)

Since I had read the two books [14, 15], I succeeded in formulating a new set of differential equations for RGD (by approximating the integral form of the radiative transfer equation by differential equations). One the papers of G. B. Whitham[2] (W-3) guided me in understanding the physical implications of various terms in my equations. It was also of great value in my learning various aspects of linear and nonlinear waves. When I presented a draft of a long paper to PLB, he commented that it was too intuitive (Whitham's style) and asked me to mathematically justify most of the steps. I took a month and presented the results again. He split the paper, big in size, in two papers, first on linear waves in RGD and second one on nonlinear waves in RGD. Since I was working in a project from ministry of defence, the first one was sent for publication in Defence Science Journal (1967) and the the second one [38] in the Journal of IISc; for some reason PLB desired quick publications[3]. Same year, I discussed shock structure in great detail, which was published later in 1969 in Quart. J. Mech. and Appl. Maths; see [40].

I showed that the optical thickness of the medium plays an important role on the nature of the waves to be observed at various spatial scales. A radiative shock is actually a gas dynamics shock across which radiation energy and radiation pressure are continuous and this surface of discontinuity is embedded in a radiative shock structure joining two uniform states. In certain cases the strength of the embedded discontinuity may tend to zero giving a continuous profile in the shock structure. My RGD equations formed a hyperbolic system of partial differential equations with five distinct (and finite) characteristic velocities, the

[2] I have mentioned four papers of Whitham as W-1, W-2, W-3 and W-4, see later in Section 4.

[3] I think, PLB wished quick publication of my first few papers because he wished that I should get selected as lecturer in IISc early. After the interview, maybe in September 1967, PLB came to my room and asked me to go for a walk with him. When we had walked a little, he said "Phoolan, you have stood first in the list of selected candidates in the interview today, promise me that you will not leave India". He could not find free time to go through my thesis till October, 1967. I submitted my thesis in November, 1967 and Ph.D. degree was awarded in June, 1968.

signal velocity being comparable to the speed of light. This work was mentioned in Annual Review of Fluid Mechanics, **3**, 1971 by Vincenti and Traugott for the derivation of a new set of equations for RGD with finite speed of light and for examples of nonlinear waves in RGD.

2.5 Phase Plane Diagram

The paper [40] "a class of one dimensional steady state flows and general shock structure problem in RGD" required drawing of the integral curves of an autonomous pair of nonlinear ordinary differential equations giving a qualitative picture of the phase plane. Maybe there exists today soft wares to do this in minutes, but I had noticed the diagram on the cover page of Chandrasekhar's books [15]. There were five different cases and for each case I first drew the complicated isoclines and on each isocline I drew the line elements of the integral curves with the help of a scale and a set square. I joined the line elements to approximately draw curves giving steady solutions. Each figure took about one to two weeks. But it was thrilling to see that, at the end, I was able to discuss all steady flows, which also included the various cases of shock structures.

The diagram on the cover page of [15] represents a phase plane with a singular point, which is a spiral point. There are more attractive diagrams in side the book on pages 138 and 143 with two singularities. Each of the five phase planes, I drew, has three singular points and gives a global picture of integral curves to discuss all types of steady solutions. It was quite complex but I could join the two states at infinity uniquely either through the sonic singularity or jumping across the sonic singularity. I also obtained an interesting approximate expression for the solution with an embedded gas dynamics shock in a RGD shock structure between two states at infinity at $x = \pm\infty$; see [39]. This simple result, leading to a deep understanding of a physical phenomenon, gave me a lot of satisfaction.

2.6 Completing the Research for the Thesis

In April, 1967, PLB asked me to start writing the thesis excluding one of my work on numerical solution depicting one-dimensional blast wave in a tube closed at two ends[4]. In this I could capture multiple reflections of the shock from the two boundaries of the tube. My supervisor's asking me to write the thesis was a matter of some surprise since I had completed only one year and three months since I chose my first research problem and only 2 years of joining IISc as a research student but it also gave me great satisfaction.

2.7 Stability of Steady and Self-Similar Flows Near Singularities

In the middle of 1967, I realised that further research in RGD will require working with more physically realistic transport equation containing frequency dependent radiation, which for

[4]PLB said that the thesis would become too big.

me was too complex to develop simple and elegant mathematics, and I would be deprived of learning good mathematics.

After my detailed study of steady state flows in RGD in Section 2.5 had introduced me to flows near singularities, I found a beautiful paper by Kulikovskii[5] and Slobodkina (K&S) [29]. It appealed me because it dealt with equilibrium of steady transonic flows passing through a "sonic type" of singularity of an arbitrary quasilinear system of n-equations in two independent variables x and t, of the form

$$A(x, \boldsymbol{u})\boldsymbol{u}_t + B(x, \boldsymbol{u})\boldsymbol{u}_x + \boldsymbol{C}(x, \boldsymbol{u}) = 0, \tag{1.1}$$

where $\boldsymbol{u} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $\boldsymbol{C} \in \mathbb{R}^n$ with the only assumption that a characteristic velocity c is real and simple. Hence (1.1) need not be hyperbolic. The "sonic type" of singularity corresponds to a point where $c(x, u_0(x))$, the characteristic velocity c in a steady solution $u_0(x)$, vanishes (say at $x = 0$). PLB also got deeply interested. We used to study this paper together (for the first time) everyday for about one to two hours and I used to work out details[6].

By this time I had read English translation of two important Russian books: (1) B. Zel'dovich and Yu. P. Raizer *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, Vol2, 1967 and (ii) L. I. Sedov *Similarity and Dimensional Methods in Mechanics*, 1959.

The K&S transformed the approximate equation governing the perturbations of a steady solution in the neighbourhood of the singularity to another equation, which governed not only the evolution of perturbations but all steady solutions. Therefore, they could study all steady flows near the singularities and evolution of perturbations of any one of the steady flows in the phase plane of the characteristics equations (autonomous system) of a very simple equation

$$c_t + cc_x = \alpha c + \beta x, \tag{1.2}$$

where α and β are constants. Such a general study is possible because α and β turn out to be independent of the steady solution on which the perturbation has been made.

[5]I met Prof. Kulikovskii in 1990, see a footnote in Section 18

[6]This was in 1969, and PLB knew that he would be leaving IISc forever. He was quite concerned about my future at IISc as he felt that no one at IISc would understand the quality of my research. When he was in Egypt in an advisory committee of higher education there, he wrote a letter in his own hand writing to me giving me a hint to apply for an Assistant Professor's position in IISc but I was did not understand the hint as I had completed less than two years as Lecturer. On his return, he was looking at the applications and he did not find mine in the bundle. He was very angry with me and called me to his room. He handed over a note to take it to the Registrar, who immediately issued an application form and thus my application for the next higher post was submitted before PLB left IISc to take the position of the V.C. of Rajasthan University in Jaipur. The interview took place after 2 years and I was selected in absentia as Assistant Professor in 1970 but I joined it on 13th July 1972 after 1 year 8 months on my return from U.K.. 13 is supposed to be an unlucky number, which I did not believe.

As far as I know, this is the only approximate equation which governs not only the evolution of perturbations but also the basic steady state on which perturbations have been created. It has many novel features (see [12]), I need not describe them here.

We used K&S theory to prove that all steady flows in RGD are stable in the neighbourhood of the sonic point, derived conditions for appearance of discontinuities in perturbations and finally extended this theory to study stability of self-similar flows. We also showed that all imploding strong shocks are stable for radially symmetric perturbations. The paper [12] was communicated to PRS, the editor accepted it but instructed that it was very condensed and needed more explanation for easy reading. When we revised the paper, the number of pages became one and half times more. Tagare and I [79] used the method of [12] to show that the similarity solutions of C. Hunter (JFM - 1963) were unstable and hence not physically relevant.

This was a further encouragement to my learning more theoretical aspects of hyperbolic systems and research in new mathematical structures in these equations. This comment is important because in those days, research in applied mathematics in India was understood as research in an area of application of the type (2) in Section 0 i.e., without creativity in mathematics. The work of the type (3b) was rare in India. It was only good luck for me that in applied mathematics I got PLB as my research supervisor, who was basically a mathematician and he was happiest in learning and in creativity in mathematics.

Soon I got interested in extending the theory of K&S to a general system of quasilinear equations for the case when a real characteristic velocity c is multiple of multiplicity s. I constructed about ninety examples from which I could get an idea about the axioms required for the extension: the eigenspace of the real eigenvalue c should be complete i.e., there should be s linearly independent eigenvectors (without any restriction on other eigenvalues, which need not even be real). When we started working, the algebraic expressions were too complex but after very long calculations we were able to transform the perturbation equations to a system of s PDEs

$$\frac{\partial \pi_i}{\partial t} + c\frac{\partial \pi_i}{\partial t} = \sum_{r=1}^{s} g_{ir}\pi_r + h_i x, \quad i = 1, 2, \dots, s, \tag{1.3}$$

where c is expressed in terms of the new state variables $\pi_1, \pi_2, \dots, \pi_s$ as

$$c = Rx + \sum_{r=1}^{s} c_{w_r}\pi_r, \tag{1.4}$$

with g_{ir}, h_i, R and c_{w_r} as constants. It was really nice to see that very long and complex calculations lead to a set of s simple and beautiful equations with expressions of the coefficients independent of the particular steady solution, on which perturbation was made.

These equations govern not only the perturbations but also all steady solutions in the neighbourhood of a "sonic type" of singularity at the point $x = 0$, thus the steady solutions and their stability can be studied in $(s+1)$-dimensional phase space of $x, \pi_1, \pi_2, \ldots, \pi_s$. We also presented examples of some applications. Completion of the theory in this paper, while my wife Mandra and me were guests of PLB in Jaipur, was a moment of great happiness and satisfaction. PLB had left IISc and was the Vice Chancellor of Rajasthan University.

2.8 Creation of New Research Problems

At this stage it is worth mentioning that I never worked in a fashionable area but created my own research problems all throughout my research career. 1965 - 1970 was the period, when intensive research was going on in KdV equation, soliton theory (by C. S. Gardener, J. M. Green, M. D. Kruskal[7] and R. M. Miura) and associated general theory by Peter Lax. I was reading some of the papers and collected about 30 reprints, which played an important role[8] in preparation of PLB's book [11]. However, it was impossible for me to make any good contribution to this area, though my own research student S. G. Tagare started working on solitons and later I wrote (jointly with PLS) a review article (published in 1975) on KdV equation. There is an advantage in creating good research problems in a new area because there is no competition. But, for me, there was a disadvantage also because there were no followers except my own small group, which included just a few mathematicians from U.K. and Germany. Once we showed that a method of calculating the shock position by Whitham required serious modification. The well known scientist, whom I had met earlier and who was once keen to my visit to California Institute of Technology to work together, was furious. I shall describe this incident later in Section 17.

2.9 Stability of Transonic Flows on an Aerofoil

Without knowing the full history of transonic controversy, which I shall describe later, I decided to extend K&S theory to a hyperbolic system in multi-space dimensions (represented by (1.12) below) first as a mathematical problem and then wished to study a nonlinear pulse moving slowly in a transonic flow next to the surface of an aerofoil moving with high subsonic velocity, say from right to left. K&S dealt with a localised one dimensional pulse, each point of which moved with a velocity depending on the value of the amplitude at that point. In one space dimensional case each moving point on a pulse is a wavefront. In multi-dimensions (muli-D, by this I shall mean multi-space dimensions) we consider a localised

[7]Kruskal was my guest for a week at IISc during the middle of nineteen eighties.

[8]When I went to Mehta Research Institute (now renamed as Harish-Chandra Research Institute) at Allahabad, I handed over all the reprints to PLB. He took interest in the KdV equation, tried to promote it in India by holding a month long lecture workshop in 1976 (with only two resource persons, he and me) and wrote a book [11]. This book is available (with permission from OUP) by typing (please do not copy and paste) in a browser $https://drive.google.com/file/d/1Ffohxed1I1ZAPYR54rebewpENIn_cIUu/view$ Much later, academician Vladimir E Zakharov (see a footnote in item 3 in Section 21.3) was my guest at IISc and told me that he was surprised to see the first book (a good book - he emphasised) from a country where no contribution to the subject was made. Soon after its publication, the book was translated into Russian.

pulse generated by a quasi-one dimensional nonlinear wavefronts, each front is almost plane with normal direction varying slowly along the front.

Since October 1970 I was a post doctoral fellow at University of Leeds, UK, and Prof. F. A. Goldsworthy (FAG), with whom I was associated since October, 1970, gave me free hand to work on my problems[9]. I had an independent room of my own overlooking a very big sloping lawn in the middle of the university buildings. I such a beautiful setting, I could concentrate reading books (manly Courant and Hilbert [18]) and papers, and on thinking deeply about the the conditions for waves to be trapped in a supersonic pocket above an aerofoil[10]. Christmas vacation started, my wife Mandra and daughter Deepika had not yet joined me at Leeds and I did not have good friends to spend time with. I used to go to the university and work almost alone in the department. In such a desolate situation of Christmas vacation, one day it occurred to me that the trapping of a multi-D pulse is possible if the pulse is generated by fronts almost normal to the streamlines and all components of the ray velocity (or bicharacteristic velocity[11]) vanish at some point on the sonic line. I also realised that a pulse trapped in the neighbourhood of a point on sonic line (downstream boundary of the supersonic pocket) across which a flow decelerates from supersonic state to subsonic state, a wavefront convex to the negative direction of the fluid velocity will tend to become plane normal to the stream lines, and that concave to fluid velocity will become more curved and will move out of the supersonic pocket.

This was a a moment of supreme happiness for me. I walked to my apartment (a single room with a kitchen but a shared bathroom) crossing the Hyde park and took some food. Nature started welcoming me with the first snowfall of my life and I walked for an hour in the snow with a thrilling experience. I just did not think of my research that day. Next day I checked and rechecked my realisation. Then, I made an assumption that there exists a real characteristic velocity c of the system and for a particular value of unit normal $\boldsymbol{n}^*$ all components of the ray velocity corresponding to c vanish at a critical point $\boldsymbol{x}^*$ in the

[9]FAG welcomed me in his department and later accompanied me to the tea room, where I met Prof. T. G. Cowling, who said "Dr. Prasad, your second paper with Prof. Bhatnagar will appear in PRS next month.". Cowling is the joint author (with Chapman) of the comprehensive and mathematically elegant book "The Mathematical Theory of Non-Uniform Gases" first published in 1939. PLB taught a course at Harvard University based on this formidable book during 1951-52 and also wrote the lectures in the form of a book, which he presented to me in 1976 just before his death. The notes formed the basis of a book published by Chapman and Hall.

I purchased the 2nd edition of Chapman-Cowling's book in 1969 from my Calcutta University Prize money. A later edition of the book devotes one chapter on BGK model. In summer 1972, Dept. of Applied Math. of Leeds University organised a farewell party to Prof. & Mrs. Cowling, Mandra and me. It was a very touching moment. In November, 1998 I visited the Department and gave a seminar. Though FAG was very old and rarely visited the Department, he came to listen to my talk.

[10]FAG also gave me a nice paper of Varley and Cumberbatch, written in his department but it was not of much use to me.

[11]To be defined and explained later.

steady flow on the sonic line. This implied that the wave front velocity c vanishes at $\boldsymbol{x}^*$ and the wave is trapped. Then I derived for the perturbations on a multi-D steady solution $\boldsymbol{u}_0(\boldsymbol{x})$ a simple approximate PDE

$$w_t + (c_\xi \xi + c_w w) = Kw, \tag{1.5}$$

where constant coefficients c_ξ and K depend on the state of the steady solution $\boldsymbol{u}_0(\boldsymbol{x})$ at $\boldsymbol{x}^*$, c_w is another constant independent of the steady solution and w represents a measure of the small amplitude perturbation. This equation could be treated essentially as a PDE in two independent variables: time t and spatial coordinate ξ, which is distance from $\boldsymbol{x}^*$ in the normal direction $\boldsymbol{n}^*$.

However, there was also a great disappointment. I worked a lot for about a month but unlike the work of K&S, it was not possible for me to find a transformation to transform the equation to one in which the coefficients were independent of the state of the particular steady solution (on which the perturbation is made) at $\boldsymbol{x}^*$. Thus the great advantage of K&S theory, that all steady solutions as well as all perturbations could be studied in just one phase plane of (ξ, w), was lost.

For a transonic flow in a polytropic gas, there was a surprise. It turned out that $c_\xi = -K$, which implied neutral stability. The approximate equation became

$$w_t + (c_w w - K\xi) = Kw, \tag{5'}$$

I deduced from this beautiful equation the neutral stability property of steady transonic flow. FAG[12] read the manuscript carefully and the paper was published in JFM [41].

Prof. Paul Germian, a French academician reviewed my paper in Mathematical Reviews (MR 0347219 (49#11939), 76.35) with a comment "the work on the classical problem of the stability of transonic flows is now completed by a convincing mathematical analysis". This paper was especially mentioned in the opening lecture of Symposium Transonic - II, 1975 by M. T. Landhal. N. Apazidis and Lasser, SIAM J. Sci. Comp., 1999, have used this work to show that computation of transonic flows have inherent difficulties and hence additional care is required.

Much later, in 1993, when I wrote to Germain that I was in Kaiserslautern, he invited me for a colloquium in the University of Paris.

2.10 The Transonic Controversy

Germain's review mentioned above requires some explanation. For references, please see [41]. In 1947 A. R. Kantrowitz concluded that in a Laval nozzle a continuously accelerating

[12]There was a Ph.D. student, who had come from East Pakistan, working with FAG. FAG was quite disappointed with him and desired that he should submit a thesis for just M.Phil. and return to his country. One day FAG asked me to look after the student, who worked with me and completed his thesis for M.Phil.. The student was a Reader in Dacca University and reflected the poor state of education in East Pakistan (now Bangladesh) in 1970.

one dimensional flow through speed of sound was stable but a decelerating flow was unstable. Y. H. Kuo's investigation in 1951 on the stability of two dimensional flows agreed with Kantrowitz's results and this result was not in contradiction with early experimental results. Then came a result from a highly regarded mathematician C. S. Morawetz in 1956 and 1964 on the non-existence of a neighbouring steady transonic flow on an aerofoil. There was a conclusion that it was not possible to design a shock free transonic aerofoil and it retarded the progress of design of a high subsonic aircraft with a transonic pocket. Only after G. Y. Nieuwland (1966) and Spee (1971) at NLR (Netherlands Aerospace Laboratory) experimentally observed and argued that the mathematical result was too strict and an aircraft with slowly moving weak shocks in the transonic region can be designed. It was at this time I showed that Spee's results at NLR were mathematically justified.

2.11 Departure from Leeds

Though FAG was very nice to me and I made a number of very good friends (in fact family friends), I was a bit tired of living in UK and I was missing IISc. I did not complete the full term of Commonwealth Scholarship (as PDF) and I returned to Bangalore and joined IISc on 13th June, 1972.

3 Further Work on Propagation of a Nonlinear Pulse in a Transonic Flow

I kept on coming to this problem again and again. I shall like just to mention the author's names, titles and publication details here.

1. Prasad, P. and Krishnan E. V. Nonlinear wave propagation in a two-dimensional steady transonic flow. *J. Fluid Mech.*, 82, 17-28, 1977.

2. Chandrashekar, D. and Prasad, P. Transonic flow of a fluid with positive and negative non-linearity, *Physics of Fluids*, A3, 427 - 438, 1991.

3. Prasad, P. Upstream propagating curved shock in a steady transonic flow. Proc. IUTAM Symposium Transsonicum IV, 2-6 September, 2002, Goettingen, Germany; Ed. H. Sobieczky, Series "Fluid Mechanics and Applications", 73; Kluwer Academic Publishers, Dordrecht-Boston-London, ISBN 1-4020-1608-5, 25-32, 2003.

4 Moving Towards a Long Term Project

In the following, I will describe the development of three major topics.

1. Weakly Nonlinear Ray Theory (WNLRT)

2. Shock Ray Theory (SRT)

3. Kinematical Conservation Laws (KCL)

Gerald B. Whitham had four papers one each in 1956 (W-1) and 1957 (W-2) and two in 1959 (W-3, W-4), a good discussion of the results in all these papers are available in his well known book [81]. Since these four papers greatly influenced my research, let mentioned their titles here

1. W-1: *On propagation of weak shock waves*, J. Fluid Mech. **1**, 290-318, 1956,

2. W-2: *A new approach to problems of shock dynamics, Part I, Two-dimensional problems*, J. Fluid Mech., **2**, 146-171, 1957,

3. W-3: *Some comments on wave propagation and shock wave structure with application to magnetohydrodynamics*, Comm. Pure Appl. Math., **12**, 113-158, 1959 and

4. W-4: *A new approach to problems of shock dynamics, Part II. Three-dimensional problems*, J. Fluid Mech., **5**, 369-386, 1959.

Whitham got instant fame with publication of these papers with deep physical ideas and they helped me a lot to understand physics but the derivation of the results in these papers were intuitive and it became my aim to derive them mathematically. I have already mentioned W-3 while describing my results on RGD.

5 A Few Equations and Results Used in the Descriptions of My Work

I shall refer to these equations but shall try to present results in such a way that an understanding these equations will not be needed.

5.1 Euler equations of a polytropic gas

The paper W-1, W-2 and W-4 of Whitham basically dealt with waves governed by Euler equations, let me write here the system of conservation laws from which they can be derived for smooth solutions

$$\boldsymbol{H}_t + \langle \nabla, \boldsymbol{F} \rangle = 0 \tag{1.6}$$

where $\boldsymbol{H}$ and $\boldsymbol{F}$ are defined by

$$\boldsymbol{H} = \begin{bmatrix} \rho \\ \rho q_1 \\ \rho q_2 \\ \rho q_3 \\ \rho(e + \frac{1}{2}\boldsymbol{q}^2) \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \rho \boldsymbol{q} \\ \rho(q_1^2 + p/\rho,\ q_1 q_2,\ q_1 q_3) \\ \rho(q_2 q_1,\ q_2^2 + p/\rho,\ q_2 q_3) \\ \rho(q_3 q_1,\ q_3 q_2,\ q_3^2 + p/\rho) \\ \rho \boldsymbol{q}(e + \frac{p}{\rho} + \frac{1}{2}\boldsymbol{q}^2) \end{bmatrix}. \tag{1.7}$$

Here, ρ is the mass density, $\boldsymbol{q} = (q_1, q_2, q_3)$ the fluid velocity, p the pressure, a is sound velocity in the medium given by

$$a^2 = \gamma p/\rho \tag{1.8}$$

and γ is the ratio of specific heats, assumed to be constant. The specific internal energy (denoted by e) and the relation between pressure, density and specific entropy (denoted by σ) for a **polytropic gas** are given by

$$e = \frac{p}{(\gamma-1)\rho} \text{ and } p = A(\sigma)\rho^\gamma, \text{ with } A(\sigma) = (\gamma-1)\exp\left(\frac{\sigma-\sigma_0}{c_v}\right). \tag{1.9}$$

5.2 System of conservation laws

My aim became to understand deeply propagation of nonlinear wavefronts (NLWF) and shock fronts (SF)[13] in gas dynamics by first developing general theories for waves governed by a hyperbolic system of conservation laws with source terms in multi-space dimensions:

$$\boldsymbol{H_t}(\boldsymbol{u}) + \boldsymbol{F}^{(1)}_{x_1}(\boldsymbol{u}) + \boldsymbol{F}^{(2)}_{x_2}(\boldsymbol{u}) + \cdots + \boldsymbol{F}^{(d)}_{x_d}(\boldsymbol{u}) + \boldsymbol{C}(\boldsymbol{x}, t, \boldsymbol{u}) = 0. \tag{1.10}$$

where $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{H} \in \mathbb{R}^n$ and $\boldsymbol{F}^{(d)} \in \mathbb{R}^n$. When the source term $\boldsymbol{C}(\boldsymbol{x}, t, \boldsymbol{u}) = 0$), this system is called a **system of conservation laws**.

Let

$$A(\boldsymbol{u}) = \langle \boldsymbol{\nabla_u}, \boldsymbol{H} \rangle \quad \text{and} \quad B^{(\alpha)}(\boldsymbol{u}) = \langle \boldsymbol{\nabla_u}, \boldsymbol{F}^{(\alpha)} \rangle, \tag{1.11}$$

then differential form of (1.10) becomes

$$A(\boldsymbol{x}, t, \boldsymbol{u}))\boldsymbol{u_t} + B^{(\alpha)}(\boldsymbol{x}, t, \boldsymbol{u})\boldsymbol{u}_{x_\alpha} + \boldsymbol{C}(\boldsymbol{x}, t, \boldsymbol{u}) = 0, \tag{1.12}$$

the sum over α is on $(1, 2, \cdots, d)$.

5.3 Hyperbolic systems in multi-dimensions

Assume that $A \neq 0$. We define the system (1.12) (and hence also the system of conservation laws (1.10)) to be hyperbolic in a domain D of the **space-time** with t as time-like variable if, given an arbitrary unit vector $\boldsymbol{n}$, the characteristic equation

$$Q(\boldsymbol{x}, t, \boldsymbol{u}, \boldsymbol{n}) \equiv \det\left[n_\alpha B^{(\alpha)} - cA\right] = 0 \tag{1.13}$$

has n real roots (called eigenvalues) and eigenspace is complete at each point of D.

The normal velocity c of a wavefront in the medium governed the hyperbolic system (1.12) is a root of the characteristic equation (1.13). We need to make a clear distinction between a wavefront Ω_t in the physical space and a characteristic surface Ω in the space-time (i.e., $(\boldsymbol{x}, t)$-space). When a wavefront moves in the physical space, its successive positions

[13] Whitham did not distinguish between NLWF and SF in W-1, W-2 and W-4, see their definitions and a clear distinction between these two in section 1.8 of [51].

cover a domain in it. In the space-time, it forms a surface Ω. Similarly a ray is a curve in the physical space traced by a point on the wavefront moving with the ray velocity (see (1.17) below). A bicharacteristic is a curve in the space-time traced by a point ($\boldsymbol{x}$,t) when $\boldsymbol{x}$ moves with the ray velocity.

5.4 Bicharacteristics

In 1970 I could find reference to this word only in [18] as "Lemma on Bicharacteristics[14]" In gas dynamics, ray velocity $\boldsymbol{\chi}$ of a point on a wavefront is the vector sum $\boldsymbol{q}+\boldsymbol{n}a$ of the fluid velocity $\boldsymbol{q}$ and the velocity of the sound in normal direction $\boldsymbol{n}$ of the wavefront. A ray is the curve traced by a point $P_t = \boldsymbol{x}(t)$ on the wavefront Ω_t in $\boldsymbol{x}$-space moving in the direction of $\boldsymbol{\chi}$. A bicharacteristic is a curve traced by successive positions of the point $P = (\boldsymbol{x}(t), t)$ in $(\boldsymbol{x}, t)$-space, i.e., space-time as t varies. Now we state a very important theorem (first proved in my book [47], a more elegant proof of the theorem in the form sated below is in [57]). But we need we define a few terms before that.

We denote the eigenvalues, satisfying (1.13), as

$$c_1, c_2, \ldots, c_n \tag{1.14}$$

[14]On my return from Leeds I found IISc very different. In 1969 PLB had made an appointment of an excellent young mathematician in computer science and combinatorics: E. V. Krishnamurthy, who was later awarded SS Bhatnagar award in 1978 (the first faculty of mathematics "then applied mathematics (Appl. Math.) department in IISc" who got this award). The Director Satish Dhawan had appointed in IISc more than 50 new scientists - two in Appl. Math. department. Most of them were well established in their areas of research and this started a beginning of a new era in research and teaching in IISc. In addition in 1970 selection committee there were appointments of four lecturers along with two promotions as assistant professors (I was one). As the youngest member in the department I found the department charged with a new spirit and I had to cope with it. Along with this change, Dhawan started a new programme IISc-TIFR programme in applications of mathematics with the help of the distinguished mathematician K. G. Ramanathan of TIFR. One newly appointed professor (who left IISc forever) and a lecture started discrediting older members of the department and their contribution to the Appl. Math. department is actually negative. However, I must write more about this newly appointed lecture - he understood mathematics very well, he was an excellent teacher and very helpful when some one was in distress - had he devoted fully to research, he would have been a great addition. The other three lectures appointed in 1970 turn out to be of the type of researchers I have mentioned in (2) of Section 0. Thus all new five appointments made by Dhawan to Appl. Math. Department were disappointing in the long run and they were partly responsible in my difficulty in making the department a research centre with researchers of the type 3(a) and (3c) of Section 0. Though, I must say that after my first seminar on bicharacteristics equations, which I gave after my return from UK, K. B. Athreya told me personally "you are the only person in the department with whom I can talk to". I shall say more about my struggle in the department later but let me point out that by 1980, both the Appl. Math. department (due to its internal problems) and the IISc-TIFR programme (due to indifference of and misunderstanding of applied mathematics by TIFR Bombay) had gone down significantly and could not take off for a long time. One day Prof. Ramanathan, retired, quite old and disappointed, met me by chance in TIFR Bombay in 1991 and told me "Prof. Prasad, I made a mistake, I should have taken your help in IISc-TIFR programme from IISc side".

and left and right eigenvectors by $\boldsymbol{\ell}^{(i)}$ and $\boldsymbol{r}^{(i)}$, which satisfy

$$\boldsymbol{\ell}^{(i)}\ (n_\alpha B^{(\alpha)}) = c_i \boldsymbol{\ell}^{(i)} A, \quad (n_\alpha B^{(\alpha)})\boldsymbol{r}^{(i)} = c_i A \boldsymbol{r}^{(i)}. \tag{1.15}$$

Suppose an eigenvalue $c_i(\boldsymbol{x}, t, \boldsymbol{u}, \boldsymbol{n})$ is repeated p_i times in the set (1.14), completeness of eigenspace at each point of D implies that the number of linearly independent left eigenvectors (and hence also right eigenvectors) corresponding to c_i is p_i. Each of the left and right eigenvectors $\boldsymbol{\ell}^{(i)}$, $\boldsymbol{r}^{(i)}$ is unique except for a scalar multiplier. It also implies that we have a hyperbolic system (1.12) with **characteristics of uniform constant multiplicity**.

Theorem 5.1. *Let $\boldsymbol{\chi}$ be the ray velocity of a wavefront Ω_t with normal $\boldsymbol{n}$, governed by the hyperbolic system (1.12), moving with the normal velocity (i.e., eigenvalue) c, then*

$$c = \langle \boldsymbol{n}, \boldsymbol{\chi} \rangle. \tag{1.16}$$

The ray equations of the wavefront Ω_t are given by

$$\frac{dx_\alpha}{dt} = \frac{\boldsymbol{\ell} B^{(\alpha)} \boldsymbol{r}}{\boldsymbol{\ell} A \boldsymbol{r}} = \chi_\alpha \tag{1.17}$$

and the time rate of change of the the $\boldsymbol{n}$ is given by

$$\frac{dn_\alpha}{dt} = -\frac{\ell}{\boldsymbol{\ell} A \boldsymbol{r}} \boldsymbol{\ell} \left\{ n_\beta \left(n_\gamma \frac{\partial B^{(\gamma)}}{\partial \eta^\alpha_\beta} - c \frac{\partial A}{\partial \eta^\alpha_\beta} \right) \right\} \boldsymbol{r} = \psi_\alpha, \quad \textit{say}, \tag{1.18}$$

where $\frac{\partial}{\partial \eta^\alpha_\beta}$ is given by

$$\frac{\partial}{\partial \eta^\alpha_\beta} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta}. \tag{1.19}$$

Further, the system (1.12) implies a compatibility condition on a characteristic surface Ω (a surface generated by the motion of the wavefront Ω_t in space-time) in the form

$$\boldsymbol{\ell} A \frac{d\boldsymbol{u}}{dt} + \boldsymbol{\ell}(B^{(\alpha)} - \chi_\alpha A) \frac{\partial \boldsymbol{u}}{\partial x_\alpha} + \boldsymbol{\ell} C = 0, \tag{1.20}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi_\alpha \frac{\partial}{\partial x_\alpha} \quad \textit{and} \quad \boldsymbol{\ell}(B^{(\alpha)} - \chi_\alpha A) \frac{\partial}{\partial x_\alpha} \tag{1.21}$$

are derivatives in directions tangential to Ω.

Courant and Hilbert (1962) had only the equation (1.17) and called it Lemma on Bicharacteristics. In my paper [42], I had (1.18) but in the definition of $\frac{\partial}{\partial \eta^\alpha_\beta}$ I took $\boldsymbol{n}$ to be the normal of the linear wavefront because I had not deduced the evolution equation (1.18) for $\boldsymbol{n}$. This was because Whitham was too famous and Gubkin [21] said “he has derived Whitham’s theory”. But I intuitively felt that it required deeper investigation. This set forth my future direction of research.

(1.17) gives n ray velocities $\boldsymbol{\chi}^{(i)}$ corresponding to n eigenvalues in (1.14) of the hyperbolic system. Let us consider a curved pulse generated by a particular family of wavefronts corresponding to an eigenvalue $c_i = c$, say. Let the one parameter family of wavefronts Ω_t in the pulse be represented by an equation $\varphi(\boldsymbol{x}, t) = constants$, then the normal velocity c of the wavefront is related to its ray velocity $\boldsymbol{\chi}$ by

$$c \equiv -\frac{\varphi_t}{\mid \nabla\varphi \mid} = \langle \boldsymbol{n}, \boldsymbol{\chi} \rangle, \qquad where \quad \boldsymbol{n} = \nabla\varphi / \mid \nabla\varphi \mid. \tag{1.22}$$

Thus we see that the function $\varphi(\boldsymbol{x}, t)$ satisfies an equation, called eikonal equation,

$$\varphi_t + \langle \boldsymbol{\chi}, \nabla\varphi \rangle = 0. \tag{1.23}$$

Therefore, it follows that for wavefronts in a particular mode, the eikonal equation (1.23) replaces the characteristic equation (1.13). It may be very difficult to derive the equations (1.17) and (1.18) of the Theorem 5.1 from (1.13) but I derived them easily from (1.23), see Section 4.3.3 of [57].

6 Weakly Nonlinear Ray Theory (WNLRT)

Another moment of supreme happiness and satisfaction in research for me began with the publication of my paper [42] in 1975. In [42], my main aim was to mathematically justify Whitham's theory in W-1, which was used successfully for the first time in computing the signature of a sonic boom produced by a supersonic plane. In [42] I derived a quasi-linear equation, which is an approximation of perturbation equations in the neighbourhood of a bicharacteristic curve for a weak pulse governed by a general system of first order quasi-linear partial differential equations in $d + 1$ independent variables $(x_1, x_2, \dots x_d, t)$ and derived Gubkins result as a particular case. I also discussed the form of the approximate equation describing the waves propagating upstream in an arbitrary multidimensional transonic flow.

There are four aspects rays for construction of a wavefront with the help of rays (equivalent to Huygens' method, see Ramanathan (1985) [67], PP (1987) [44] & (2013) [53]).

1. a linear ray (Huygens stated his method for light rays), when the ray velocity $\boldsymbol{\chi}_\alpha$ in (1.17) is constant and the wave moves in a medium having constant state,

2. a linear ray changes it direction due to variation of the properties of the medium along a wavefront according to (1.18),

3. there is a contraction or elongation of a nonlinear ray due to nonlinearity (this effect is more clear for the model equation $u_t + uu_x = 0$ governing an one-dimensional pulse), and

4. in a nonlinear medium, such as isentropic gas, there is a nonlinear diffraction of a ray due to variation of the wave amplitude along the wavefront.

It is surprising all important mathematicians[15] Keller (1954), Whitham (1956), Choquet-Bruhat (1969), Parker (1969, 1970), Hunter & Keller (1983) and Hunter (1995) could not see that their nonlinear propagation theories took into account only the effect (3) and could not capture the nonlinear diffraction mentioned in (4) which is always present in a multi-dimensional nonlinear wave propagation (in a mode with genuine nonlinearity). I too missed it in paper [42] in 1975 but later we used the equation (1.18) for a small amplitude nonlinear wave and captured nonlinear diffraction along with nonlinear elongation or contraction of rays. This important paper [66] of mine and Renuka Ravindran was jointly with my student Ramanathan[16], who was an excellent researcher in theory as well as computation. The numerical results obtained by him beautifully showed resolution of a caustic by a genuine nonlinearity in the characteristic field (see also his thesis [67]). I used figures of his thesis to show the resolution of a caustic and appearance of singularities of a nonlinear wavefronts in many of my publications starting from [47] and [48].

Working in the Institute of Prof. H. Neunzert[17] I wrote a paper [48] explaining details of weakly nonlinear ray theory.

John K Hunter[18] desired to understand our WNLRT. I visited him in 1996 at UC Davis, USA. He was convinced of our WNLRT but desired a proof of convergence of a series expansion leading to the theory. We worked for about two months but such a proof looks almost impossible (can we say impossible in mathematics!) because convergence of even simpler expansion of Choquet-Bruhat has not been proved. Hunter gave up but I worked further on the asymptotic derivation and published the results in [50] completing whatever I could do on this beautiful theory - WNLRT. I also wrote all our results up to 2000 in a book [51]. In my opinion, this is the most satisfactory derivation of our WNLRT and I discussed the limiting case in which Choquet-Bruhat theory follows.

---

[15]for references see [51] and [57]

[16]M T Ramanathan moved to USA even before he could complete writing the second joint paper with me and finally joined civil service in USA

[17]Neunzert, trained in mathematical analysis, set up an excellent research group in applications of mathematics at Kaiserslautern, Germany. He put one of his faculty Dr. Martin Reissel to understand and use my work on weakly nonlinear ray theory. Martin worked with me for two months and Neunzert fully satisfied with the correctness of the theory, asked me to give a colloquium. While introducing me he said "Prasad's theory corrects the mistakes committed by many important mathematician's in past". I also initiated one of his scientists, Jens Struckmeier, in developing a kinetic scheme for a general hyperbolic system - which was achieved there for the first time. Jens is Professor at Hamburg University.
Martin is Professor of Numerical Mathematics at the University of Applied Sciences in Aachen. It is nice that I had correspondence with him in 2020 after 26 years.

[18]A student of famous J. B. Keller earned a good name in rigorous treatment of geometrical optics and asymptotic analysis of nonlinear hyperbolic equations.

7 Intensive Involvement in Research and National Activities 1976-1987

Soon after writing my paper 1975 [42], I went to Allahabad to work at the Mehta Research Institute (MRI), which was taken over by DAE in the mid nineties and later renamed as Harish-Chandra Research Institute (HRI).

7.1 International Conference on Mathematics Education

In 1976, PLB organised an international conference on mathematics education at Bharwari, where MRI was to have its permanent campus. Sir James Lighthill was the main guest. We were just a few members at MRI and we worked day and night, the conference was highly appreciated. PLB also organised the one month long summer course about which I shall write a little later.

7.2 Associate Professor

In August 1977 all members and students of the department gathered for tea after a lecture, E. V. Krishnamurthy (Chairman of the Department) took this opportunity to announce my promotion as Associate Professor. This was interesting because so far (exception may be C. Devanathan long back in 1964) no one got this promotion after first review.

7.3 Editing PLB's Book

In 1977-78 I was deeply involved in editing PLB's book for OUP mentioned in subsection 7.11.

7.4 Director of UGC Course

In 1977, I received a letter from UGC to organise a workshop on "Differential and Integral Equations" with me as the Director of the course and send a proposal for it. With the help of a few of my colleagues I made a proposal for a 5 week course, it was a big event with 50 participants and almost all Indian experts in the subject as faculty. The lectures were published [43] by the Department of Appl.Maths., I.I.Sc., in July 1978, (506 pages). The course was more effective than the Indo-US bi-national conference in emphasising proper teaching of ODE, PDE and integral equations in India because the participants were carefully selected university and college teachers teaching the subject. Outcome of this course were three books, one by S. C. Dev, another by Ram Mohan Rao and the third was by PP & RR [62].

7.5 PP & RR book "Partial Differential Equations" [62]

Its international Edition was by John Wiley Sons, New York, 1984 and Indian edition was by Wiley Eastern in 1985. M. J. Lighthill wrote the "Foreword". The topics in Chapters 1 and 3 were dealt in a refreshingly good style. The book was a recommended text book for the Mathematical Tripos course at Cambridge University, U.K.. A copy of the first edition of the book is available on:
http://www.math.iisc.ernet.in/~prasad/prasad/book/PP-RR_PDE_book_1984.pdf

7.6 Research Group 1978-1987

For quite some time Renuka Ravindran (Renu) was not very happy with working on rheology and was looking forward to work in PDE. It was an opportunity for me to get an excellent co-worker in theory of nonlinear waves and hyperbolic PDE. She not only participated in editing PLB book but also used my WNLRT to study turning of a trapped nonlinear wavefront in a transonic region, her paper[19] was published in JFM (1979). I had a DST project "Mathematical Theory of Nonlinear Waves"[20], in which I could appoint two bright PDFs N. Kalyanasundaram[21] (NKS) and B. J. Venkatachala[22] (BJV). NKS worked with me and Renu on surface elastic waves and I asked BJV to write a review in a simpler language of the paper "Lacunas for hyperbolic differential operators with constant coefficients. I", Acta Mathematica, **124**: 109189, 1970 by Atiyah, Bott, Garding and Lars. Our research group at IISc was small but, in my opinion, quite good. It consisted of Renu, me; PDFs NKS and BJV; and students E. V. Krishnan, T. M. Ramanathan and R. Srinivasan.

7.7 Member of NBHM and Many Visitors to Our Department

I was very active in mathematical activities in India - member of National Board for Higher Mathematics (NBHM) for 11 year since 1985 and member of various committees of UGC, DST, CSIR etc. There were visitors during this period like M. D. Kruskal and academicians Vladimir E. Zakharov and Vladimir I Arnold, with the last two I developed good friendship. Arnold liked our WNLRT and sent his student (already Ph.D. with him - I am forgetting

[19] While accepting the paper, M J Lighthill wrote to Renu "this is the last paper I am accepting for JFM as the editor".

[20] Which also provided me grant to keep a personal secretary and I had this opportunity in various projects till 1915

[21] A PhD in electrical Communication Engineering but very well trained in mathematics.

[22] A PhD - his research supervisor had put him to work in Appl. Math. of type 2 in Section 0., and he was looking forward to work in analysis of PDE. Later in 1986, he met me and told me that he was very disappointed in teaching due to the poor quality of students in M.Sc. in Kuvempu University, Shimoga. I knew that he was an excellent person in creating and solving IMO level of problems. I was able to appoint him in MO Cell in my department and he finally retired as a professor in the cell - but as an employee of DAE

his name) from Moscow to work in our group[23] for 3 months. There was a brilliant mathematician, namely Ranjan Roy[24] from Beloit College, USA. He worked in our group at IISc for one year.

7.8 Recognition in India

I became Professor in 1982. I was awarded SS Bhatnagar Prize in 1983; became Fellow of National Academy of Sciences, India in 1977, Fellow of the Indian Academy of Sciences in 1985 and Fellow of the Indian National Science Academy in 1987. I was able to get the Prize and Fellowships in the first year of nomination itself.

7.9 Mathematical Olympiad (MO) activity in India

In 1968, only 12 countries participated in International Mathematical Olympiad (IMO) and there was a visionary in India, PLB at IISc who organised the first ever MO in India in January 1968 and the next in December 1968. PLB left IISc in 1969 and this activity at IISc stopped. I, with the help of some members of the Department of Appl. Math. of IISc, again started MO in 1978 and it became an official activity of IISc. A highlight of this activity was a one-day lecture program on the culture of mathematics for those who qualified in the IISc MO examination. The MO lecture program of IISc caught the attention of NBHM and NBHM requested me to conduct training program for the IMO, which started at IISc in 1986. India first participated in the International Mathematical Olympiad (IMO) in 1989 and then there was no turning back. I was the Coordinator of the School Committee of NBHM to plan MO activity in India. I was also the leader of the Indian team in IMO-1991 where India ranked within the first ten countries for the first time.

7.10 Visit to MRI 1975-76

PLB desired my company and wished that both of us should work together. He wished that I should join (i) Rajasthan University, (2) Himachal Pradesh University, (3) Delhi Engineering College and finally Mehta Research Institute (MRI) at Allahabad, which he joined in 1974 as the founding director. I reached MRI[25] in October with my family. He

[23] He took the equations of our WNLRT, used Arnold's method and found rays and wavefronts with cusps. This was physically unrealistic and against the numerical results of Ramanathan. We did not send this paper for publication.

[24] Prof. Ranjan Roy has published papers in differential equations in the complex domain, Kleinian groups, Riemann surfaces, fluid mechanics, analytic number theory, and the development of mathematics. We started learning many important topics in mathematics with his help. About Ranjan see $https://www.beloit.edu/live/profiles/275-ranjan-roy$

[25] (1) B R Seth's (one of the senior most applied mathematicians in India) visited MRI and highly appreciated J N Kapur for his work in diverse areas of applied mathematics. I was listening, and when BRS left, PLB told me "see what BRS is talking? JNK was my first and best research student but he has not achieved

was also concerned about my future at IISc - which he clearly expressed when I decided to return to IISc, he said "Phoolan, no one at IISc will understand the quality of your work".

7.11 PLB's Book Published by OUP

This book [11] is the result of lectures in a one month long summer course funded by Indian Mathematical Society (IMS), and the secretary, P. C. Vaidya[26], was very happy. In the course there were only two speakers (1) PLB - who spoke on Dispersive & KdV Equations and (2) me - who covered Hyperbolic Equations and Nonlinear Waves, both teaching more than 2 hours each day. Editing this book was a moment of satisfaction in learning. PLB passed away soon after writing this book and I spent about a year carefully editing with the help of V G Tikekar, Renuka Ravindran and Swarnalata Prabhu. Sir James Lighthill (MJL), Lucasian Professor at Cambridge University, had written the a "Foreword" and after I sent the final copy for perusal, MJL added one paragraph containing "I am deeply grateful to Dr. Phoolan Prasad . . ". Publication of the book has a very troubled history[27].

---

what he could have in research because for his diversion in large number of areas with trivial work".

JNK indeed was a brilliant mathematician, a great orator, excellent teacher and very good organiser. He set up the mathematics department at IIT Kanpur, which was the best mathematics department amongst all IITs and produced the largest number of good mathematicians in the country for many years. He appreciated good applied mathematics and was responsible for republishing my article "Applied Mathematics" mentioned as item No.2 in Section 27.2. It was he who proposed Mathematics Olympiad to be organised by NBHM. He was very good to the students of PLB and was very affectionate to me.

(2) PLB, as the first Director of MRI, became very critical of research of the type 2 in section 0. He told participants of MRI Summer School in 1976 to revolt against their supervisors if the supervisors used them just to increase number of their publications by research of this type. Unmindful of his heath, he was working very hard to build a group with excellent creativity in mathematics at MRI.

(3) One day sitting in PLB's office I mentioned that "Sir, there is hardly any appointments in MRI". He angrily asked me "Phoolan, tell me one name in India working in Appl. Math., I shall send the appointment letter today". I could not give a single name - this reflected the situation of Appl Math India at that time except probabilist and statisticians at ISI.

[26]Prahalad Chunnilal Vaidya (P.C.Vaidya; 23 May 1918 12 March 2010), was an Indian physicist and mathematician, renowned for his instrumental work in the general theory of relativity. Apart from his scientific career, he was also an educationist and a follower of Gandhian philosophy in post-independence India, specifically in his domicile state. Please see Wikipedia.

[27]S R Sinha (SRS), secretary of MRI, had no idea of a research institute and started treating it as his own institute. This led to a rift between him and the faculty appointed by PLB. When I decided to return to IISc, SRS wrote to the registrar of IISc to dismiss me from the service but his antic did not work. Slowly all other members of HRI left.

SRS also tried to cease the PLB's book claiming that it was the property of MRI. But I was firm, I got the permission of the eldest son Rakesh of PLB to publish it as I wished. I arranged the royalty of the book to be given to Rakesh. MJL also helped me in getting the book released from MRI.

7.12 The Year 1987 When My All Academic Activities Stopped

In February 1987 all my academic activities stopped, especially intensive learning of abstract mathematics with Ranjan Roy, when I had a massive heart attack in Kolkata (see section 27) and after that my work slowed down very much[28].

8 Some Additional Work on Trapped Transonic Pulses and Two-D KdV Equation

These are quite interesting works with E. V. Krishnan which combines the theory developed in [41] and [42]. The first paper [58] (published in 1977) shows not only slow turning of a trapped pulse in a transonic flow but also its deformation. The second paper [59] (published in 1978) deals with evolution of multi-dimension packets of surface water waves. Prof. P. G. Drazin saw this paper and sent me a telegram from UK and asked me if Krishnan would be suitable to work with him. Krishnan worked with him for two years as a PDF.

Our work using the method of bicharacteristics and transonic flow attracted the attention of the famous scientist K. Oswatisch, while he was in Germany. He was agreeable to my working with him but after retirement he moved to Vienna. When he realised that Austria could not offer a good Fellowship, he wrote to Juergen Zierep (JZ) to support me for an Alexander von Humboldt Fellowship at Karlsruhe. When I was in Karlsruhe, Oswatisch even made my visit to Netherlands Aerospace Laboratory possible.

9 Study of Modal Wave Problems

Renu and me were inspired by Hayes'[29] classification of waves as local and modal waves and we worked on two problems ([61] and [68]):

The surface water waves are "modal" waves in which the "physical space" (t, x, y, z) is the product of a propagation space (t, x, y) and a cross space, the z-axis in the vertical direction. We derived a new set of equations in [61] for the 2-dimensional long waves in shallow water in the propagation space. When the ratio of the amplitude of the disturbance to the depth of the water is small, these equations reduce to the equations derived by

[28] My wife Mandra suddenly flew to Kolkata. My recovery in Kolkata was very slow, one of the reasons was that our the two children (16 an 11 years) were alone in our isolated house in Bangalore. Mandra and I returned to Bangalore in the end of April and unlike in Kolkata Mandra missed the moral protection of my two elder brothers and my former teachers. She had to work alone to look after me.

[29] See Wikipedia, Wallace D. Hayes (b. September 4, 1918 d. March 2, 2001) was a professor of mechanical and aerospace engineering at Princeton University and one of the world's leading theoretical aerodynamicists. On March 20, 2001, The New York Times reported "Wallace Hayes, 82, Aeronautics Expert, Dies". He was quite familiar with my work and we exchanged many letters. In about 1990, he sent me a paper from Brown University with a comment "You are the only person who will understand this work." Was the work of a person of his eminence also affected sometimes by Whitham or his supporters?

Whitham (1967) by the variational principle. In the neighbourhood of a wave front, this equation reduces to the multidimensional generalisation of the KdV equation derived by Shen & Keller (1973). After some elaborate discussion, we presented a general theory of approximating a multi-dimensional hyperbolic system of quasi-linear equations containing small dispersion terms following one of the modes.

The second modal wave problem was a mathematical analysis of nonlinear waves in a fluid filled viscoelastic tube [68]. The problem is formulated in terms of pseudo-differential operators (which originated in the mid 1960s) in propagation space. The mathematics is quite intricate and many results have been discussed. We examined the existence and stability Stokes waves in this system.

10 Study of a Complex Surface Elastic Waves

NKS, Renu and me studied nonlinear propagation characteristics of surface acoustic waves on an isotropic elastic solid [27]. The solution of the harmonic boundary value problem for Rayleigh waves was obtained as a generalised Fourier series whose coefficients are proportional to the slowly varying amplitudes of the various harmonics. The infinite set of coupled equations for the amplitudes, when solved, exhibit an oscillatory slow variation signifying a continuous transfer of energy back and forth among the various harmonics. It is interesting that the evolution of the shape of the pulse is close to that of a simple model first order equation. The paper with Kalyanasundaram is continued to be cited regularly even today.

11 Kinematics of a Multi-Dimensional Shock of Arbitrary Strength in an Ideal Gas

In 1980 I went to the University of Karlsruhe as a Humboldt Fellow in the Institute of Prof. Jurgen Zierep (JZ)[30]. The well known Prof. S. I. Pai (whose books were followed by students of PLB in IISc) and I shared a large room facing each other from two sides of a big table. There was a very peaceful atmosphere (like that in 1970-72 at Leeds) free from teaching and official work at IISc. I wanted to develop a theory for shock propagation in

[30]JZ was most respected expert in fluid dynamics of his time in Germany and Austria etc. He was a very handsome, tall, soft-spoken and kind person. Though older to me by about 25 years, he treated me affectionately and with respect. We developed family friendship with JZ and Pai. JZ helped me to settle down in Karlsruhe, his wife Elizabeth furnished our empty apartment with very good furniture, I was told that they were kept in her house. JZ took us with our children to a day outing to Swartzwald. Whenever we visited Germany, we went to the house of JZ and sometimes we stayed in his house. I had lost contact with JZ since some time and when we desired to meet Zierep family during our visit in 2017, a friend of ours near Karlsruhe found out that Ziereps were in an old home and JZ could hardly talk in English now. His wife, Elizabeth, remembered us, their visit to Bangalore and listing to Deepika singing on harmonium and Amritanshu accompanying her on tabla in our house.

multi-space-dimensions. I realised that I needed a partial differential for the shock surface $\mathcal{S}(\boldsymbol{x}, t) = 0$, I called it shock manifold $\mathcal{S}$ in space time, like the characteristic equation (1.13) or (1.23) for the nonlinear wavefront. I had read Hayes' important paper [25] and Renuka and me used it in our work on a modal wave problem [61, 68]. Hayes' work also encouraged me to find the Shock Manifold Equation (SME) for $\mathcal{S}$. I used to discuss the problem with JZ (he loved mathematical elegance) and he took a keen interest in it.

I derived the shock manifold equation (SME) for the Euler equation (1.7) and showed that even though the SME is not unique, two different SMEs give the same set of shock rays (which could be unambiguously defined with the help of the characteristics of a SME). I also derived a beautiful form of SME for a weak shock in gas dynamics. Starting from a given position of the initial shock and distribution of shock Mach number and gradient of the flow on it, I claimed that the successive positions of a weak shock and the complete flow behind it can be determined by SME combined with the WNLRT. This was a moment of supreme happiness and satisfaction in research.

I wrote the paper, gave it to JZ on Friday evening. On Monday morning JZ walked into my office and said "Prof. Prasad, may I accept your paper for publication in my journal" (he was the editor of Acta Mechanica). I said "I have discussed the results many times with you before I wrote it and hence it should be joint paper." JZ reply was "the main ideas are yours and you will be the sole author". JZ was very generous to people who worked in his Institute[31]. This led to the publication of the paper "Kinematics of a Multi-Dimensional Shock of Arbitrary Strength" [45].

Weak shock ray theory is useful also in physical systems other than gas dynamics, hence in 2014-15 I extended my 1982 theory and gave a complete proof (see [55]) of a theorem for an arbitrary hyperbolic system of conservation laws (1.10) in d-space dimensions:

Theorem 11.1. *For a weak shock, the shock ray velocity is equal to the mean of the ray velocities of nonlinear wavefronts just ahead and just behind the shock, provided we take the wavefronts ahead and behind to be instantaneously coincident with the shock front. Similarly, the rate of turning of the shock front is also equal to the mean of the rates of turning of such wavefronts just ahead and just behind the shock.*

12 First Application of WNLRT and Kinematics of a Shock Front

The kinematics of a multi-dimensional shock can be used to track the successive positions of a shock only if the states on the two sides of the shock is known. If the shock propagates in a known state (as usually the case is), we can calculate the state behind a weak shock

[31] Unfortunately, one of his own students, whom he helped to succeed him, was very unkind to him. I was sad to see this when I visited the Institute in 2002.

by WNLRT and then we track the shock fully. In [66] we first developed a general theory and then studied propagation of a weak shock produced by a circular cylinder when it is introduced suddenly in a gas in uniform motion. Ramanathan continued extensive computation in his thesis [67] and captured resolution of a caustic by genuine non-linearity. He also interpreted the results as Huygens' method of wavefront and shock front construction by nonlinear rays and shock rays. It was a moment of great satisfaction when I observed that the numerical algorithm developed by us for kinematics of a shock front really works.

13 Waves on Interface of a Clear Liquid and a Mixture

I visited the Institute of Prof. Wilhelm Schneider[32] (Willi) at the Technical University of Vienna for 4 months in summer 1982. He and his student Schaflinger were conducting experiments on the sedimentation process in a mixture in a vessel with an inclined wall and observed a boundary layer of clear liquid moving rapidly upward beneath the inclined wall. They also observed very conspicuous waves on the interface of a clear liquid and the mixture and Willi asked me to give a mathematical explanation of the waves.

For two weeks I carefully observed the experiments and read relevant literature. My mathematical model (at one place Willi helped me) of the phenomena consisted of a very complex system of 9 equations and 3 boundary conditions. After long calculations the dispersion relation of the system showed that "the waves are caused by a discontinuity in the gradient of the tangential velocity of the clear liquid and the mixture, across the interface". In the case of small concentration of the particulate phase in the mixture, the model gave a dispersive wave running upward along the interface in the direction of the flow in the boundary layer and the effect of finite concentration was to introduce damping.

I obtained these results when Willi and Sonia were at their holiday home in the Alps mountains. When I told him the results on phone, he did not believe me. Soon I joined them but we did not discuss the results there but enjoyed the village, beauty of the hills including viewing a glacier and climbing a hill which required a whole day. When we returned to Vienna, Willi looked at the result and asked me to give a seminar, which was very good in his opinion. He said that we shall publish a joint paper including the experimental results.

[32] I first met Willi in Bangalore when he visited Renuka Ravindran and next in Vienna when I went to give a seminar on an invitation from Alfred Kluwick, who worked on nonlinear waves in gas-dynamics. Willi was a very friendly person and soon we developed an excellent family friendship, Willi's wife Sonia, and their children Wolfgang and Peter mixed freely with Mandra, Deepika and Amritanshu. Sonia was an excellent ballerina and used to perform at the one of the leading opera houses in the world, namely Vienna Opera. Once when Mandra and I visited Vienna in 1994 (third visit), Willi told my wife sadly "Mandra, I spent time with Sonia only for two hours in the last 24 hours". Later when Sonia suffered from cancer, Willi looked after her with great devotion and after the recovery Willi did not have to complain again. When Willi came to know that I had visited Germany in 2017, he wrote "Phoolan, I am very unhappy with you that you did not inform me of your visit. A lecture by Phoolan Prasad on any topic is welcome".

After a year, Willi wrote to me that the wave speed found by me did not agree with the experimentally found speed and I was free to publish the paper alone. I sent it to JFM, it was soon accepted but the Editor wrote to me "I have carefully modified those sentences in your paper where there was even a hint that the results would have some agreement with experimental results".

My results were so surprising and the computations so complex that scientists at Vienna and Stanford even suggested initially that I must have made some mistake in very long calculations. However, in 1987 I received a letter from Willi, which contained a sentence "Phoolan! I am very happy today, Andreas Acrivos[33] gave a seminar in my institute today and mentioned that your results were correct and the experiments of Schaflinger were carried out at a different length scale". Indeed, Shaqfeh and Acrivos [74] at Stanford University arrived at the same waves in one limiting case and confirmed that the waves discovered by me were the basic dispersive waves in the phenomenon. This is a beautiful example of a mathematical explanation of a newly discovered physical phenomenon important from the point of view of industrial application and the derivation in summer 1982 was a moment of supreme happiness and gave a great satisfaction. However, I do not consider it a work in applied mathematics as there is no mathematical creativity in it - it is of the type of (3b) in Section 0.

14 WNLRT and Shock Ray Theory Interpreted as Huygens' Method

Christiaan Huygens proposed in 1676-1678 that all points of a wavefront of light can be considered to be point sources of spherical secondary wavelets and after a time t the new position of the wavefront is an envelope of these secondary wavelets. Validity of this method (or principle) for a linear hyperbolic system has been discussed in a great detail in [18]. The rays of the system play an important role in this proof. We shall discuss this again in Section 25 from my paper [53]. Ramanathan discussed it in his thesis [67] and I wrote a semi-popular article "Extension of Huygens' construction of a wavefront to a non-linear wave-front and a shock-front" in 1987 in Current Science [44]. There is another, equally well known method, namely Fermat's method of wavefront construction (1650), and what interested me was a proof for the equivalence of the two classical methods.

[33] Andreas Acrivos is Professor of Chemical Engineering, Emeritus, at Stanford and The Albert Einstein Professor of Science and Engineering, Emeritus, and former director of the Levich Institute at the City College of the City University of New York. He is recognized on an international scale as a leading Chemical Engineering educator and Fluid Dynamicist for his numerous fundamental contributions to laminar boundary layer theory, asymptotic expansions, heat & mass transfer, the rheology of suspensions and, more recently, the influence of electric fields on the motion of small particles.

15 Poor Academic Atmosphere in My Department

I am going to describe a dark period of my department for a long period of about 20 years since 1977, which also affected my academic work. I wish to be truthful, straightforward and frank. Some readers may think that I am too impolite. But what I write in this section is very important also for many academic institutions in India. Mathematics departments of some national institutions in India have been or are in a dark period for much longer duration (one of them for about 55 years) but my department has not only recovered but has turned out to be one of the best departments in the country. Directors of the institutions, where faculty in mathematics departments are not involved in creativity in mathematics, should first assess and then take decisions, which may be quite unpleasant to the senior faculty, to appoint good mathematicians (which would not be easy) in larger numbers.

By 1978 Prof. Atreya had left for USA - not for a good reason and Prof. E. V. Krishnamurthy (see Wikipedia, he held several positions working for many institutions in India, Australia, USA, Europe and other nations) had also left. K. P. Sinha had given up the Chairmanship of our department. In this situation the next senior most person professor (NSMP) was appointed as the Chairman. He was quite a helpful person in personal matters but continued working in engineering mathematics (without any creativity in mathematics and without any motivation for real applications). During his period, the department was dominated by people doing research of the same type. NSMP took a very heavy load of teaching and students continued flocking to his courses to get good grades. There was also a competition to attract Ph.D. students. Degree was assured, without much preparation in basics as well as advanced topics, within a specific time and without working on challenging problems. Even some very good students joined them but could not get high academic recognitions. NSMP did not encourage any appointment. One day Prof. V. G. Tikekar (VGT) and I met the Director of IISc and requested him to change the Chairman, but when we went to his farewell party on his retirement, he took me on one side and said “Prof. Prasad, I am sorry I could not implement your request”.

The reputation of the department kept on sliding down till Prof. C. N. R. Rao (CNR), as the Director, started making changes. The first thing CNR wished to change was the Chairman of the Department. For this, in 1987 he asked VGT, but VGT had already accepted a Fellowship to visit USA on sabbatical and then after sometime, CNR appointed Prof. Renuka Ravindran as the Chairman. After VGT returned from USA, he appointed VGT as the Chairman of the Department in 1990. VGT continued[34] as the Chairman till 1994 and then he retired in 1995. Second thing which CNR did was to change the name of the department to “Department of Mathematics” in 1990. That time “Applied Mathematics” was looked down as a discipline of mathematics. In the beginning, I was

[34]On one occasion, VGT received invitation for another position in India but CNR suggested that he should continue as the Chairman and should not leave IISc.

hurt by this change because I always considered that "applied mathematics is a creativity in mathematics for a deeper understanding of an area of application" but I soon realised that it was necessary to do this.

Now it was possible to attract good mathematicians and CNR held an interview for faculty positions in mathematics. He selected six mathematicians in 1992 at a junior level, which started changing the nature of the research in the Department.

I was the Chairman of the Department during 1996 - 2000. There were 8 professors other than me (I was the senior-most but younger to all of them); 6 of them were keen to support appointments only in their outdated areas of research and the other two later became indifferent to quality. VGT retired and though I had support of newly appointed faculty, I was alone in the Committee of Professors (COP) to take the department forward. However, two successive directors and the divisional chairman ignored the other professors and we kept on trying quality appointments in the department. Last blow to the troublesome professors came, when the Director of IISc ignored them and in 2002 he appointed as Chairman, an associate professor G. Rangaragan, bypassing all of them. After that they could not be effective and they retired one by one.

For his own evaluation for promotion from associate professorship to professorship, Rangarajan could not chair the COP, but it was a good luck for the department that none of the eight professors, who were against the younger faculty, was the senior-most professor. The Director asked me to chair the committee. Evaluation was fair and confidential report on Rangarajan's work must have been very good. Now Rangarajan is the Director of IISc and we are looking forward to him for an excellent future of IISc and I hope he will continue to give special attention to the Department of Mathematics.

Eight professors, except one, were against me because, though younger to all of them, not only I had got SSB Prize and was elected Fellow of all Indian science academies by 1987, but even after 1990, I was offered MSIL Chair Professorship (1993-96), Honorary Professorship and DAE Raja Ramanna Fellowship[35]. They were also against me because all the younger faculty and I were keen to intensive training of students before they registered for Ph. D. under supervision of a faculty. They did not agree to even four basic courses. With great difficulty, we could start Integrated Ph. D. programme.

16 Shock Ray Theory

In my paper on kinematics of a shock front [45] (see also sections 11 and 12), I defined shock rays but I did not think of deriving transport equations along a shock ray. I found it in Maslov's paper [36] (and also in Grinfel'd's paper [20]). I was giving a series of lectures, in TIFR Centre at IISc in a workshop sponsored by NBHM, on Maslov's theory. One day my

[35]After the meeting of the selection committee, a high official of DAE called me on phone and told "Your selection as Raja Ramanna Fellow was unanimous in the committee."

student R. Srinivasan[36] (RS) met me in my office and said "Sir, there is a mistake in the proof of Maslov's lemma." We checked, corrected the mistake and derived the correct form of the infinite set of compatibility conditions along a shock ray of the full Euler equations (1.6), (1.7), (1.8) and (1.9). There had been lots of work on shock waves in gas dynamics since 1860, but the existence of an infinite system of compatibility conditions along shock rays [45] came after 120 years from Maslov and Grinfel'd. Reading Maslov's compatibility condition was a great enlightenment but finding the correct form of them was a moment of supreme happiness and satisfaction in research. This paper showed that Whitham's shock dynamics, derived intuitively, had a serious deficiency and needed corrections.

17 Controversy with Whitham

As soon as Whitham saw our paper in Proc. Indian Acad. Sci (Math. Sci.), he wrote a paper "On shock dynamics" and sent it to the same journal, which was published after some revision in Vol. 96, 1987. He criticised our result in very uncomplimentary language and tried to justify his old theory. I was very unhappy that the editor, though my friend, did not give me a chance to reply and publish it next to Whitham's paper. This shows how non-experts in India are influenced simply by a very well known researcher in western world. One faculty in my department was very happy and mentioned the criticism of our work to many. When he came (with sarcastic smiling) to me to show the criticism, I showed him a letter I had received from the editor of JFM. It was a long letter but essence was "I am very sorry that you have withdrawn your paper from JFM, referee's reports are very positive and I was going to write to you". After going through this letter, my colleague stopped talking about Whitham's criticism.

This was the time I was just recovering from a massive heart attack I had on 9th February, 1987 in a review committee meeting of the Department of Applied Mathematics of the Calcutta University, where I had collapsed after signing the report to revive the UGC

[36] After joining the Department RS was looking for an area of research in which he could learn good mathematics. On advice of Ramanathan he chose to work in my supervision - it was not easy for me to get a student in the dark period of the department due to offer of getting Ph.D. easily as described in Section 15. RS was a brilliant student but wrote only one research paper with me [76]. But he was eccentric, proud and looked down upon other students. He also created a situation in which the Chairman, NSMP wished to cancel his registration for Ph.D.. I prevailed upon him and mentioned that RS deserved Ph.D. at least. I asked him to write his thesis with just one paper, which he had completed in the first year of his admission. The paper was sent to JFM but there some delay. Then, I withdrew the paper and submitted it to Proc. Indian Acad. Sci (Math. Sci.) and after a few days the Editor, Prof. M. S. Narasimhan (who had a room in our building) walked into my office and said "I accept your paper". RS got a job in BITS Pilani but lost after a couple of years. After a long time, he called me on my phone and asked for help in getting a job. He looked mentally unbalanced and there is no trace of him today. Whenever I think of this brilliant student of mathematics, my heart goes out for him.

Centre of advanced study. Renu gave me a lot of support[37] as she was fully aware that we were right. She helped me to write a paper [46], which showed that Whitham's criticism was not correct and his theory can give more than 100% error - in fact much more error. The paper was published in the same journal but after referring, which caused some unjustified delay.

17.1 We were let down not only by Whitham but many of his supporters

There are a large number of scientists who deeply feel that Whitham's work can not be challenged. Let me give an example of the editor of the European Journal of Applied Mathematics. We had submitted a paper on 2-D KCL [60] to this journal, which was accepted by David Creighton, an associate editor. I gave a lecture in Oxford University Computing Laboratory, on invitation from my collaborator and friend K. W. Morton. Morton also invited the John Ockendon[38](John) for lunch, it was surprising that John repeatedly remarked that all that was there in the paper contained nothing new and was already done by Whitham. He repeatedly raised the unnecessary questions in correspondence and being fed up, we did not pursue its publication. This was the first and basic paper on KCL theory.

Even for a paper [6] with Aun which we submitted to SIAP on July 2011, I think one of the referee kept on delaying and after a few reminders we received reply from the editor on 23 April 2013 "Based on these materials and my own reading, I find that your paper may merit publication after minor revision. Please respond to the referee suggestions and submit a suitably revised version. Include a response to the reviewers' comments with your revision." There were 4 or 5 referees and the paper received very good reports from all except the 1st referee who could not find any mistake, wrote a very long report highlighting the results but desired comparison with Euler's numerical solution. We replied that this was unnecessary as it had been done a few times earlier first by by Kevlahan [28] and later by me with Monica and Baskar, see Baskar's paper in JFM, **523**, 174-198, 2005, figures 1, 2, 5, 8 and 9. These figures show excellent agreement of our method with numerical solutions of Euler equations. He kept on insisting knowing very well that computation with 3-D Euler equations would take a long time to do. Arun was too busy to take it up and write a good programme suitable to highlight the fine features, hence we gave up publication in SIAP. The paper [6] is available on my webpage and arXiv.

I do not know whether Whitham read Kevlahan's work in Section 19.1 below and my work with Renu, Sangeeta, Monica, Baskar and Arun. If he came to know about these,

[37] As soon as the department received a telegram from Calcutta, Renu booked tickets for my wife to fly to Calcutta. She showered plenty of affection to our children - she used to take them to her home weekly and did so many good things. Our children fondly remember the affection and kindness of all members of Renu's family.

[38] The first Director of the Oxford Centre for Collaborative Applied Mathematics (OCCAM) and a former Director of the Smith Institute for Industrial Mathematics and System Engineering

he should have apologised to me for his 1987 criticism. Whitham's Ph.D. supervisor, Lighthill[39], who promoted Whitham greatly commented to me "Now I understand your theory" see Section 19.3 below.

17.2 What would have happened had Whitham accepted our theory and highlighted our work?

That is obvious - international fame and leading to many applications, which we tried to do by our small group and limited resources. A person like the editor of European J. of Appl. Math. would not have commented in the presence of two authors of a paper, see Section 17.1. National attention would have increased our group with many competent mathematicians and engineers leading to a large output of good research work.

18 Renu Took a Great Challenge

The sad episode with Whitham and my very precarious health made Renu to take the great challenge. She and her family supported us in every respect and she worked very hard with Sundar[40] and Lazarev[41] and me. In two papers [63] and [69] Renu and me proposed a new theory of shock dynamics (NTSD) alternative to Whitham's shock dynamics. To emphasise the need of NTSD, we first took the simplest conservation law

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{1.24}$$

and derived infinite system of compatibility conditions along a shock path and truncated at the nth stage. We showed that our NTSD gave results very close to exact solutions, where Whitham's shock dynamics failed. In [70], we explicitly showed the step where Maslov's lemma was to be corrected and derived the two compatibility conditions[42] along the shock

[39] In one important place, MJL referred Whitham as his best student.

[40] S. Sundar completed his Ph.D. at Kaiserslautern in the excellent group of Prof. Neunzert. After working with us as a PDF, he joined IIT Madras. He is a good manager also and currently member of many national bodies.

[41] M. P. Lazarev (MPL) is a Ph.D. student of M. A. Grienfel'd (MAG). MAG worked initially in the Institute of Physics of the Earth, Moscow and later migrated to USA. Prof. V. K. Gaur and Dr. R. N. Singh invited MAG in a DST exchange program and he visited me at IISc. When I went to Moscow for a month in an exchange programme of academies, MPL was the young scientist fixed by the USSR Academy to look after me. I had an office in Steklov Mathematical Institute of Russian Academy of Sciences, where my host, Prof. Kulikovskii, Head of the Fluid Dynamics Division, welcomed me with open arms (see Section 2.7 for the influence which the paper of K&S had on my work). MPL accompanied me to various places including the residences of Maslov, Zakharov, his institute, Bolshoi Theatre, Leningrad etc. I gave lectures in many institutes and Moscow University. Next year MPL visited IISc and worked with me for a year.
In Steklov Institute, I was thrilled to notice the name plate Vladimir I. Arnold outside an office. Hardly did I imagine that I shall develop family friendship with this great mathematician, see Section 19.2

[42] These involved very long and complex calculations.

rays for Euler equations (1.6) - (1.8) in three space dimensions. In papers with Sunder (also with Lazarev and a student Singh (1995)), we worked out new applications of the NTSD. A highlight of the collaboration with Lazarev [31] was use of tensor analysis and we followed Grinfeld's[43] method to derive the explicit form of infinite set of compatibility conditions for Euler equations (1.6) - (1.8). Both Lazarev and Renu were well versed with tensor analysis but I was not.

My collaboration with Renu was very productive, some of which I shall describe in section 2-D KCL.

19 My Visits to UK in 1989 and 1992

In 1989, I visited UK on INSA-Royal Society exchange programme. I visited six universities starting from Cambridge, Department of Applied Mathematics and Theoretical Physics (DAMTP) (as guest of Prof. David G. Creighton (DGC)) and ending with Mathematical Institute, Oxford (as a guest of the well known Thomas Brooke Benjamin[44] (TBB)). In my lecture and at lunch, I met the well known expert in numerical solution of hyperbolic equations Keith William Morton (affectionately called Bill). I requested Bill to visit India and helped us in numerical analysis, which he accepted and visited IISc four times as my guest. The family friendship (Bill and his wife Pat) continues even today and we are in regular correspondence. Bill tried to set up a numerical analysis group at IISc, to which the director of IISc, many professors of IISc and scientists at NAL were keen, but my department was not ready, see Section 15. Bill and his research collaborator (almost like his student) Maria Lukacov'a Medvidov'a (Maja) played a very important role in training my students for about 20 years. I shall describe our collaboration in research in later sections, particularly in Section 23.

D G Crighton (DCG), head of the DAMTP, Cambridge University, worked in an area very close to mine and had joined as professor at Leeds University after I left. DCG invited me as a visiting Professor at DAMTP in 1992. In this visit, I had the privileges of a Fellow of the St. John's College - a contact which I enjoy even today. I had an office at DAMTP and also a room at St. John and a big house (of St. John) nearby. My wife Mandra and I very fondly remember the activities of the College, its special dinners and beautiful setting of the city of Cambridge full of colleges.

I started writing my first research monograph "Propagation of a curved shock and

[43] In early 2010th, Grinfel'd wrote to me from USA thanking me for giving so much importance to his work in my papers and my book.

[44] Benjamin was an excellent host and treated me with great affection. Among many important contributions, his name is associated with Benjamin-Bona-Mahony equation. Jerry Lloyd Bona, a former student of Benjamin visited me at IISc and also at TIFR-CAM atleast three times. Our family friendship grew and he gave me an open invitation to visit his department in Chicago. I planned this visit in 2014 but unfortunately I had a stroke in Austin and I had to cancel my trip and the two lectures in Chicago.

nonlinear ray theory", published by Longman Higher Education in 1993. I also gave a series of 5 lecturers on WNLRT and shock ray theory (SRT) and their use in solving weak shock propagation problems as completed in my work with Ramanathan and Srinivasan. In this lecture I did not cover 2-KCL (though I included its full detail in the above book). One student, N.K.R. Kevlahan[45] of Prof. Julian Hunt met me after one of the lectures and asked "May I use your theory in solving shock propagation in turbulence?".

19.1 Kevlahan's thesis and paper

I gradually explained our theory over a period of one month and he completed his thesis and also wrote a paper "The propagation of weak shocks in non-uniform flow" [28], which was published in JFM in 1996. I quote from his paper

"A new theory of the propagation of weak shocks into non-uniform, two-dimensional flow is introduced. The theory is based on a description of shock propagation in terms of a manifold equation together with compatibility conditions for shock strength and its normal derivatives behind the shock. This approach was developed by Ravindran & Prasad (1993) for shocks of arbitrary strength propagating into a medium at rest and is extended here to non-uniform media and restricted to moderately weak shocks. The theory is tested against known analytical solutions for cylindrical and plane shocks, and against a full direct numerical simulation (DNS) of a shock propagating into a sinusoidal shear flow. The test against DNS shows that the present theory accurately predicts the evolution of a moderately weak shock front, including the formation of shock-shocks due to shock focusing. The theory is then applied to the focusing of an initially parabolic shock, and to the propagation of an initially straight shock into a variety of simple flows (sinusoidal shear, vortex array, point-vortex array) exhibiting some fundamental properties of turbulent flows."

In section 5 he writes "In this section we describe shock propagation in terms of compatibility conditions on a shock manifold in space–time. This method was first proposed by Grinfel'd (1978) and Maslov (1978) and has since been developed for arbitrary shock strength and uniform flows ahead of the shock by Ravindran & Prasad (1993). The shock manifold equation describes how the shape of the shock evolves."

He concludes by saying "The novel shock manifold/compatibility condition method of Ravindran & Prasad (1993) is used to describe shock propagation." and later adds "Phoolan Prasad kindly explained his theory of compatibility conditions on a shock ray."

[45]Nicholas Kevlahan became Professor in the Department of Mathematics and Statistics, McMaster University, Canada. He kept contact with me for many years and asked one of his students, who returned to India, to meet me in IISc. He is a very active researcher.

19.2 Meeting Arnold again

One day, I suddenly met Arnold at the porter's gate of St. John's College. He greeted me warmly and the first question he asked was "What is your son doing?". When I told him that Amritanshu is doing Ph.D. in the University of Chicago, there was a great happiness and satisfaction on his face. One day during his visit to IISc, he had spent one whole morning talking to Amritanshu during a long walk in the campus of IISc. He was deeply interested in nurturing mathematical talent of young students in USSR. Mandra and me had also noted how he kept on working next to the bed of his wife Elya, who was briefly admitted to the Health Centre of IISc. We have a photograph of Arnold, Elya and our two children in our house.

After a brief discussion at the porter's gate, Arnold invited us for dinner at his residence, provided by Trinity College. When we reached his residence, we were greeted warmly, a highlight of the dinner was a few preparations of mushroom, freshly picked from the forest by both of them.

19.3 Meeting Sir James Lighthill (MJL)

PLB and MJL were good friends. MJL had known me since 1970. But I first met him at the conference on mathematics education in 1976 hosted by Mehta Research Institute. When PLB was quite concerned about the future of MRI, MJL pointed out that Phoolan would look after it. We also had a lot of correspondence during publication PLB's book [11] and my book [62] with Renu. In 1992, both Mandra and I went from Cambridge to meet him at the Imperial College, London and we spent the whole day with him and his wife Nancy (both sharing the same room). MJL must have known the controversy of my work with Whitham, whom he had mentioned as his best student in a lecture he gave in a workshop organised by the Royal Society of London. After he heard my lecture on WNLRT and SRT, he said "Now I understand your theory".

Unlike his student Whitham, MJL was very considerate and encouraged all good researchers. Let me mention just one incidence. Kalyanasundaram submitted his thesis for Ph.D. in the Department of ECE and the external examiner was Whitham, who withheld approval of the thesis in spite of the correct (in my opinion) explanation. I advised his research supervisor (he used to consult me) to request for an alternative examiner, who would be MJL. Kalyanasundaram, an extremely bright engineer with very good training in mathematics, got hid Ph.D. and joined me as PDF (see Section 7.6).

20 Visiting K. W. Morton at Oxford and an Application of the Bicharacteristic Formulation

Bill Morton had visited IISc for the second time in summer of 1992, and we had completed working on 2-D KCL, which I shall describe in the next section. We went to Oxford basically for a family visit to Mortons. Pat picked up us from the bus station, but there was a big surprise when Mandra and I reached her house. I found Bill working with Prof. Gerald Warnecke and his young collaborator Maria Lukacov'a Medvidov'a (Maja), who was to submit a thesis for habilitation. Bill working with younger people was not surprising but what was surprising was that they were working on development of a numerical method based on bicharacteristic formulation, which Bill first saw in our paper [65] at Bangalore. Bill was looking for an evolution operator for multi-D hyperbolic equation and when he read our paper, he came excitedly to my room and told me that he has now found it. The joint work of Maja, Bill and Gerald appeared in 2000 referring to our work in [35]. The topic "evolution Galerkin methods for hyperbolic systems" has been a very active area of research, in which we also joined with a paper [2] in 2009. Maja and Bill wrote a review article on the topic in 2010 [34].

21 2-D Kinematical Conservation Laws (KCL) ([47, 51, 57])

When Bill Morton visited IISc in the summer of 1992, we were fully ready with the differential forms of the WNLRT (or NTSD) in a polytropic gas. For a front Ω_t propagating in 2-D (x, y)-space we introduced ray coordinates (ξ, t), where constant values of t give the successive positions of the propagating curve Ω_t at different times and values of $\xi =$ constant give rays. Ω_t is represented as

$$\Omega_t : x = x(\xi, t), \quad y = y(\xi, t). \tag{1.25}$$

21.1 Weakly Nonlinear Ray Theory in 2-space Dimensions in Conservation Form

Consider first the WNLRT in a polytropic gas (an isotropic medium). Let the unit normal of Ω_t be $\boldsymbol{n} = (n_1, n_2)$. Denoting the wave amplitude ω of Ω_t by m, the x and y components of the ray equations associated with Ω_t are

$$x_t = n_1 m = m\cos\theta, \quad y_t = n_2 m = m\sin\theta \tag{1.26}$$

where θ is the angle which normal to Ω_t makes with the x-axis. Here we have used appropriate non-dimensional variables. The bicharacteristic equations (1.17) and the compatibility condition (1.20) along the bicharacteristic reduce respectively the following equations

$$\theta_t + \frac{1}{g}m_\xi = 0, \quad g_t - m\theta_\xi = 0, \tag{1.27}$$

and

$$m_t + \frac{m-1}{2g}\theta_\xi = 0, \tag{1.28}$$

where $g = \sqrt{x_\xi^2 + y_\xi^2}$. Given initial position a nonlinear wavefront and amplitude distribution on it, we could use equations (1.26) - (1.28) to tract Ω_t and get distribution m on it at any time t.

However, we knew from the work of Ramanathan that this was possible as long as singularities in the form of kinks did not appear on Ω_t. Bill, Renu and me started working finding conservation forms of the equations (1.27) and (1.28) but there was no clue for about a month. One day, Renu came from her home and showed us some manipulations of the equations (1.27) to get

$$(g\sin\theta)_t + (m\cos\theta)_\xi = 0, \quad (g\cos\theta)_t - (m\sin\theta)_\xi = 0. \tag{1.29}$$

This was a moment of great joy. I showed later that (1.29) can be obtained by equating $x_{tx} = x_{xt}$ and $y_{tx} = y_{xt}$ from (1.25). That was great and we called (1.29) kinematical conservation laws (KCL).

The fact that these are physically realistic conservation forms of (1.27) is discussed in Section 3.3.3 of [51] and Section 6.3 of [57].

We transformed the transport equation (1.28) to a form

$$(m-1)^2 e^{2(m-1)} g = f(\xi), \tag{1.30}$$

where $f(\xi)$ is to be determined from the initial position of Ω_0 and m_0 on it.

My student K. Sangeeta joined us in doing excellent computation with these equations and obtained many results including resolution of caustics, thus extending the results of Ramanathan.

It was this paper that we communicated to European Journal of Applied Mathematics and John Ockendon withheld its publication in spite of recommendation of Crighton.

Later I replaced the equation (1.30) by the conservation law

$$\left\{(m-1)^2 e^{2(m-1)} g\right\}_t = 0. \tag{1.31}$$

Thus the final form of our WNLRT consists of (1.26) and three conservation laws (1.29) and (1.31).

21.2 NTSD or Shock Ray Theory in 2-space Dimensions in Conservation Form

I cut the history short and I just mention that starting from the papers [63], [69] and [70] we also developed 2-D SRT, which I write here only for a weak shock, in the form (here M

is non-dimensional shock strength or shock Mach number, G the metric associated with the coordinate ξ and Θ is the angle which shock normal $\boldsymbol{N}$ makes with the x-axis):

$$(G\sin\Theta)_t + (M\cos\Theta)_\xi = 0, \quad (G\cos\Theta)_t - (M\sin\Theta)_\xi = 0, \tag{1.32}$$

$$\left\{(M-1)^2 e^{2(M-1)} G\right\}_t + 2M(M-1)^2 e^{2(M-1)} G\mathcal{V} = 0 \tag{1.33}$$

and

$$\left\{e^{2(M-1)} G\mathcal{V}^2\right\}_t + (M+1)e^{2(M-1)} G\mathcal{V}^3 = 0, \tag{1.34}$$

where $\mathcal{V}$ is a measure of the gradient in the normal direction of the pressure or density just behind the shock and takes into account of the effect of interactions of the nonlinear waves which catch-up with the shock from behind. Note that we have used only the first two compatibility conditions along the shock ray - the first compatibility condition (1.33) represents energy transport equation. The papers where these equations are derived are available in books [51] and [57]. These books also give all about these equations.

21.3 Some important results from 2-KCL based SRT and NTSD

In this, three students K. Sangeeta, A. Monica and S. Baskar took part. Baskar was very dedicated, excellent in computation and worked on many different types of applications.

1. Sangeeta extended the work, which she had started with Bill, Renu and me on WNLRT, and it was published jointly with me in JFM in 1999 [64]. She first drew the figure of a linear caustic and showed that in all cases the nonlinear rays deviate away from the centre line of the linear caustic leading to the resolution of the caustic. She also found finite amplitude everywhere on the nonlinear wavefronts. I had noted resolution of the caustic in the much cited 1976 experimental results of Sturtevant and Kulkarni in [78] and it became my aim to see this computationally after developing a convincing theory. I have made some more definite statement in section 9.1 of [57] that a caustic will always be resolved by genuine nonlinearity, which could not be seen so clearly in experiments in [78].

   We also discussed propagation of a nonlinear wavefront of periodic shape where we observed formation, propagation and multiple interaction of kinks on it and the long term behaviour that a periodic shaped nonlinear wavefront will tend to become plane as $t \to \infty$.

2. Monica concentrated on the numerical solution of the SRT equations and did very comprehensive numerical work. All the results of Sangeeta for WNLRT were extended to weak shock propagation and in addition, she also compared the results of one case with numerical solution full gas dynamic equations. The paper was published in JFM in 2001 [37].

Roberto Natalini's desired to give a proof: See footnote 41 in Section 25 about Natalini. He wrote a review in Mathematical Reviews of our results on geometrical shape of shocks with Monica. Then he wrote to me "I have always appreciated your work. I would like to know if it is possible to prove the results obtained on the geometric shapes of the shock in your paper by a rigorous mathematical analysis of original Euler's equations or some approximate system". He wished to do this, but I think, the proof has not come so far.

3. **Now I start describing a series of results which Baskar and me completed.**

   - There were two research students: Nitin Potdar from Mumbai and Jeremie Szeftel[46] of Ecole Normale Superieure, Lyon, who worked with Renu and me only for a short time. I gave them a small project, which included finding the elementary solutions[47] of the equations of WNLRT in (ξ, t)-plane and map these solutions on a nonlinear wavefront. We called the images of elementary solutions in (ξ, t)-plane on (x, y)-plane as elementary shapes (an entirely new idea) on the fronts. The results were beautiful but both Nitin and Jeremie left without completing the full work. This was to some extent completed by Baskar. In order that the work of Nitin and Jeremie does not get lost, I published a popular article "Geometrical features of a nonlinear wavefront" in Current Science, **79**, No. 7, 2000 and also included in my book [51].

     Baskar and me took up this investigation for a system more general than the equations of WNLRT. We first solved Riemann problem, discussed all elementary solutions, their images on the fronts (i.e., elementary shapes) and all possible interactions of these elementary shapes. These results were published in IMA Journal of Applied Mathematics, 2004 [8].

   - There was an INI conference on water waves, Isaac Newton Institute, Cambridge, 28-31, August 2001 and WoW conference, Churchil College, Cambridge, 3-5 September 2001. Zakharov[48] invited me to this conference and I had again

[46] He is now Director, CNRS at the Laboratoire Jacques-Louis Lions de Sorbonne Universite'.

[47] Discussion of elementary solutions and their interaction for polytropic gas had been discussed in the classic book [16].

[48] Vladimir E. Zakharov, is a Soviet and Russian mathematician and physicist. He is currently Regents' Professor of mathematics at The University of Arizona, director of the Mathematical Physics Sector at the Lebedev Physical Institute, and is on the committee of the Stefanos Pnevmatikos International Award. Zakharov's research interests cover physical aspects of nonlinear wave theory in plasmas, hydrodynamics, oceanology, geophysics, solid state physics, optics, and general relativity. He was my guest at IISc and I visited his residence in Moscow in 1990. Zakharov told me that he did not accept the highest award in USSR (probably Lenin Prize), because he felt that the prize should have given to Shabat also jointly with him. He also commented on the PLB's book [11] that he was surprised to find the first book on the subject appeared from India, from where no contribution to the subject was made. He had recommended the Russian translation of the book in Moscow University courses.

interaction with Julian Hunt, research supervisor of Kevlahan (who worked with me at Cambridge during 1992, I have already described a part of his thesis in Section 19.1.). At this conference I presented "kinematical conservation laws applied to study geometrical shapes of a solitary wave", which was published in the proceedings of "Wind Over Waves II: Forecasting and Fundamentals", [7]. This is very interesting paper, where KCL theory is applied to a phenomena not governed by a hyperbolic system but a dispersive system of equations. This was possible because the crest line of a curved soliton is a moving curve where KCL applied. We derived some old result of J. W. Miles (1977), who was also present at the conference. But most important part of this paper was derivation of geometrical shapes of the curve soliton with kinks.

- With Baskar, I worked on "propagation of curved shock fronts" published in JFM, 2005, [9]. We reformulated the conservation forms (1.33) and (1.34) of the two compatibility conditions and used very convincing analysis. We calculated successive positions of a shock front produced by a curved piston, both concave and convex to the forward directions of motion, and compared the results with those obtained by Whitham's shock dynamics and numerical solution of the Euler equations. Results by our new theory of shock dynamics were very close to the Euler's solutions but those by Whitham's shock dynamics were not.

  We also calculated waves produced by an explosive placed in a container in the shape of a square, did comparisons and showed a very important result "the genuine nonlinearity of the Euler equations makes the fronts acquire asymptotic circular shapes much before than the asymptotic circular shapes of linear wave-front drawn by Huygens' method".

  There were some more interesting results in this comprehensive paper.

- The next research I took with Baskar was formulation of the problem of sonic boom by a manoeuvring aerofoil as a one parameter family of Cauchy problems and calculation of the leading shock and nonlinear waves following it in the sonic boom. In 2004, I was at Katholieke University, Leuven, Belgium for 6 months about which I shall write later. From there, I went to the Institute for Analysis and Numerics[49], where I formulated this problem and it was published in Proceedings of Indian Academy of Sciences: Mathematical sciences, 2006,

[49] In the University of Magdeburg, I worked in the research group of Prof. Gerald Warnecke, the young professor I met in the house of Bill Morton in 1992. Gerald had made an excellent arrangement for our stay in the guest house and we became good family friends (his wife Christiane and his two sons). On 60th birthday of Gerald, I attended "Conference on Recent Advances in Analysis and Numerics of Hyperbolic Conservation Laws from 8th-10th September 2016, at Magdeburg and spoke on "KCL theory and application to sonic boom", which I did while I was at Magdeburg in 2004-05. Christiane spent a lot of time with Mandra, when we were busy in the Conference. At the conference, I met the well known expert in hyperbolic equations, namely Constantine M. Dafermos, who had referred to some of our work in his book [19] "Hyperbolic

[10]. Derivation of the Cauchy data for this one-parameter Cauchy problems for a manoeuvring aerofoil were very involved and required deep understanding of physics. Manoeuvring aerofoil was either accelerating or decelerating or moving on a curved path. We got many new results about the position and geometrical shape of the leading shocks both from the upper and lower surfaces of the aerofoil and finer details of formation and propagation of the kinks on the shocks.

Preliminary results were presented by me at the 10th International Conference on Hyperbolic Problems: Theory, Numerics and Applications, Osaka, Japan, September 13-17, 2004 and published in its proceedings, **1**, 287-294, 2005, Yokohama Publishers, Japan. The conference started with an inaugural lecture by C. M. Dafermos, see the footnote 48 below. The results were also presented by Baskar at the 17th International Symposium on Nonlinear Acoustics including the International Sonic Boom Forum, Penn State University, USA, 18-22 July 2005 and appeared at AIP Conference Proceedings, **838**, 611 - 614, 2005.

- Based on the large number of quality publications, Baskar was selected as Post-Doctoral Fellow in 2004 at Laboratoire de Modelisation en Mecanique, Universite Pierre et Marie Curie, Jussieu, Paris for two years and directly as Assistant Professor at IIT Bombay in 2006.

21.4 Two Research Monographs: One in 1993 and Another in 2001

I was keen to put some of my research work in a consolidated form and hence I wrote the following books.

1. Propagation of a Curved Shock and Nonlinear Ray Theory published in 1993 [47].

2. Nonlinear Hyperbolic Waves in Multi-dimensions [51]. Some information on the book is available on $http://math.iisc.ernet.in/\sim prasad/prasad/book/book7.htm$

I wrote the third research monograph published in 2018, see Section 26.2.

22 IISc Mathematics Initiative (IMI)

In November 2003, Prof. G. Rangarajan, Chairman of our department wrote to me about IMI. The first proposal, which was accepted, was given by me in July 2004. It was "Scientific Computation, Numerical Analysis and Applications" with an organising committee

Conservation Laws in Continuum Physics" as contribution to differential geometry. All expenses of the travel for this conference as well as visits to Aachen and Mainz were met by AvH Foundation (5th visit on AvH support). I had visited Aachen may times, where Prof. J. Ballmann, Mechanics Department and Prof. Sebastian Noelle, Institut fuer Geometrie und Praktische Mathematik, had interest in our work since a long time.

consisting of 5 members including me. IMI was a yearly programme over a year focussed on one topic but since the time was short for the first year, we kept the duration from January 01, 2005 to July 31, 2005. I made the model plan, which consisted of (i) Workshop 1: February 14 - 25, 2005 and (ii) Workshop 2: July 04 - 15, 2005 to prepare the same set of participants followed by (iii) International Conference: July 18 - 21, 2005. The conference was attended by experts from many countries. The success of this IMI programme was a great satisfaction for me and it was very useful to me, my co-workers and my research students, of course all participants benefited. The IMI programme in following years followed the pattern set by me and later on was held in other institutions in India in the name of **National Mathematics Initiative**, thanks to the organising skill of Rangarajan.

23 3-D Kinematical Conservation Laws (KCL) ([57]) and Some Other Work

After Bill Morton's visit, we had another distinguished visitor from Oxford University Mathematical Institute, namely Mike Giles in 1994 and we started working on the formulation of 3-D KCL i.e., the conservation form of equations of evolution of a surface in 3-D physical space. We were quite successful and I wrote a joint paper with Mike and Renu. When we sent it for publication, there were unusual comments and since I became too busy with applications of 2-D KCL, I could not spend much time on it. Renu had lost interest in continuing research on this problem.

In December 2004, when I was in USA, I received an email from K. R. Arun, a student from IIT Madras with very good academic achievements. He desired to do research in my supervision. I was on my sabbatical leave and I asked him to wait for admission process. He was selected and finally started working with me from August, 2005. I found him to be a very fast learner and capable of collaborating with many researchers at the same time. In the IMI international conference described above, Maria Lukacov'a Medvidov'a (Maja) participated and soon we got a DST-DAAD collaborative research grant under which Arun visited Technische Universitaet Hamburg-Harburg, Germany twice during the period 2006-07. In collaboration with Maja he learned many aspects of numerical analysis of hyperbolic conservation laws.

Now I start describing a series of results in which Arun was involved.

- Arun (with Maja, S. V. R. Rao and me) developed a genuinely multi-dimensional relaxation scheme for hyperbolic conservation laws. This is Based on a new discrete velocity Boltzmann equation, which is an improvement over previously introduced relaxation systems in terms of isotropic coverage of the multi-dimensional domain by the foot of the characteristic, a finite volume method is developed in which the fluxes at the cell interfaces are evaluated in a genuinely multi-dimensional way, in contrast to the traditional dimension-by-dimension treatment. This algorithm was tested on some bench-mark test problems for hyperbolic conservation laws. The work was presented at the 7th Asian Computational Fluid Dynamics Conference, Bangalore, November 26-30, 2007. Electronic Publication, paper No. 19.4, pages 1029-1039.

- Arun (with Marcus Kraft, Maja and me) developed a finite volume evolution Galerkin method for hyperbolic system of conservation laws. The subject started with Bill Morton noticing our work of 1982 mentioned in Section 20. In this we generalised previous methods for hyperbolic systems with spatially varying flux functions. The goal was to develop a genuinely multi-dimensional numerical scheme for wave propagation problems in a heterogeneous media. We illustrated the method for acoustic waves in a heterogeneous medium but the results can be generalised to more complex systems. The finite volume evolution Galerkin (FVEG) method is a predictor-corrector method combining the finite volume corrector step with the evolutionary predictor step. In order to evolve fluxes along the cell interfaces we use multi-dimensional approximate evolution operator. The latter is constructed using the theory of bicharacteristics under the assumption of spatially dependent wave speeds. To approximate a heterogeneous medium a staggered grid approach is used. Several numerical experiments for wave propagation with continuous as well as discontinuous wave speeds confirm the robustness and reliability of the new FVEG scheme.
  A survey article on "Finite Volume Evolution Galerkin Methods" by M. Lukacova-Medvidova and K. W. Morton was published in Indian J. Pure Appl. Math., 2010 [34] and is available at
  http://www.math.iisc.ernet.in/~prasad/prasad/book/2010~Mariia~Morton~INSA~paper.pdf

- As far as I am concerned, most satisfying work which Arun did with me is to complete the theory of 3-D kinematical conservation laws (KCL) - to study evolution of a surface Ω_t in $\mathbb{R}^3$, which I had started with Mike Giles and Renu in 1994. The work started when I was in USA for three months and we used to discuss the results on e-mails. In this we assume that the motion of the surface is isotropic and discuss various properties of these 3-D KCL. These are the most general equations in conservation form, governing the evolution of Ω_t with singularities which we call kinks and which are curves across which the normal $\mathbf{n}$ to Ω_t and amplitude w on Ω_t are discontinuous. From KCL we derive a system of six differential equations and show that the KCL system is equivalent to the ray equations of Ω_t. The six independent equations and an energy transport equation (for small amplitude waves in a polytropic gas) for an amplitude w (which is related to the normal velocity m of Ω_t) form a completely determined system of seven equations for 3-D WNLRT in a polytropic gas.

  We could determine eigenvalues of the system of 3-D WNLRT by a very novel method and found that the system has two distinct nonzero eigenvalues and five zero eigenvalues and the dimension of the eigenspace associated with the multiple eigenvalue 0 is only 4. For an appropriately defined m, the two nonzero eigenvalues are real when $m > 1$ and purely imaginary when $m < 1$. Finally we gave some examples of evolution of weakly nonlinear wavefronts. The results were published in Wave Motion in 2009 [4] and Applied Mathematics and Computation in 2010 [5].

I improved some deductions and proofs and also extended the theory later, which are available in my 2016 paper [56] and 2018 book [57]. Here I dealt with m-D KCL for propagation of a surface in a space of arbitrary dimensions $d \geq 2$.

- With Lukocova, me and S. V. R. Rao, Arun worked on application of 3-D KCL based WNLRT showing many interesting features of propagation of a three dimensional nonlinear wavefront, which was published in SIAM J. Appl. Math. in 2010 [3].

- Arun worked independently on an elegant numerical scheme for 3-D front propagation and control of Jordan mode.

  Abstract: The KCL based front propagation equations form a weakly hyperbolic system of seven conservation laws with an additional vector constraint, each of whose components is a divergence-free condition. This constraint is an involution for the system of conservation laws, and it is termed a geometric solenoidal constraint. The analysis of a Cauchy problem for the linearised system shows that when this constraint is satisfied initially, the solution does not exhibit any Jordan mode. For the numerical simulation of the conservation laws we employ a high resolution central scheme. The second order accuracy of the scheme is achieved by using MUSCL-type reconstructions and RungeKutta time discretisations. A constrained transport type technique is used to enforce the geometric solenoidal constraint. The results of several numerical experiments are presented, which confirm the efficiency and robustness of the proposed numerical method and the control of the Jordan mode.

  This work was published in a very good journal SIAM J. Sci. Comput. Sci. in 2012 [1].

- **My final aim of application of 3-D KCL theory** was to study successive positions of a shock front in 3-space dimensions, formation and propagation of kink lines on it and highlight its intricate geometrical features. In fact we formulated KCL theory basically to understand this phenomenon. I started this work with Arun. Here is a brief description:

  The 3-D KCL, derived purely on geometrical considerations, form an under-determined system of conservation laws. The 3-D KCL system is closed by using two appropriately truncated transport equations (see the NTSD i.e., new theory of shock dynamics in Section 18) from an infinite hierarchy of compatibility conditions along shock rays from [70]. The resulting governing equations of this KCL based 3-D shock ray theory leads to a weakly hyperbolic system of eight conservation laws with three divergence-free constraints. The conservation laws are solved using a Godunov-type central finite volume scheme, with a constrained transport technique to enforce the constraints. The results of extensive numerical simulations reveal several physically realistic geometrical features of shock fronts and the complex structures of kink lines formed on them. A comparison of the results with those of a weakly nonlinear wavefront shows that a weak shock front and a weakly nonlinear wavefront are topologically the same. The major important differences between the two are highlighted in the contexts of corrugational stability and converging shock fronts.

I have described the difficulty in publication of this paper in Section 17.1 paragraph 2. Actually we first wrote informally to JFM editor about this paper on September 1, 2010 and in his assessment this good paper was too mathematical for JFM. Finally I advised Arun (he was very reluctant) to put it on arXiv:
http://arxiv.org/abs/1709.06791
I also put it on my webpage: http://math.iisc.ernet.in/prasad/prasad/preprints/

- With Arun I also wrote some review articles, which I do not wish to describe as it will increase the length of this section too much.

- Based on the large number of quality publications in very short time, Arun was awarded Alexander von Humboldt Fellowship in 210 to work at Institut fuer Geometrie und Praktische Mathematik, RWTH-Aachen University, Germany, even before he got his Ph.D. degree. He also awarded "The Martin Forster Medal" in 2012 for the best thesis in the Mathematical Sciences Division of IISc. After returning from Germany, he joined IISER Thiruvananthapuram (IISER TVM).

24 MHD and an Attempt to Solve an Ill-posed Cauchy Problem for Elliptic Equations

Two interesting events took place in the year 2005.

24.1 Admission of Hari Shankar Gupta (Hari)

In the interview of research students in 2005, there was a student Hari, who tried to answer questions slowly but what he answered showed that he was capable of doing research. He made it clear that both in B.Sc. and M.Sc. courses, there was not much opportunity to learn in the class and hence he tried to learn on his own. He was admitted and I took the responsibility of supervising his Ph.D. work.

When we reduced the sonic boom problem to one parameter family of Cauchy problems, the nonlinear wavefronts and shock fronts originating from the front part the aerofoil were governed by hyperbolic systems (on which I had worked successfully with Baskar, see the fourth item in Section 21.3) and those originating from the rear part were governed by elliptic systems of equations.

Solution of the elliptic Cauchy problem was important for getting the full signature of the sonic boom. I started training Hari to work on this problem.

24.2 Visit of Sergei V Pereverzev (SVP) from Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Linz, Austria

I had visited RICAM (that time the name was Institute of Industrial Mathematics) in 1994 to give a colloquium on an invitation of its Director Heinz W. Engl, who asked me whether my SRT can be used to solve an inverse problem on focussing of shock waves. That time I had no idea but after my work between 2000-2005 I could have answered positively giving a hint.

During his visit to IISc-TIFR programme, SVP visited me and I explained our Cauchy problem for the elliptic system of the sonic boom for an open domain $(0 < \xi + t)$ in the ray coordinates (ξ, t)-plane with Cauchy data prescribed on $\xi + t = 0$. This is an ill posed problem, which is difficult to solve numerically - more so because our problem was for quasilinear equations. After a year, I received a letter from SVP with a copy of a paper and asking me become co-author. My reply was that I only proposed the problem and it was his group who worked on the theory and computation and I requested him only to thank me instead. I am happy that he accepted my request and published the paper "Hui Cao and Sergei V Pereverzev, *The balancing principle for the regularization of elliptic Cauchy problems*, Inverse Problems **23**, 1943-1961, 2007".

24.3 Hari's work on the ill-posed problem

I have always tried to work on open research problems with unknown outcomes but in an area in which I felt confident. I had no experience on a Cauchy problem for an elliptic system therefore, I was a bit hesitant to suggest Hari to work on this problem. However, I had full assurance for collaboration with excellent mathematicians. Firstly, the well known French applied mathematician Olivier Pironneau told me in a programme at IISc that this is a routine problem. My good friend M. Vanninathan from TIFR having excellent expertise in the theory of such problems also assured all help and started guiding Hari. Another expert Jean-Pierre Raymond from Laboratoire de Mathematiques pour lindustrie et la Physique, Toulouse took interest and invited Hari to work in his group. He wrote on March 25, 2008 that Hari Has made progress. On return from Toulouse we found that the work done so far was not sufficient for Ph.D. thesis.

Hari was very depressed and I too felt very bad for a sincere and good student since I was responsible for it. Then I invited SVP to Bangalore and he visited with one of his co-workers. Hari worked with SVP and a good progress was in solving a number of problems. Here is the abstract of one the his papers "Iterated Tikhonov regularization for numerical solution of elliptic inverse problems" published in Inverse Problems in Science and Engineering published in 2012 [22]:

"We have used optimal control approach with weaker space to recover lacking data at the part of boundary where no data is available from the knowledge of Cauchy data on the other part of the boundary for an elliptic Cauchy problem. This optimal control formulation is viewed as a Tikhonov regularization method with initial guess. The balancing principle is used in order to choose an appropriate regularization parameter for Tikhonov method. Finally, numerical experiments highlight the efficiency and robustness of the proposed method in the model context of linear second order elliptic equation."

Another work of Hari was published as "A numerical study of variable coefficient elliptic Cauchy problem via projection method", International Journal of Computer Mathematics, in 2012 [23].

24.4 Hari's work on magnetohydrodynamics equations

I spent 6 months in 2003 at Katholieke University, Leuven, Belgium as a guest of Professor S. Poedt, who provided me a visiting position[50]. Looking at the interest of Poedt, I took up bicharacteristic formulation of the MHD equations. Calculations were very complex and I hoped that someday I shall have manpower to do computation. I asked Hari to complete one aspect of the problem and our paper was published as "A bicharacteristic formulation of the ideal MHD equations" Journal of Plasma Physics [24]. I just quote the last sentence of the abstract "In spite of little longer calculations, the final four equations (three ray equations and one transport equation) for the fast magneto-acoustic wave are simple and elegant and cannot be derived in these simple forms by use of a computer program like REDUCE". REDUCE is used when calculations are too complex for manual calculation. My student Chandrashekar had used REDUCE successfully and did complex computation in 1998-90 [17].

[50] Visit was possible due to effort made by Dipankar Banerjee, who treated us in Leuven as elderly members of his family consisting of his wife Tisha, a daughter and son. Dipu and Tisha had lived for many years in the first floor unit of our house in Bangalore and their distinguished parents also used to visit and stay in that flat. They belong to the family of the mathematician Sir Ashutosh Mukherjee CSI, FRSE, FRAS, FPSL, MRIA, Vice Chancellor of Calcutta University. Tisha is daughter of the well known scientist A. N. Bhaduri. Dipu is now the Director of Aryabhatta Research Institute of Observational Sciences (ARIES), Nainital.

24.5 Hari completed his thesis work

With enough good work, Hari submitted his thesis and received Ph.D. degree in 2011. Hari had excellent training in mathematics and scientific computation but he knew that because of the smaller number of publications he will have difficulty in getting a job in a good research institute. Hence he worked hard and, during 3 years of stay with me after Ph.D., he improved his skills in probability models, statistical inference, exploratory data analysis, time series analysis, stochastic calculus, stochastic differential equations, the Black-Scholes world for financial derivatives; numerical methods and Monte Carlo simulation for derivative pricing. Finally he got a very good job in industry. He is well settled now as a Senior Data Scientist, SAP Labs, Bangalore and his wife Swati is a Senior Manager HCL, Bangalore. I am still in close contact with Hari. They have two children.

It gives me a great satisfaction to narrate the story of struggle in the education and research of a very bright, self-learned and hard working young man, who never gave up and finally he and his wife (who supported him very well in his most difficult days) form a model family. Hari spends a lot of time in teaching online courses without any honorarium.

25 Equivalence of Huygens' and Fermat's Methods of Wavefront Construction 1994 - 2013

In 1993 there were two visitors to the institute of Prof. Neunzert, first Prof. M. A. Anile[51] from the University of Catania and second Anile's former student Prof. Giovanni Russo, from the University of L'Aquila, who later shifted to Catania. I had common interest with them since they had worked in WNLRT (of course different from my WNLRT) almost following Choquet-Bruhat.

I ask Russo to work on equivalence of Fermat's and Huygens' methods of wavefront construction (see Section 14). It is surprising that no one had looked into the equivalence of these two classical methods in more than 350 years. We can see three distinct methods of construction of successive positions of a wavefront in a *physical system*.

1. Huygens' method (1676-78) using envelops of secondary spherical wavefronts,
2. Eikonal equation, a first order PDE[52] derived from the system of governing equations of a medium - we usually take a hyperbolic system but it can be crest line of a curved solitary wave (see the second work with Baskar in the Subsection 16.3), and
3. Fermat's principle (1650) using variational formulation.

In the first two, concept of wavefront is basic, in (1) rays do not appear at all and in second (2) a ray is a derived concept. In the third, the concept of rays is basic from which

[51] **Revisit to Austria and Only Visit to Italy:** Anile also took interest in my work and invited me to Catania for about 2 months, which I accepted from February 1994. This provided Mandra and me an opportunity to visit the whole of Italy except the very northern part. From Kaiserslautern, we travelled to Vienna to meet my former co-worker Willy, then to Linz on invitation from Heinz W. Engl, then to ICTP, Trieste on invitation from M. S. Narasimhan. The journey from Linz to Trieste by train in January is unforgeable, we could see snow laden Alps and people skating below. We visited Venice for a day and another day to Grotta Gigante, world's second deepest cave. Then we started our journey to Rome, where I gave seminars in the Institutes of Errico Presutti and Roberto Natalini in the University of Rome. Presutti and Natalini are important applied mathematicians in Italy, Natalini uses a lot of abstract mathematical tools and I had correspondence with him later many times - see the item 2 in Section 21.3 on his comment for a proof of our SRT results. From Rome we went to the Department of Mathematics and Computer Science, University of L'Aquila, where Russo worked. L'Aquila is a beautiful medieval town with very narrow streets. We traced back our journey from L'Aquila to Rome and from there to Catania in Sicily, which is the largest island in the Mediterranean Sea and one of the 20 regions of Italy. Anile provided us with accommodation in a big house (actually too big for just two of us and a little cold also) in Aci Castello, a beautiful town a little north of Catania. While travelling between Aci Castello and Catania we could see Mount Etna, in the mornings the sky used to be clear with smoke coming out of the mouth of Etna and by evenings the sky used to be cloudy. One day Anile took us to visit to Etna and we reached almost up to the rim of the crater of the volcano. The house in which we stayed was on the seashore and when large waves used to hit the shore, water droplets came on the balcony of the house. We ate mainly fish (fresh from sea), rice, freshly baked bread, tasty aubergine and oranges. We visited many beautiful towns and Roman historical ruins in Sicily.

[52] Theory of first order PDE started in 1760 with Euler and D'Alembert

a wavefront is constructed.

Huygen's method

All points of a wavefront can be considered to be point sources of spherical secondary wavelets and after a time t the new position of the wavefront is an envelope of these secondary wavelets.

Fermat's Principle

A ray in a medium from a point P_0 to another point P_1 chooses a path such that the time of transit is stationary. This defines a ray.

- Use of rays - With the help of rays we can construct the wavefront Ω_t at time t from a wavefront Ω_0 at $t = 0$. We call this *Fermat's method.*
- This method is widely used in geophysics and many other areas.

It has been shown in Section 3.2.6 of my book [51] that the Fermat's principle is valid in a medium in steady motion and that it is ill-posed for a medium in unsteady motion. Then I formulated an **Extended Fermat's Principle** valid in a medium in unsteady motion. My paper [53] also contains this. We use this extended principle in some results mentioned below.

If we like to prove equivalence, we shall ask where? We need a medium, in which the wavefront or shock front propagates. Our natural choice will be a medium governed by a hyperbolic system of equations. Now the problem of equivalence reduces to proof of two steps

- **Step 1** Fermat's principle $\Longleftrightarrow$ Rays from the eikonal equation of a hyperbolic system.
- **Step 2** Rays from the eikonal equation of a hyperbolic system $\Longleftrightarrow$ Huygens' method of wavefront construction.

Step 2 has been discussed in great detail in [18] which inspired me to write related results in [51] in sections 3.2.2 and 3.2.3.

I have also discussed many results (based also on the work I did with Russo) in a paper [53] published in 2013. Russo and I have also written a paper (still unpublished) "Equivalence of Fermat's and Huygens' Principles in a Polytropic Gas" and this paper contains calculations, which are not only quite difficult but also the steps which can not be easily perceived. The results so far derived by us are briefly:

- There is no doubt about the equivalence of Fermat's and Huygens' methods for a wavefront in an isotropic medium (like light waves and sound waves in a gas at rest).
- **Not proved and still open:** Rays from Fermat's principle $\Longleftrightarrow$ Rays from the eikonal of a hyperbolic system.

- When the coefficients of the hyperbolic system depend on t and we have an unsteady solution, then
  Rays from the eikonal equation of a hyperbolic system $\Longrightarrow$ Huygens' method of wavefront construction.
- When the coefficients of the hyperbolic system do not depend on t and we have a steady solution, then Rays from the eikonal equation of a hyperbolic system $\Longleftrightarrow$ Huygens' method of wavefront construction.
- **Theorem** (by PP & Russo): For a polytropic gas the ray equations obtained by Euler-Lagrange variational equations of extended Fermat's principle are indeed the rays of Euler equations of a polytropic gas in an unsteady motion.
  The converse of this theorem even for a polytropic media remains open.

Derivation of our theory by Extended Fermat's Principle is stated briefly 1994 in [48] and is explained in detail in section 3.2.7 of [51]. The theories developed by Whitham (1957, 1959); Choquet-Bruhat (1969); Hunter and Keller (1983); Hunter, Majda and Rosales (1986) etc., do not satisfy Fermat's principle.

I was delighted to give an exact derivation of our theories using Extended Fermat's Principle. **The realisation that that our WNLRT and SRT are consistent with Fermat's Principle has given me great happiness and satisfaction. It is like a personal triumph for me.**

26 Two Papers in 2016 and a Book in 2018

Publication of the following two papers and a book gave me a lot of satisfaction.

26.1 Two papers in 2016:

I wished to work out the most general forms of some basic ideas running in a large number of my research papers. They are included in my papers

1. Ray equations of a weak shock in a hyperbolic system of conservation laws in multi-dimensions (arbitrary spatial dimension d), [54]

   From abstract: It gives a complete proof of a theorem for a hyperbolic system of conservation laws, which states that "for a weak shock, the shock ray velocity is equal to the mean of the ray velocities of nonlinear wavefronts just ahead and just behind the the shock, provided we take the wavefronts ahead and behind to be instantaneously coincident with the shock front. Similarly, the rate of turning of the shock front is also equal to the mean of the rates of turning of such wavefronts just ahead and just behind the shock".

2. Kinematical conservation laws in a space of arbitrary dimensions, [56]

   Abstract: In a large number of physical phenomena, we find propagating surfaces which need mathematical treatment. In this paper, we present the theory of kinematical conservation laws (KCL) in a space of arbitrary dimensions i.e., d-D KCL, which are equations of evolution of a moving surface Ω_t in d-dimensional $\boldsymbol{x}$-space, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. The KCL are derived in a specially defined ray coordinates $(\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_{d-1}), t)$, where $\xi_1, \xi_2, \ldots, \xi_{d-1}$ are surface coordinates on Ω_t and t is time. KCL are the most general equations in conservation form, governing the evolution of Ω_t with physically realistic singularities. A very special type of singularity is a kink, which is a point on Ω_t when Ω_t is a curve in $\mathbb{R}^2$ and is a curve on Ω_t when it is a surface in $\mathbb{R}^3$. Across a kink the normal $\boldsymbol{n}$ to Ω_t and normal velocity m on Ω_t are discontinuous.

26.2 Publication of a book in 2018:

I also wished to collect all results of our group on KCL and its applications after the publication of my 2001 book [51]. These are in a consolidated form in the book "Propagation of Multi-Dimensional Nonlinear Waves and Kinematical Conservation Laws", Springer, 2018 [57]. First thirteen pages and list of references are available on
http://www.math.iisc.ernet.in/~prasad/prasad/book/2018_PP_book_cover_&_first_12_pages.pdf

27 Satisfaction in Mathematics Education: Nurturing Mathematics Talent, Teaching and Writing Popular and Review Articles

This is not a part of my research. My own education up to M.Sc. has gone through a very turbulent period and I desired only to be a school teacher. I am surprised that I could reach a good institute for research. This is mainly because my brother took utmost care of my education, my private teacher (when I was in class 10) taught me to learn on my own and I was fortunate to go to excellent colleges. See the article "My Education" at
http://www.math.iisc.ernet.in/~prasad/prasad/Education.pdf
Therefore, I got great satisfaction in the activities mentioned in this heading. I list some of them here:

27.1 Mathematics Education:

I list here a few of these.

- **Mathematics Olympiad:** I have already mentioned about it briefly in Section 7.9. Details are available in my article "Mathematics Olympiad in India" published in

Azim Premji University's *At Right Angles*, November 2018, Problem Corner, available at
*http://publications.azimpremjifoundation.org/*1769/1/17_*Phoolan_MathsOlympiadsIndia.pdf*
Interestingly, the first ever article in India on the topic was written by V. G. Tikekar and me, and it appeared on the central page of the news paper Deccan Herald 19, October 1978, as "Mathematics Olympiad".
It shall also remind that the first ever MO was organised at IISc in 1968 by P. L. Bhatnagar, who conducted it to many parts of India as the President of Association of Mathematics Teachers of India (AMTI). This was continued by PLB's first Ph.D. student J. N. Kapur, who also made it a NBHM activity. Finally MO activity was restarted at IISc by students of PLB and training for IMO became reality at IISc from 1986. Many members of NBHM worked hard to made it successful, see the article in the above link.

- **Meeting the Prime Minister Rajiv Gandhi and requesting him setting up special schools for bright students:** I met the Prime Minister Rajiv Gandhi in 1984 (probably in November). After the SSB Prize award function, he mixed freely with awardees and but paid a lot of attention to my ideas. He asked me to write those to him after returning to Bangalore. When I wrote, he replied in a few lines signed personally with PMs seal and it was delivered to me by a special postal service. My discussion with him and my letter to him may have had some influence in setting up Jawahar Navodaya Vidyalayas (JNV) in 1985 by an initiative taken personally by the PM. To start with, two JNVs were established during 1985 at Jhajjar (Haryana) and Amravati (Maharashtra). The foundation stone of Jhajjar JNV was laid down by Hon Prime Minister, Sh. Rajeev Gandhi himself in April, 1985.

- **Attempt to draw attention of PDE teachers to importance of soliton theory:** In the last five years I have been trying to promote teaching of the theory of Solitons, Kotreweg de Vries (KdV) equation and, soliton theory and algebraic geometry in Indian universities. I covered lectures on this topic in many workshops, particularly in IAS sponsored discussion meeting mentioned in item number 6 in Section 27.2 below. My own lecture is available at
*https://web-japps.ias.ac.in:*8443*/SEP/pdf_files/P_Prasad_KdV*%20*E_&_solitons.pdf*

27.2 Review Articles and Articles on Mathematics Education:

I have written at least 50 such articles. I mention here just a few of these articles:

1. **Mathematics education and research - Radio Interview**, AIR, Bangalore 12 June 1985, 9.16 p.m. - 9.30 p.m. interviewed by Prof. Mythily Ramaswamy. It is a long interview touching mathematics education and research available on

http://math.iisc.ernet.in/ ~ prasad/prasad/radiointer1985.pdf
The interview also deals with my idea of applied mathematics, which basically means what I have written in (3c) in Section 0.

2. **Applied Mathematics:** News Bulletin of Calcutta Mathematical Society, 1988, 11, 1-7. It was reprinted, on request by Prof. J. N. Kapur, in Mathematical Education, 1989, 5, 205 - 209. Also a Pre-conference paper in the National Conference in Development of mathematics, organised by NBHM at Bombay, July 22 - 24, 1990. It is available on
http://math.iisc.ernet.in/ ~ prasad/prasad/appliedmaths1989.pdf
This article has been appreciated by both pure and serious applied mathematicians.

3. **The agony of a talented student:** The problem and some suggestions for solution. Published in the Souvenir for the Annual Meeting of the Association of the Mathematics Teachers of India, 1986. It is available on
http://math.iisc.ernet.in/ ~ prasad/prasad/1985_AGONY_TALENTED _STUDENT_problem_suggestions.pdf

4. **Popular Article on Genuine Nonlinearity and Shocks:** Every undergraduate and graduate Natural Scientist and Psychologist at St. Johns is a member of the Larmor Society of St Johns College, Cambridge. Society asked me to explain one of my research work to undergraduate students in November, 1992. I tried to explain genuine nonlinearity and shocks avoiding differential equations and using the language of physics. It was highly successful. I repeated a modified version of this lecture in IISc Institute Colloquium in 1995 and based on this I wrote an article, which was published in Resonance in two parts in 1997 [49].

5. **P.L. Bhatnagar and the BGK Model:** 19th P.L. Bhatnagar Memorial Award Lecture - 2006, delivered at 72nd Annual Conference of the Indian Mathematical Society at Jabalpur, 27 - 30 December, 2007. Available on *http://math.iisc.ernet.in/ ~ prasad/prasad/plbhatnagar.pdf*

6. **First Order Partial Differential Equations: a simple approach for beginners:** This in my opinion is a very good article prepared to make the topic interesting to students, which students find quite difficult. It points out very deep ideas with the help of simple examples and also has a new topic "Existence and propagation of singularities on a characteristic curve". It is available on
http://math.iisc.ernet.in/ ~ prasad/prasad/preprints/2013_140528_first_ order_PDE_characteristics_only.pdf

7. **Workshops to Integrate Differential Equations with Areas of Applications:** In order to integrate teaching of differential equations with areas of applications and

to highlight some topics in physics which almost all students, teachers and researchers in differential equations should know, I and A. K. Nandakumaran jointly organised two meetings: Discussion Meeting on Some Important Equations of Physics at Orange County, Coorg, 22 - 25 February 2018 and Discussion Meeting on Teaching of Differential Equations in India at IAS - Academy Fellows Residency, Jalahalli, Bengaluru, 26 - 28th February, 2018. The reports with 10 lectures are available at webpage of IAS:
http://web-japps.ias.ac.in:8080/SEP/Discussion_Meeting.jsp
About 40 mathematicians, PDFs (prospective faculty) and scientists attended this meeting.

8. **Little Mathematical Treasures and Compact Course:** As the President of the Ramanujan Mathematical Society during 2002-2003, I started many activities such as Compact Courses for post-graduate students and RMS Series on Little Mathematical Treasures. LMT was published jointly by the Ramanujan Mathematical Society and the Universities Press: *http://www.ramanujanmathsociety.org/math_treasures.html* and I was its Editor-in-Chief for many years.

28 Honorarium to Janakiammal Ramanujan by INSA

In 1988 I was at INSA, when the Executive Director met me and ask if I knew that Mrs. Janakiammal Ramanujan, the wife of Srinivasa Ramanujan, was going through a great financial difficulty. He also added that if I appeal to INSA, she would be given some regular honorarium will be given. I did appeal to INSA and INSA agreed to pay Rs. 1,500/- to Mrs. Janakiammal Ramanujan per month.

29 Area of my Research

In the year books of the the three Indian academies, my specialisation has bee mentioned as “Theory of Wave Propagation, Partial Differential Equations and Fluid Dynamics”. I have worked only in a very limited parts of these three topics. However, it gave me a great satisfaction when one of the most important experts in conservation laws, Constantine M. Dafermos, has referred to our work at a few places in his 2016 book [19] “Hyperbolic Conservation Laws in Continuum Physics” as contributions to differential geometry.

Acknowledgement: I planned writing this article, during the lockdown in India which started in March 2020. That would not have affected writing this article very much, even though my wife Mandra and I started coping with all household work and safe shopping without any support. Then suddenly my old back pain and giddiness came back and

my working hours came down from about 8 hours daily to an average of just 1 hour daily.

With a feeling that I may not be able to complete this article, I sent the rough draft, with many mistakes, to my former student K. R. Arun to revise it. I am thankful to him for formatting the LATEX file, which gave a good shape to this article.

But I was left with making the corrections on my own, which took a very long time and it still lacks a reasonable perfection. The article has too many cross references in to various sections. Arun again came to my help and enabled the command "tableofcontents" so that contents appear in the beginning.

In the circumstances described above, despite her RA disease, Mandra took the full responsibility of running the household along with taking care of a reasonably big garden. She had always done this since 1969 but her support during the difficult days of increasing incidences of COVID-19 has also been very valuable (see also Section 7.12 for even more difficult days).

From our children, Deepika and Amritanshu, we get at least two calls a day since the beginning of COVID-19 started in India and this gives us a lot of solace and strength.

I thank my former colleague Prof. V. G. Tikekar for valuable comments on this article.

This is November 2020, my giddiness has been controlled and my back pain is a little better.

Bibliography

[1] K. R. Arun, *A Numerical Scheme for Three-dimensional Front Propagation and Control of Jordan Mode*, SIAM J. Sci. Comput., **34**, B148-B178, 2012.

[2] Arun, K.R., Kraft, M., Lukacov'a Medvidov'a, M., and Phoolan Prasad, *Finite volume evolution Galerkin method for hyperbolic conservation laws with spatially varying flux functions*, Journal of Computational Physics, **228**, 565590, 2009.

[3] K. R. Arun, M. Lukacova-Medvidova, Phoolan Prasad and S. V. Raghurama Rao, *An application of 3-D kinematical conservation laws: propagation of a three dimensional wavefront*, SIAM. J. Appl. Math., **70**, 2604-2626, 2010.

[4] Arun K R and Phoolan Prasad, 2009, 3-D Kinematical Conservation Laws (KCL): Equations of evolution of a surface, Wave Motion, 46, 293-311.

[5] K. R. Arun and Phoolan Prasad, *Eigenvalues of Kinematical Conservation Laws (KCL) based 3-D weakly nonlinear ray theory (WNLRT)*, Applied Mathematics and Computation, **217**, 2285-2288, 2010.

[6] Arun K R, and Phoolan Prasad, *Propagation of a three-dimensional weak shock front using kinematical conservation laws*, 2012, available on
$http://math.iisc.ernet.in/\sim prasad/prasad/preprints/3d\_kcl\_srt.pdf$
When the paper was ready, authors first wrote to JFM editor about this paper on September 1, 2010.
2017 Available on arXiv at: http://arxiv.org/abs/1709.06791

[7] S. Baskar, and Phoolan Prasad, *Kinematical conservation laws applied to study geometrical shapes of a solitary wave*, Proceedings of Wind Over Waves II: Forecasting and Fundamentals. Ed. S. Sajjadi & J. Hunt, Horwood Publishing Limited, England 2003, 189-200. This is the proceeding for INI conference on water waves, Isaac Newton Institute, Cambridge, 28-31, August 2001 and WoW conference, Churchil College, Cambridge, 3-5 September 2001.

[8] S. Baskar and Phoolan Prasad *Riemann problem for kinematical conservation laws and geometrical features of nonlinear wavefronts*, IMA Journal of Applied Mathematics, **69**, 391-420, 2004.

[9] S. Baskar and Phoolan Prasad, *Propagation of curved shock fronts using shock ray theory and comparison with other theories*, J. of Fluid Mechanics, **523**, 171-198, 2005.

[10] S. Baskar and Phoolan Prasad, *Formulation of the problem of sonic boom by a maneuvering aerofoil as a one parameter family of Cauchy problems*, Proceedings of Indian Academy of Sciences : Mathematical sciences, **116**, 97-119, 2006.

[11] P. L. Bhatnagar, *Nonlinear Waves in One-dimensional Dispersive Systems*, Oxford Mathematical Monograph Series, Oxford University Press, 1979.

[12] P. L. Bhatnagar and Phoolan Prasad, *Study of self-similar and steady flows near singularities, Part I*, Proc. Roy. Soc., London, **A 315**, 1970, 569-584.

[13] P. L. Bhatnagar and Phoolan Prasad, *Study of the self-similar and steady flows near singularities, Part II- a case of multiple characteristic velocity*, Proc. Roy. Soc., London, **A.322**, 1971, 45-62.

[14] S. Chandrasekhar, *Radiative Transfer*, Dover Publications, 1960; this is slightly revised version of OUP published in 1950.

[15] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Chicago University Press, 1939; also published by Dover in 1957.

[16] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, 1948.

[17] Chandrashekar, D. and Prasad, P. *Transonic flow of a fluid with positive and negative nonlinearity*, Physics of Fluids, **A3**, 427 - 438, 1991.

[18] R. Courant and D. Hilbert, *Methods of Mathematical Physics, vol 2: Partial Differential Equations*, Interscience Publishers, 1962. C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Spinger, 2016.

[19] C. M. Dafermos, *Hyperbolic Conservation Physics Laws in Continuum Physics*, Spinger, 2016.

[20] Grinfel'd, M A (1978), Ray method for calculating the wavefront intensity in nonlinear elastic material, *PMM J. Appl. Math. and Mech.*, **42**, 958-77.

[21] K. E. Gubkin, *Propagation of discontinuities in sound waves*, J. Appl. Math. and Mech. 22 (1958), 787-793.

[22] Hari Shanker Gupta, *Iterated Tikhonov regularization for numerical solution of elliptic inverse problems*, Inverse Problems in Science and Engineering, 2012

[23] Hari Shanker Gupta, *A numerical study of variable coefficient elliptic Cauchy problem via projection method*, International Journal of Computer Mathematics, **89**, pp. 795-809, 2012, DOI:10.1080/00207160.2012.659426

[24] H. S. Gupta and Phoolan Prasad, *A bicharacteristic formulation of the ideal MHD equations*, Journal of Plasma Physics, **77**, 169-191, 2011

[25] W. D. Hayes, *Kinematic wave theory*, Proc. Roy. Soc.(London), **A 320**, 209-26, 1970.

[26] L. Howarth, *Modern developments in fluid dynamics: High speed flow*, Vol I and II, OUP, (1953).

[27] Kalyanasundram N., Renuka Ravindran and Phoolan Prasad, *Coupled amplitude theory of nonlinear surface acoustic waves*, J. Acoustical Society of America, **72**, 488-93, 1982.

[28] N. K. R. Kevlahan, *The propagation of weak shocks in non-uniform flow*, J. Fluid Mech. **327**, 167-97, 1996.

[29] A. G. Kulikovskii and F. A. Slobodkina, *Equilibrium of arbitrary steady flows at the transonic points*, PMM J. Appl. Math. Mech., **31**, 623-630, 1967.

[30] Lazarev, M.P., Phoolan Prasad and Singh S.K., *Approximate solution of the one-dimensional piston problem*, ZAMP, **46**, 752-771, 1995.

[31] Lazarev, M.P., Renuka Ravindran and Phoolan Prasad, Shock propagation in gas dynamics: explicit form of higher order compatibility conditions. Acta Mechanica, 1998, 126, 139-155.

[32] M. J. Lighthill, *Introduction to Fourier Series and Generalised Functions*, Cambridge University Press. 1958.

[33] P. D. Lax, *Hyperbolic system of conservation laws II*, Comm. Pure Appl. Math., **10**, 537-66, 1957.

[34] Lukacov'a Medvidov'a M., Morton K. W. *Finite Volume Evolution Galerkin Methods*, Indian J. Pure Appl. Math., **41**, 329-361, 2010.

[35] Lukacov'a Medvidov'a M., Morton K. W., Warnecke G. *Evolution Galerkin methods for hyperbolic systems in two space dimensions*, Math. Comp., **69**:1355–1384, 2000.

[36] Maslov, V P, *Propagation of shock waves in an isentropic non-viscous gas*, *J. Sov. Math.*, **13**, 119-163,,1980, (Russian publication 1978).

[37] A. Monica and Phoolan Prasad, *Propagation of a curved weak shock*, J. Fluid Mech., **434**, 119-151, 2001.

[38] Phoolan Prasad, *Nonlinear waves in radiation gas dynamics*, J. Ind. Inst. of Sci. **49**, 1967, 148-178.

[39] Phoolan Prasad, *A class of one dimensional steady state flows and general shock structure problem in radiation-gas-dynamics*, Quart. J. Mech. and Appl. Maths., **22**, 1969, 333-353.

[40] Phoolan Prasad, *Effect of radiation on shock structure when temperature is not very high*, Def. Sci. J., **19**, 1969, 213-222.

[41] Phoolan Prasad, *Nonlinear wave propagation on an arbitrary steady transonic flow*, J. Fluid Mech., **57**, 1973, 721-737.

[42] Phoolan Prasad, *Approximation of the perturbation equations of a quasilinear hyperbolic system in the neighbourhood of a bicharacteristic*, J. of Mathematical Analysis and Applications, **50**, 1975, 470-482.

[43] Phoolan Prasad (Editor) *LECTURES ON DIFFERENTIAL AND INTEGRAL EQUATIONS was prepared in collaboration with 8 authors and formed the material of lectures in the UGC sponsored SPECIAL ADVANCED LEVEL COURSE ON DIFFERENTIAL AND INTEGRAL EQUATIONS*, PP was the Director of the course. The lectures were published by the Department of Appl.Maths., I.I.Sc., in July 1978, (506 pages).

[44] Phoolan Prasad, *Extension of Huyghen's construction of a wavefront to a non-linear wave-front and a shock-front*, Current Science, 56, 50-54, 1987.

[45] Phoolan Prasad, *Kinematics of a multi-dimensional shock of arbitrary strength in an ideal gas*, Acta Mechanica, **45**, 163-176, 1982.

[46] Phoolan Prasad, *On shock dynamics*, Proc. Ind. Academy Sci. (Math. Sci.), **100**, 87 - 92, 1990.

[47] Phoolan Prasad, *Propagation of a Curved Shock and Nonlinear Ray Theory*, Pitman Researches in Mathematics Series, **292**, Longman, Essex, (1993).

[48] Phoolan Prasad, *A nonlinear ray theory*, Wave Motion, **20**, 21-31, 1994, also Berichte der Arbeitsgruppe Technomathematik, No.101, 1994, Universitat Kaiserslantern, Germany.

[49] Phoolan Prasad, Nonlinearity, *Conservation Law and Shocks, Part I: Genuine Nonlinearity and Discontinuous Solutions*, RESONANCE- Journal of Science Education by Indian Academy of Sciences, Bangalore, **2**, No. 2, 8-18, 1997 and *Conservation Law and Shocks, Part II: Stability Consideration and Examples*, **2**, No.7, 8-19, 1997.

[50] Phoolan Prasad, *An asymptotic derivation of weakly nonlinear ray theory*, Proc. Indian Academy Sciences (Mathematical Sciences), **110**, 4, 431-447, 2000.

[51] Phoolan Prasad, *Nonlinear Hyperbolic Waves in Multi-dimensions*. Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, **121**, 2001.

[52] Phoolan Prasad, *Upstream propagating curved shock in a steady transonic flow*, Proc. IUTAM Symposium Transsonicum IV, 2-6 September, 2002, Goettingen, Germany; Ed. H. Sobieczky, Series "Fluid Mechanics and Applications", 73; Kluwer Academic Publishers, Dordrecht-Boston-London, ISBN 1-4020-1608-5, 25-32, 2003.

[53] Phoolan Prasad, *Fermat's and Huygens' Principles, and hyperbolic equations and their equivalence in wavefront construction*, Neural, Parallel, and Scientific Computations, 21, 305-318, 2013.

[54] Phoolan Prasad, *Ray equations of a weak shock in a hyperbolic system of conservation laws in multi-dimensions*, Proc. Indian Acad. Sci. (Math. Sci.), 126, No.2, May 2016, 199-206, DOI 10.1007/s12044-016-0275-6

[55] Phoolan Prasad, *Ray equations of a weak shock in a hyperbolic system of conservation laws in multi-dimensions*, Proc. Indian Acad. Sci. (Math. Sci.), **126**, 199-206, 2016.

[56] Phoolan Prasad, *Kinematical Conservation Laws in a Space of Arbitrary Dimensions*, Indian J. Pure Appl. Math., **47**(4), 641-653, 2016.

[57] Phoolan Prasad, *Propagation of Multi-Dimensional Nonlinear Waves and Kinematical Conservation Laws*, Springer, Springer Nature Singapore, 2018, DOI 10.1007/s12044-016-0275-6

[58] Phoolan Prasad and Krishnan E.V., *Nonlinear wave propagation in a two-dimensional steady transonic flow*, J.of Fluid Mech., 82, 1977, 17-28.

[59] Phoolan Prasad and Krishnan E.V., *On multi-dimensional packets of surface waves*, J. of Physical Soc. of Japan, 44, 1028-1032, 1978.

[60] Phoolan Prasad, Renuka Ravindran, and Morton, K.W., - Conservation forms of the nonlinear ray equations. (Accepted for publication in European Journal of Applied Mathematics, but the revised version could not be prepared), Tech. Report No.2, 1992, Dept. Math, Indian Institute of Science, Bangalore.

[61] Phoolan Prasad and Renuka Ravindran, *A theory of nonlinear waves in multi-dimensions with special reference to surface-water-waves*, J.Inst. Math. and Applcs., **20**, 9-20, 1977.

[62] Phoolan Prasad and Renuka Ravindran, *Partial Differential Equations*. Wiley Eastern Ltd and John Wiley & Sons, 1984.

[63] Phoolan Prasad and Renuka Ravindran, *A new theory of shock dynamics, Part - II numerical solution*, Applied Mathematics Letters, **3**, 107 - 109, 1990.

[64] Phoolan Prasad and K. Sangeeta , *Numerical simulation of converging nonlinear wavefronts*, J. Fluid Mechanics, 385, 1-20, 1999.

[65] Reddy A.S., Tikekar V.G. and Phoolan Prasad,*Numerical solution of hyperbolic equations by method of bicharacteristic*, J. Mathematical and Physical Sciences, **16**, 575-603, 1982.

[66] Ramanathan, T.M., Phoolan Prasad and Renuka Ravindran, *On the propagation of a weak shock front: Theory and application*, Acta Mechanica, **51**, 167-177, 1984.

[67] Ramanathan, T M, *Huygens Method for Construction of Weakly Nonlinear Wavefronts and Shock fronts with Application to Hyperbolic Cautics*, Ph. D. Thesis, IISc, 1985.

[68] Renuka Ravindran and Phoolan Prasad, A mathematical analysis of nonlinear waves in a fluid-filled visco-elastic tube, Acta Mechanica, **31**, 253-280 , 1979.

[69] Renuka Ravindran and Phoolan Prasad, *A new theory of shock dynamics, Part-I: analytic considerations*, Applied Mathematics Letters, **3**, 77 - 81, 1990.

[70] Renuka Ravindran and Phoolan Prasad, *On infinite system of compatibility conditions along a shock ray*, Q.J.Appl.Math. and Mech., **46**, 131-140, 1993.

[71] Renuka Ravindran, Sunder S. and Phoolan Prasad, *Long time behaviour of the solution of system of equations from new theory of shock dynamics*, Computers Math. Applc., **27**, 91-104, 1994.

[72] P. L. Sachdev and Phoolan Prasad, *A numerical procedure for shock problem using artificial heat conduction*, J.Phys. Soc., Japan, **21**, 2715-24, 1966.

[73] H. Schlichting *Boundary Layer-Theory*, McGraw-Hill, 1955.

[74] Shaqfeh ESG, Acrivos A. *Enhanced sedimentation in vessels with inclined walls: Experimental observations*. Phys. Fluids. **30**, 1905-1914, 1987. DOI: 10.1063/1.866205

[75] V. I. Smirnov *Course of Higher Mathematics, Vol 4: Integral equations and partial differential equations*, Pergamon Press, 1964.

[76] Srinivasan, R. and Phoolan Prasad, On the propagation of a multi-dimensional shock of arbitrary strength, Proc. Indian Acad. Sciences, Mathematical Sciences, 1985, 49, 27 - 42.

[77] J. J. Stoker, *Water Waves The Mathematical Theory With Applications*, Interscience Publishers, 1957.

[78] B. Sturtevant and V. A. Kulkarni, *The focusing of weak shock waves*, J. Fluid Mech., **73**, 651-71, 1976.

[79] S. G. Tagare and Phoolan Prasad, *Instability of the self-similar flow into a cavity*, Proc. Ind. Acad. Sci., **A71**, 1970, 219-224.

[80] J. VonNeumann and R. D. Richtmyer, *A Method for the Numerical Calculation of Hydrodynamic Shocks*, Journal of Applied Physics 21, 1950.

[81] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons, New York, 1974.