Research Interest and Open Research Problems

Most important aspect of these problems from the point of view of students is described in the section 11

1 Weakly nonlinear ray theory (WNLRT) and shock ray theory (SRT)

The nonlinear effects in the caustic region has been a difficult problem, which remained unsolved till publication of Prasad's book "Propagation of a curved shock and nonlinear ray theory"and two papers in JFM, one by Prasad and Sangeeta (1999) and another by Monica and Prasad (2001). We started working on this problem in 1977, soon after publication of the experimental results of Sturtevant and Kulkarni in JFM in 1976. In order to simulate these experimental results, which showed resolution of the Hyperbolic caustic by nonlinearity, we developed what we call a weekly nonlinear ray theory (WNLRT) and a shock ray theory (SRT) over a number of years. In 1911, Sommerfeld and Runge derived the transport equation along rays for the amplitude of a wave using small amplitude high frequency approximation for a linear wave. Attempts were made by Keller (1954), Whitham (1956), Choque-Bruhat (1969) and Parker (1969) to incorporate in the transport equation the nonlinear effects. Unfortunately, though the transport equation derived in these attempts was correct, even the mathematically elegant but formal perturbation method of Choque-Bruhat missed the non-linear diffraction of the rays which plays equally important role in the caustic region (or on large length scales). Our WNLRT developed since 1975 (see section 4.1, Prasad, 2001) accounted for this nonlinear diffraction effect correctly and we obtained for the first time a coupled system of ray equaitons and the transport equaition.

Original derivation of the shock ray theory starts from the derivation of an infinite system of transport equations along a shock ray (Grienfeld (1978) and Maslov (1978)) and truncation of this system at a suitable stage so that we are left with a finite number of transport equations (Ravindran and Prasad, 1990). This also leads to a coupled system of ray equations and transport equations taking care of the nonlinear dffraction of shock rays. It is interesting and thrilling to observe the consistency of the two theories WNLRT and SRT: SRT can be derived from WNLRT.

The solution of the system of equations of two theories (equations are in differential forms) may develope singularities at some critical times even if the Cauchy data is smooth. To find a solution after the critical time, we need conservation form of the equations. The ray equations of the two theories can be put into conservation forms. These conservation forms turn out to be geometric and physically realistic - what we describe in section 3 as kinematical conservation laws (KCL). The transport equations can also be put into conservation forms. Thus, we have a system of conservation laws for the equations of the WNLRT and a system of conservation laws with source terms for the equations of

SRT. The two theories have been very successful in discovering many finer geometrical properties of nonlinear wavefronts and shock fronts and in showing the resolution of a linear caustics by nonlinearity. The SRT has given results which agrees well with the experimental results and with full numerical solution of the original Euler equations. Both theories are also computationally efficient as the dimension of the problem is reduced by one. There are many open mathematical questions associated with these theories and they need mathematical proofs, which are quite challenging.

2 Fermat's principle and Huygen's Method

Huygen's method (1690) was stated for the construction of the successive positions of a wavefront and Fermat's principle (1650) difines rays which can also be used to construct successive positions of a wavefront. In space free from obstacles, it should be possible to prove that both the theories give the same wavefront as can be seen easily for the wave equaiton in a multi-dimensions. The equivalence has been proved only recently for elasticity equations (1992) and for the Euler's equations of gasdynamics. A proof of equivalence of the two theories for a general system of hyperbolic equaitons is still open (Prasad, 2001, page 106).

3 Kinematical conservation laws (KCL) for a moving surface

Associated with a given moving surface Ω_t in m-dimensional space $(\underline{x}) = I\!R^m$, we can associate a $m - 1$ parameter family of curves, called rays. Using purely geometrical considerations, we can derive a system of conservation laws (KCL) in a ray coordinate system. For example, for $m = 2$ the ray coordinate system consists of two independent variables ξ and t such that $\xi = $ constant give rays and $t = $ constant give successive positions of the moving curve Ω_t. Let m be the normal valocity of Ω_t, g be the metric associated with the coordinate ξ and let θ be the angle which normal to Ω_t makes with the x-axis, then KCL for an isotropic motion of Ω_t is given by (first obtained by Mortan, Prasad and Ravindran in December, 1991)

$$(g \sin \theta)_t + (m \cos \theta)_\xi = 0$$

$$(g \cos \theta)_t - (m \sin \theta)_\xi = 0$$

KCL is a purely mathematical system of equations and gives not much information on the propagation Ω_t unless some dynamics of Ω_t is introduced. Hence KCL is an underdetermined system of 2 equations in three unknowns m, θ and g. The nature of the curve, say whether Ω_t is a nonlinear wavefront or a shock front or the crest line of a long curved solitary wave, determines the additional closure relations (Baskar and Prasad

(2001), Prasad (2002, 2004)). KCL with additional relations has been very successful in solving a large number of different problems. Extension of the above two equations to the case when the motion of Ω_t is not isotropic is available in Prasad(2001).

Extension of the KCL from $m = 2$ to $m = 3$ and higher has many open problems, which needs to be resolved before they can be used in answering topological questions regarding the propagation of Ω_t in many areas of applications.

It also appears that KCL would provide an alternative or probably a better method than the well known level set method (LSM). We first need to see if the viscosity solution of the LSM formulation give the same jump relations as those obtained by KCL.

4   KCL in other geometries

What will be the form of KCL in other geometries such as non-Euclidean geometry and fractal geometry? If we are able to write KCL and give additional information on the dynamics of Ω_t, we hope to get many interesting results.

5   Sonic Boom

Though the sonic boom phenomenon arose with the advent of supersonic flights of aircrafts in early 50's of the 20th century, it has become a hot topic of research today. Since observers and buildings on the ground are affected by the loud noise associated with the shocks in the boom, the undesirable effect of a sonic boom is regarded as a nuisance in the society and is to some extent harmful to the health of living creatures and human being. There has been extensive research in sonic boom by engineers. But a survey of literature in this field shows that the mathematical aspects of the problem need much more development. Recently we have worked on a new mathematical formulation of a sonic boom, which we describe in the next paragraph.

The structure os a sonic boom produced by a simple aerofoil at a large distance from its source consists of a leading shock (LS), a trailing shock (TS) and a one parameter family of nonlinear wavefronts in between the two shocks. We have shown that the leading shock is governed by a hyperbolic system of equations in conservation form and the system of equations governing the TS has a pair of complex eigenvalues. Similarly, each nonlinear wavefront originating from the points on the front part of the aerofoil is govrned by a hyperbolic system of conservation laws and those originating from points on the back part is governed by a system of conservation laws, which is elliptic. Consequently, we expect the geometry of the TS to be smooth and topologically different from the geometry of the LS. This seems to have been observed in a very limited computation with Euler equations.

The above mathematical formulation opens up a new direction of research which we believe, will be both mathematically and computationally challanging as we encounter now an ill posed Cauchy problem (or an initial value problem) for a system which exhibits elliptic nature in those two eigenvalues, which are of relevant to us.

6 Analysis of multi-dimensional ideal magnetohydrodynamic (MHD) equations

MHD is a commonly used word whether the medium is incompressible or not. Infact, we shall deal with the equations of an ionized gas neglecting dissipative processes. The ideal MHD equations in 3-space dimensions form a system of 8 equations in 4 independent variables (x_1, x_2, x_3, t). Though formidable from the point of view of mathematical analysis, the system of MHD equations were well established about 50 years back and have beautiful mathematical structure. It has only recently been suggested that the intermediate shocks, which were supposed to be unstable, are not unstable. This has been seen mainly in careful numerical solution of MHD equations. The whole problem about the admissibility of intermediate shocks is mahtematically open and is very challanging.

The MHD equations are not strictly hyperbolic, it has four distinct types of waves: fast waves, slow waves, Alfen waves and waves moving with the fluid. These waves are also not isotropic even in the medium at rest. The MHD bow shock have been seen to have dimpled shape experimentally and this has been recently verified by numerical computation. We have tried to work out a formulation for this and we hope our formulation would explain the appearance of the dimple.

We have just completed a bicharacteristic formulation of the unsteady MHD equations - this appears to be the first formulation but we have done it only for a two space dimensional case. Our formulation discovers new tangential components of the ray velocity. Extension of this to 3 space variables needs to be worked out. We also need to analyse the effect of the tangential component of the ray velocity on the MHD wave propagation.

7 Bicharacteristic formulaiton of the equations of the general theory of relativity

The equations of GTR has been formulated as a system of hyperbolic equaitons. Bicharacteristic formulation and WNLRT for this systm will be interesting. This year - 2004, an exact solution of the GTR equations have been obtained by Glimm and Temple for a spherically symmetric case showing almost all aspects of the big bang theory but with an important conclusion that our expanding universe is bounded by a shock. One needs to understand whether nonlinearity in the system explains formation of a finite size black

hole instead of a point singularity.

8   A ray theory for a curved shock of arbitrary strength

This is an important physical problem but extremely difficult from the point of view of developing a suitable ray theory. We have made reasonable progress in this direction.

9   Kink Structure

Kinks are idealized geometrical objects - points on a moving curve Ω_t. In reality kinks are limits of certain curved parts of Ω_t where the curvature is very large. One can take Ω_t as a physically realistic curve, say a weakly nonlinear wave front or a shock front in gasdynamics and try to formulate a new system of equaitons by adding viscosity terms. The modified system should have smooth solution and its viscosity limit should give those kinks which are physically realistic. It appears that the higher order derivative terms or terms containing product of first order derivatives, neglected in our formulation of SRT would be suitable viscosity terms.

10   Dynamics of a coalescing bubles

Two coalescing bubles meet along a cureve. Is it possible to study the dynamics of such bubles with the help of KCL? Mathematical modelling, analysis and computation will be very interesting.

11   Scientific computation and numeraical analysis

The research problems mentioned above have two aspects: (i) open mathematical questions and (ii) extensive numerical computation with real physical mathematical models. A student working on the second aspect will become an expert in various aspects of scientific computation. While doing this, he will learn both theory and actual computation in numerical analysis. IISc has an excellent group of scientists invloves in computation and mathematicians who specialize in those aspects which are required for training a young student as a numerical analyst. This is important for career point of view, a large number of institutions, universities, R & D establishments, growing number of industries and private companies in India are looking for mathematicians specialized in scientific computation and numerical analysts. Production of Ph.D. degree holders with this specialization is alomost nil in India. Even Indian mathematicians, who go abroad (say USA)

do not specialize in this area leaving a great scarcity of such trained mathematicians in India. Therefore, a student trained in scientific computation and numerical analysis has a very bright prospect of employment.

References

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