CHAPTER 2

Independent random variables

2.1. Product measures

Definition 2.1. Let μ_i be measures on $(\Omega_i, \mathscr{F}_i)$, $1 \le i \le n$. Let $\mathscr{F} = \mathscr{F}_1 \otimes \ldots \otimes \mathscr{F}_n$ be the sigma algebra of subsets of $\Omega := \Omega_1 \times \ldots \times \Omega_n$ generated by all "rectangles" $A_1 \times \ldots \times A_n$ with $A_i \in \mathscr{F}_i$. Then, the measure μ on $(\Omega, \mathscr{F})$ such that $\mu(A_1 \times \ldots \times A_n) = \prod_{i=1}^n \mu_i(A_i)$ whenever $A_i \in \mathscr{F}_i$ is called a *product measure* and denoted $\mu = \mu_1 \otimes \ldots \otimes \mu_n$.

The existence of product measures follows along the lines of the Caratheodary construction starting with the π-system of rectangles. We skip details, *but in the cases that we ever use, we shall show existence by a much neater method* in Proposition 2.8. Uniqueness of product measure follows from the $\pi - \lambda$ theorem because rectangles form a π-system that generate the σ-algebra $\mathscr{F}_1 \otimes \ldots \otimes \mathscr{F}_n$.

Example 2.2. Let $\mathscr{B}_d, \mathbf{m}_d$ denote the Borel sigma algebra and Lebesgue measure on $\mathbb{R}^d$. Then, $\mathscr{B}_d = \mathscr{B}_1 \otimes \ldots \otimes \mathscr{B}_1$ and $\mathbf{m}_d = \mathbf{m}_1 \otimes \ldots \otimes \mathbf{m}_1$. The first statement is clear (in fact $\mathscr{B}_{d+d'} = \mathscr{B}_d \otimes \mathscr{B}_{d'}$). Regarding $\mathbf{m}_d$, by definition, it is the unique measure for which $\mathbf{m}_d(A_1 \times \ldots \times A_n)$ equals $\prod_{i=1}^n \mathbf{m}_1(A_i)$ for all intervals A_i. To show that it is the d-fold product of $\mathbf{m}_1$, we must show that the same holds for any Borel sets A_i.

Fix intervals $A_2, \ldots, A_n$ and let $S := \{A_1 \in \mathscr{B}_1 : \mathbf{m}_d(A_1 \times \ldots \times A_n) = \prod_{i=1}^n \mathbf{m}_1(A_i)\}$. Then, S contains all intervals (in particular the π-system of semi-closed intervals) and by properties of measures, it is easy to check that S is a λ-system. By the $\pi - \lambda$ theorem, we get $S = \mathscr{B}_1$ and thus, $\mathbf{m}_d(A_1 \times \ldots \times A_n) = \prod_{i=1}^n \mathbf{m}_1(A_i)$ for all $A_1 \in \mathscr{B}_1$ and any intervals $A_2, \ldots, A_n$. Continuing the same argument, we get that $\mathbf{m}_d(A_1 \times \ldots \times A_n) = \prod_{i=1}^n \mathbf{m}_1(A_i)$ for all $A_i \in \mathscr{B}_1$.

The product measure property is defined in terms of sets. As always, it may be written for measurable functions and we then get the following theorem.

Theorem 2.3 (Fubini's theorem). *Let $\mu = \mu_1 \otimes \mu_2$ be a product measure on $\Omega_1 \times \Omega_2$ with the product σ-algebra. If $f : \Omega \to \mathbb{R}_+$ is either a non-negative r.v. or integrable w.r.t μ, then,*

(1) *For every $x \in \Omega_1$, the function $y \to f(x,y)$ is $\mathscr{F}_2$-measurable, and the function $x \to \int f(x,y) d\mu_2(y)$ is $\mathscr{F}_1$-measurable. The same holds with x and y interchanged.*

(2) $\int\limits_{\Omega} f(z) d\mu(z) = \int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} f(x,y) d\mu_2(y) \right) d\mu_1(x) = \int\limits_{\Omega_2} \left(\int\limits_{\Omega_1} f(x,y) d\mu_1(x) \right) d\mu_2(y).$

PROOF. Skipped. Attend measure theory class. ■

Needless to day (*self:* then why am I saying this?) all this goes through for finite products of σ-finite measures.

Infinite product measures: Given $(\Omega_i, \mathscr{F}_i, \mu_i)$, $i = 1, 2, \ldots$, let $\Omega := \Omega_1 \times \Omega_2 \times \ldots$ and let $\mathscr{F}$ be the sigma algebra generated by all finite dimensional cylinders $A_1 \times \ldots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \ldots$ with $A_i \in \mathscr{F}_i$. Does there exist a "product measure" μ on $\mathscr{F}$?

For concreteness take all $(\Omega_i, \mathscr{F}_i, \mu_i) = (\mathbb{R}, \mathscr{B}, \nu)$. What measure should the product measure μ give to the set $A \times \mathbb{R} \times \mathbb{R} \times \ldots$? If $\nu(\mathbb{R}) > 1$, it is only reasonable to set $\mu(A \times \mathbb{R} \times \mathbb{R} \times \ldots)$ to infinity, and if $\nu(\mathbb{R}) < 1$, it is reasonable to set it to 0. But then all cylinders will have zero measure or infinite measure!! If $\nu(\mathbb{R}) = 1$, at least this problem does not arise. We shall show that it is indeed possible to make sense of infinite products of Thus, the only case when we can talk reasonably about infinite products of measures is for probability measures.

2.2. Independence

Definition 2.4. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space. Let $\mathscr{G}_1, \ldots, \mathscr{G}_k$ be sub-sigma algebras of $\mathscr{F}$. We say that $\mathscr{G}_i$ are *independent* if for every $A_1 \in \mathscr{G}_1, \ldots, A_k \in \mathscr{G}_k$, we have $\mathbf{P}(A_1 \cap A_2 \cap \ldots \cap A_k) = \mathbf{P}(A_1) \ldots \mathbf{P}(A_k)$.

Random variables $X_1, \ldots, X_n$ on $\mathscr{F}$ are said to be independent if $\sigma(X_1), \ldots, \sigma(X_n)$ are independent. This is equivalent to saying that $\mathbf{P}(X_i \in A_i \; i \le k) = \prod_{i=1}^{k} \mathbf{P}(X_i \in A_i)$ for any $A_i \in \mathscr{B}(\mathbb{R})$.

Events $A_1, \ldots, A_k$ are said to be independent if $\mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_k}$ are independent. This is equivalent to saying that $\mathbf{P}(A_{j_1} \cap \ldots \cap A_{j_\ell}) = \mathbf{P}(A_{j_1}) \ldots \mathbf{P}(A_{j_\ell})$ for any $1 \le j_1 < j_2 < \ldots < j_\ell \le k$.

In all these cases, an infinite number of objects (sigma algebras or random variables or events) are said to be independent if every finite number of them are independent.

Some remarks are in order.

(1) As usual, to check independence, it would be convenient if we need check the condition in the definition only for a sufficiently large class of sets. However, if $\mathscr{G}_i = \sigma(S_i)$, and for every $A_1 \in S_1, \ldots, A_k \in S_k$ if we have $\mathbf{P}(A_1 \cap A_2 \cap \ldots \cap A_k) = \mathbf{P}(A_1) \ldots \mathbf{P}(A_k)$, we *cannot* conclude that $\mathscr{G}_i$ are independent! If S_i are π-systems, this is indeed true (see below).

(2) Checking pairwise independence is insufficient to guarantee independence. For example, suppose X_1, X_2, X_3 are independent and $\mathbf{P}(X_i = +1) = \mathbf{P}(X_i = -1) = 1/2$. Let $Y_1 = X_2 X_3$, $Y_2 = X_1 X_3$ and $Y_3 = X_1 X_2$. Then, Y_i are pairwise independent but not independent.

Lemma 2.5. *If S_i are π-systems and $\mathscr{G}_i = \sigma(S_i)$ and for every $A_1 \in S_1, \ldots, A_k \in S_k$ if we have $\mathbf{P}(A_1 \cap A_2 \cap \ldots \cap A_k) = \mathbf{P}(A_1) \ldots \mathbf{P}(A_k)$, then $\mathscr{G}_i$ are independent.*

PROOF. Fix $A_2 \in S_2, \ldots, A_k \in S_k$ and set $\mathscr{F}_1 := \{B \in \mathscr{G}_1 \; : \; \mathbf{P}(B \cap A_2 \cap \ldots \cap A_k) = \mathbf{P}(B)\mathbf{P}(A_2) \ldots \mathbf{P}(A_k)\}$. Then $\mathscr{F}_1 \supset S_1$ by assumption and it is easy to check that $\mathscr{F}_1$ is a λ-system. By the π-λ theorem, it follows that $\mathscr{F}_1 = \mathscr{G}_1$ and we get the assumptions of the lemma for $\mathscr{G}_1, S_2, \ldots, S_k$. Repeating the argument for S_2, S_3 etc., we get independence of $\mathscr{G}_1, \ldots, \mathscr{G}_k$. ■

Corollary 2.6. (1) *Random variables $X_1, \ldots, X_k$ are independent if and only if $\mathbf{P}(X_1 \le t_1, \ldots, X_k \le t_k) = \prod_{j=1}^{k} \mathbf{P}(X_j \le t_j)$.*

(2) *Suppose $\mathscr{G}_\alpha$, $\alpha \in I$ are independent. Let $I_1, \ldots, I_k$ be pairwise disjoint subsets of I. Then, the σ-algebras $\mathscr{F}_j = \sigma\left(\cup_{\alpha \in I_j} \mathscr{G}_\alpha\right)$ are independent.*

(3) *If $X_{i,j}$, $i \le n$, $j \le n_i$, are independent, then for any Borel measurable $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$, the r.v.s $f_i(X_{i,1},\dots,X_{i,n_i})$ are also independent.*

PROOF. (1) The sets $(-\infty, t]$ form a π-system that generates $\mathscr{B}(\mathbb{R})$. (2) For $j \le k$, let S_j be the collection of finite intersections of sets A_i, $i \in I_j$. Then S_j are π-systems and $\sigma(S_j) = \mathscr{F}_j$. (3) Follows from (2) by considering $\mathscr{G}_{i,j} := \sigma(X_{i,j})$ and observing that $f_i(X_{i,1},\dots,X_{i,k}) \in \sigma(\mathscr{G}_{i,1} \cup \dots \cup \mathscr{G}_{i,n_i})$. ■

So far, we stated conditions for independence in terms of probabilities if events. As usual, they generalize to conditions in terms of expectations of random variables.

Lemma 2.7. (1) *Sigma algebras $\mathscr{G}_1,\dots,\mathscr{G}_k$ are independent if and only if for every bounded $\mathscr{G}_i$-measurable functions X_i, $1 \le i \le k$, we have, $\mathbf{E}[X_1 \dots X_k] = \prod_{i=1}^k \mathbf{E}[X_i]$.*

(2) *In particular, random variables $Z_1,\dots,Z_k$ (Z_i is an n_i dimensional random vector) are independent if and only if $\mathbf{E}[\prod_{i=1}^k f_i(Z_i)] = \prod_{i=1}^k \mathbf{E}[f_i(Z_i)]$ for any bounded Borel measurable functions $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$.*

We say 'bounded measurable' just to ensure that expectations exist. The proof goes inductively by fixing $X_2,\dots,X_k$ and then letting X_1 be a simple r.v., a non-negative r.v. and a general bounded measurable r.v.

PROOF. (1) Suppose $\mathscr{G}_i$ are independent. If X_i are $\mathscr{G}_i$ measurable then it is clear that X_i are independent and hence $\mathbf{P}(X_1,\dots,X_k)^{-1} = \mathbf{P}X_1^{-1} \otimes \dots \otimes \mathbf{P}X_k^{-1}$. Denote $\mu_i := \mathbf{P}X_i^{-1}$ and apply Fubini's theorem (and change of variables) to get

$$
\begin{aligned}
\mathbf{E}[X_1 \dots X_k] &\overset{\text{c.o.v}}{=} \int_{\mathbb{R}^k} \prod_{i=1}^k x_i \, d(\mu_1 \otimes \dots \otimes \mu_k)(x_1,\dots,x_k) \\
&\overset{\text{Fub}}{=} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{i=1}^k x_i \, d\mu_1(x_1) \dots d\mu_k(x_k) \\
&= \prod_{i=1}^k \int_{\mathbb{R}} u \, d\mu_i(u) \overset{\text{c.o.v}}{=} \prod_{i=1}^k \mathbf{E}[X_i].
\end{aligned}
$$

Conversely, if $\mathbf{E}[X_1 \dots X_k] = \prod_{i=1}^k \mathbf{E}[X_i]$ for all $\mathscr{G}_i$-measurable functions X_is, then applying to indicators of events $A_i \in \mathscr{G}_i$ we see the independence of the σ-algebras $\mathscr{G}_i$.

(2) The second claim follows from the first by setting $\mathscr{G}_i := \sigma(X_i)$ and observing that a random variable X_i is $\sigma(Z_i)$-measurable if and only if $X = f \circ Z_i$ for some Borel measurable $f : \mathbb{R}^{n_i} \to \mathbb{R}$.

■

2.3. Independent sequences of random variables

First we make the observation that product measures and independence are closely related concepts. For example,

An observation: The independence of random variables $X_1,\dots,X_k$ is precisely the same as saying that $\mathbf{P} \circ X^{-1}$ is the product measure $\mathbf{P}X_1^{-1} \otimes \dots \otimes \mathbf{P}X_k^{-1}$, where $X = (X_1,\dots,X_k)$.

Consider the following questions. Henceforth, we write $\mathbb{R}^\infty$ for the countable product space $\mathbb{R} \times \mathbb{R} \times \ldots$ and $\mathscr{B}(\mathbb{R}^\infty)$ for the cylinder σ-algebra generated by all finite dimensional cylinders $A_1 \times \ldots \times A_n \times \mathbb{R} \times \mathbb{R} \times \ldots$ with $A_i \in \mathscr{B}(\mathbb{R})$. This notation is justified, becaue the cylinder σ-algebra is also the Borel σ-algebra on $\mathbb{R}^\infty$ with the product topology.

Question 1: Given $\mu_i \in \mathscr{P}(\mathbb{R})$, $i \geq 1$, does there exist a probability space with independent random variables X_i having distributions μ_i?

Question 2: Given $\mu_i \in \mathscr{P}(\mathbb{R})$, $i \geq 1$, does there exist a p.m μ on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ such that $\mu(A_1 \times \ldots \times A_n \times \mathbb{R} \times \mathbb{R} \times \ldots) = \prod_{i=1}^n \mu_i(A_i)$?

Observation: The above two questions are equivalent. For, suppose we answer the first question by finding an $(\Omega, \mathscr{F}, \mathbf{P})$ with *independent* random variables $X_i : \Omega \to \mathbb{R}$ such that $X_i \sim \mu_i$ for all i. Then, $X : \Omega \to \mathbb{R}^\infty$ defined by $X(\omega) = (X_1(\omega), X_2(\omega), \ldots)$ is measurable w.r.t the relevant σ-algebras (why?). Then, let $\mu := \mathbf{P}X^{-1}$ be the pushforward p.m on $\mathbb{R}^\infty$. Clearly

$$\begin{aligned} \mu(A_1 \times \ldots \times A_n \times \mathbb{R} \times \mathbb{R} \times \ldots) &= \mathbf{P}(X_1 \in A_1, \ldots, X_n \in A_n) \\ &= \prod_{i=1}^n \mathbf{P}(X_i \in A_i) = \prod_{i=1}^n \mu_i(A_i). \end{aligned}$$

Thus μ is the product measure required by the second question.

Conversely, if we could construct the product measure on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$, then we could take $\Omega = \mathbb{R}^\infty$, $\mathscr{F} = \mathscr{B}(\mathbb{R}^\infty)$ and X_i to be the i^{th} co-ordinate random variable. Then you may check that they satisfy the requirements of the first question.

The two questions are thus equivalent, but what is the answer?! It is 'yes', of course or we would not make heavy weather about it.

Proposition 2.8 (Daniell). *Let $\mu_i \in \mathscr{P}(\mathbb{R})$, $i \geq 1$, be Borel p.m on $\mathbb{R}$. Then, there exist a probability space with* independent *random variables $X_1, X_2, \ldots$ such that $X_i \sim \mu_i$.*

PROOF. We arrive at the construction in three stages.

(1) **Independent Bernoullis:** Consider $([0,1], \mathscr{B}, \mathbf{m})$ and the random variables $X_k : [0,1] \to \mathbb{R}$, where $X_k(\omega)$ is defined to be the k^{th} digit in the binary expansion of ω. For definiteness, we may always take the infinite binary expansion. Then by an earlier homework exercise, $X_1, X_2, \ldots$ are independent Bernoulli(1/2) random variables.

(2) **Independent uniforms:** Note that as a consequence, on any probability space, if Y_i are i.i.d. Ber(1/2) variables, then $U := \sum_{n=1}^\infty 2^{-n} Y_n$ has uniform distribution on $[0,1]$. Consider again the canonical probability space and the r.v. X_i, and set $U_1 := X_1/2 + X_3/2^3 + X_5/2^5 + \ldots$, $U_2 := X_2/2 + X_6/2^2 + \ldots$, etc. Clearly, U_i are i.i.d. U[0,1].

(3) **Arbitrary distributions:** For a p.m. μ, recall the left-continuous inverse G_μ that had the property that $G_\mu(U) \sim \mu$ if $U \sim U[0,1]$. Suppose we are given p.m.s $\mu_1, \mu_2, \ldots$. On the canonical probability space, let U_i be i.i.d uniforms constructed as before. Define $X_i := G_{\mu_i}(U_i)$. Then, X_i are independent and $X_i \sim \mu_i$. Thus we have constructed an independent sequence of random variables having the specified distributions. ∎

Sometimes in books one finds construction of uncountable product measures too. It has no use. But a very natural question at this point is to go beyond independence. We just state the following theorem which generalizes the previous proposition.

Theorem 2.9 (Kolmogorov's existence theorem). *For each $n \geq 1$ and each $1 \leq i_1 < i_2 < \ldots < i_n$, let $\mu_{i_1,\ldots,i_n}$ be a Borel p.m on $\mathbb{R}^n$. Then there exists a unique probability measure μ on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ such that*

$$\mu(A_1 \times \ldots \times A_n \times \mathbb{R} \times \mathbb{R} \times \ldots) = \mu_{i_1,\ldots,i_n}(A_1 \times \ldots \times A_n) \text{ for all } n \geq 1 \text{ and all } A_i \in \mathscr{B}(\mathbb{R}),$$

if and only if the given family of probability measures satisfy the consistency condition

$$\mu_{i_1,\ldots,i_n}(A_1 \times \ldots \times A_{n-1} \times \mathbb{R}) = \mu_{i_1,\ldots,i_{n-1}}(A_1 \times \ldots \times A_{n-1})$$

for any $A_k \in \mathscr{B}(\mathbb{R})$ and for any $1 \leq i_1 < i_2 < \ldots < i_n$ and any $n \geq 1$.