UM102 – Analysis and Linear Algebra II
2019 Spring Semester

[You are expected to write proofs / arguments with details of your reasoning, in solving these questions.]

Homework Set 7 (*Quiz on Monday, March 18*, in Tutorial Session)

From the textbook *Calculus, Vol. II* by Tom M. Apostol (2nd edition):

Newly added:

Section 8.5 (Exercises section), on pages 251–252:

Q1 (e), (i) Q6 Q9.

Section 8.3 (Exercises section), on pages 245–247:

Question 1: Prove that the following subsets of Euclidean space ($\mathbb{R}^n$ for some $n \geq 1$) are open:

(1) Q2(c): $\{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}$. (Sketch this region.)

(2) Generalize to $\{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_j| < 1$ for all $j = 1, \dots, n\}$.
(Maybe do for $n = 3$ first, then try to spot the pattern for $n = 1, 2, 3$ – the last is Q3(b) – then guess the pattern in general, and prove it!)

(3) Q2(f).

Question 2:

- Question 5 from Section 8.3 (except part (c) which was shown in class).
- Question 4(a), 7(a),(b).
  In fact, try to do these *only* using the various parts of the preceding Q5 (in other words, without any calculations!) and the definition of a closed set.
- Question 9.

Question 3: Question (2)(b) from the Exercises: *Prove using the definitions that the set* $U = \{(x, y) \in \mathbb{R}^2 : 3x^2 + 2y^2 < 6\}$ *is open.*

This is an ellipse, and this question is meant to show you that proving things are open isn't always immediate or straightforward. But soon we will use the power of continuous functions, to prove a much easier proof, for a much larger class of problems. For now, here are the steps, using just the definitions:

(1) Take a point $(x, y) \in U$, and let $E := 6 - 3x^2 - 2y^2$ be its 'error' (measuring how much inside it is from the 'boundary'). We want $\delta > 0$ such that if $\|(c, d)\| < \delta$, then $(x + c, y + d)$ is in the ellipse. (Why is this true?)

We will show that $\delta = \sqrt{12 + (E/3)} - \sqrt{12} > 0$ works.

(2) First show for this $(x, y) \in U$ that the norm of the vector $(|x|, |y|)$ is strictly less than $\sqrt{3}$.

(3) Now show that

$$\begin{aligned} 3(x + c)^2 + 2(y + d)^2 &= 6 - E + (3c^2 + 6cx + 2d^2 + 4dy) \\ &< 6 - E + 3(c^2 + d^2) + 6\ (|c|, |d|) \cdot (|x|, |y|). \end{aligned}$$

We want the left-hand side to be less than 6, *assuming/given that* $c^2 + d^2 < \delta^2$.

(4) Show that the right-hand side is $\leq -30 - E + 3(\delta + \sqrt{12})^2$. If this is < 6 then the left-hand side would also be < 6.

(5) Show that if $\delta + \sqrt{12} < \sqrt{12 + (E/3)}$ then in the previous part we do get: $-30E + 3(\delta + \sqrt{12})^2 < 6$.