UM102 – Analysis and Linear Algebra II
2019 Spring Semester

[You are expected to write proofs / arguments with details of your reasoning, in solving these questions.]

Homework Set 1 (*due by Wednesday, January 9*, in class)

Suppose $A_{m\times n}$ is an arbitrary matrix, with entries in an arbitrary field $\mathbb{F}$, and with the additional property that the matrix $A^T A$ is invertible.[1]

Now define the matrix $P := A(A^T A)^{-1} A^T$.

(1) What are the dimensions / size of the matrix P?

(2) Suppose $u \in \mathbb{F}^m$ is in the column space of A. Compute Pu.

(3) Suppose $m = n$ and A is invertible. Compute P.

(4) Verify that P satisfies two properties:
  (a) P is symmetric, and
  (b) $P^k = P$ for all integers $k \geq 1$.

(5) Verify that the matrix $I - P$ also satisfies the properties (a) and (b) mentioned in the previous part. Here, I denotes the identity matrix of the same size as P.

(6) Compute $P(I - P)$, where I is as above.

Such a matrix P is an example of a *projection* matrix.

(7) For this question, suppose $A_{m\times n}$ and $B_{n\times m}$ are matrices with entries in a field $\mathbb{F}$. Verify that AB and BA (though of possibly different sizes) have the same *trace*. Recall here that the trace of a square matrix is the sum of its diagonal entries.

Extra questions, not for submission, and only if you want to try them:
(independent of our syllabus, homeworks, exams, and grades)

Question 1. Show that the plane $\mathbb{R}^2$ is not the union – as sets – of finitely many lines. [Hint on next page.]

Question 2. For any $n \geq 1$, show that the space $\mathbb{R}^n$ is not the union of two proper subspaces (i.e., subspaces of smaller dimension). [Hint on next page.]

[1] Not for submission: Does this mean A is invertible?

Hint to Q1: Slopes.

Hint to Q2: First if either $V_1 \subset V_2$ or $V_2 \subset V_1$ then the problem is easy. (Why?)

So we can assume that V_1, V_2 are neither a subset of the other subspace. Now if $x_1 \in V_1$ is not in V_2, and $x_2 \in V_2$ is not in V_1, then find a vector that can lie in neither subspace V_1, V_2.