UM102 – Analysis and Linear Algebra II
2019 Spring Semester

Existence and uniqueness of the determinant

Here are some notes on the existence and uniqueness of the determinant function, which serve to summarise what we have seen in the past few lectures.

Definition 1. A matrix $\overline{A}$ is said to be in **reduced row echelon form (RREF)** if it satisfies the following properties:

(1) If $\overline{A}$ has one or more rows of zeros, these rows must occur at the bottom.
(2) In any nonzero row, the leftmost nonzero entry must be 1; this is called the **pivot** of the row.
(3) In the column containing a pivot entry 1, every other entry is zero.
(4) The pivot entry 1 in a nonzero row occurs strictly to the right of the pivot entry 1 in the row above it.

We will also say that a matrix A is in **reduced column echelon form (RCEF)** if the columns of A satisfy analogous axioms to above – equivalently, a simpler way to say this is to ask that A^T be in RREF.

For example, the identity and zero matrices are in RREF and also in RCEF; and the matrix

$$A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is also in RREF for any scalar a, but not in RCEF (even if $a = 0$ – why not?).

Here are some basic properties of RREF and RCEF, which you will again see if you take the linear algebra course next year:

Proposition 2 (Basic properties of RREF / RCEF)**.** *Let $A_{m\times n}$ be any matrix (with entries in some field $\mathbb{F}$).*

(1) *The row space and column space of A have the same dimension. In other words, the row rank and column rank of A are equal.*
(2) *There exists a **unique** matrix $\overline{A}$ in RREF, into which A can be converted by a sequence of row operations (i.e. via row-reduction, or Gauss–Jordan elimination).*

    (By starting with A and applying this fact to A^T, and then again taking transpose at the end, we obtain the analogous result for column operations:)

    There is a unique matrix in RCEF into which A can be converted.
(3) *If the rows of A are linearly dependent, the RREF $\overline{A}$ will have a zero row at the bottom. Similarly for the case of dependent columns.*
(4) *If the matrix A is square, and the rows – or columns – of A are linearly independent, then the RREF and RCEF of A both equal the identity matrix.*

We will require another result, this time on the interplay between column operations, their 'inverse operations', and their effects on the determinant:

Proposition 3. *There are three types of column operations that we will carry out on a square matrix:*

(1) *One can rescale a particular column by a nonzero scalar α. This multiplies the matrix determinant by α. The 'inverse operation' is to rescale the same column by $\alpha^{-1} \in \mathbb{F}$ – and this multiplies the matrix determinant by the reciprocal scalar α^{-1}.*
(2) *One can interchange two columns. This multiplies the matrix determinant by -1. The 'inverse operation' is to repeat the same swap of columns – and this multiplies the matrix determinant by the reciprocal scalar $(-1)^{-1} = -1$.*
(3) *One can add α times the ith row to the jth row. This multiplies the matrix determinant by 1 (i.e., leaves the determinant unchanged). The 'inverse operation' is to subtract α times the ith row from the jth row – and this multiplies the matrix determinant by $1^{-1} = 1$.*

With these preliminaries, we are ready for the result we proved in an 'exciting finish' to yesterday's class:

Theorem 4. *Given an integer $n \geq 1$ and a field $\mathbb{F}$, there is a unique function – called the **determinant** – which sends every square $n \times n$ matrix with entries in $\mathbb{F}$ to a scalar in $\mathbb{F}$, and which satisfies the four 'column-axioms' for the determinant map.*

In proving this result, we will freely make use of the 'basic properties of determinants' that we have proved in class.

Proof. Let A be an $n \times n$ matrix. First – we have already shown that if the columns of A are linearly dependent then $\det(A) = 0$, and this is the only possible value of $\det(A)$.

Thus, we will assume henceforth that A is a square matrix with linearly independent columns – in particular, there is (at least) one sequence of columns operations that sends A to its RCEF, which is the identity matrix Id_n. Let us label these column operations as $Op_1, \dots, Op_k$, that is:

$$A \xrightarrow{Op_1} A_1 \xrightarrow{Op_2} A_2 \xrightarrow{Op_3} \cdots \xrightarrow{Op_{k-1}} A_{k-1} \xrightarrow{Op_k} A_k = \mathrm{RCEF}(A) = \mathrm{Id}.$$

For each $1 \leq i \leq k$, let the nonzero scalar corresponding to Op_i be α_i. In other words,

$$\det(Op_1(A)) = \alpha_1 \cdot \det(A), \qquad \det(Op_i(A_{i-1})) = \alpha_i \cdot \det(A_{i-1}), \ \forall i > 1.$$

Then by Proposition 3, the scalar corresponding to the 'inverse column operation' $(Op_i)^{-1}$ is precisely α_i^{-1}. Now,

$$\det(\mathrm{Id}) = 1 \text{ by the axioms.}$$

Apply $(Op_k)^{-1}$ and use the axioms of the determinant to get:

$$\det(A_{k-1}) = \alpha_k^{-1} \cdot \det \mathrm{Id} = \frac{1}{\alpha_k}.$$

Next, apply $(Op_{k-1})^{-1}$ and use the axioms to get:

$$\det(A_{k-2}) = \alpha_{k-1}^{-1} \cdot \det A_{k-1} = \frac{1}{\alpha_{k-1}\alpha_k}.$$

Continuing this way (by induction), we finally get:

$$\det(A_1) = \frac{1}{\alpha_2\alpha_3\cdots\alpha_k}, \qquad \det(A) = \frac{1}{\alpha_1\alpha_2\cdots\alpha_k}.$$

Thus, from just the knowledge of det(Id) and the axioms+basic properties of the determinant, we obtained $\det(A)$ for any square matrix. This proves the **existence** of a determinant function.

To prove **uniqueness**: what we showed above was that if an invertible matrix A can be reduced via a specific sequence of column operations to its RCEF = Id, then this uniquely determines $\det(A)$ by the above calculations. Thus, we have to show that if we have two different sequences of column operations that each reduces A to Id, then the associated values of $\det(A)$ are the same.

So: suppose

$$A \xrightarrow{Op_1} A_1 \xrightarrow{Op_2} A_2 \xrightarrow{Op_3} \cdots \xrightarrow{Op_{k-1}} A_{k-1} \xrightarrow{Op_k} A_k = \mathrm{RCEF}(A) = \mathrm{Id},$$

$$A \xrightarrow{Op'_1} B_1 \xrightarrow{Op'_2} B_2 \xrightarrow{Op'_3} \cdots \xrightarrow{Op'_{l-1}} B_{l-1} \xrightarrow{Op'_l} B_l = \mathrm{RCEF}(A) = \mathrm{Id}.$$

are two different sets of column operations which send A to Id. Let Op_i be associated to the nonzero constant α_i, and Op'_j be associated to the nonzero constant β_j. Then by the same reasoning as above, we have two formulas:

$$\det(A) = \frac{1}{\alpha_1\cdots\alpha_k}, \qquad \det(A) = \frac{1}{\beta_1\cdots\beta_l},$$

and so we must prove that $\alpha_1\cdots\alpha_k = \beta_1\cdots\beta_l$.

To do so, we once more carry out the reverse column operations

$$\mathrm{Id} \longrightarrow A_k \longrightarrow \cdots \longrightarrow A_1 \longrightarrow\rightarrow A$$

to obtain:

$$\det(A) = \frac{1}{\alpha_1\cdots\alpha_k}.$$

Next, we *further* perform the column operations $Op'_1,\ldots,Op'_l$ sequentially on A, and get back to Id. Check for yourselves that the determinant continues to get multiplied by the constants $\beta_1,\beta_2,\ldots,\beta_l$ in that order. Thus, we have

$$\det \mathrm{Id} = 1 = \frac{\beta_1\cdots\beta_l}{\alpha_1\cdots\alpha_k}.$$

It follows that $\alpha_1\cdots\alpha_k = \beta_1\cdots\beta_l$. This proves that $\det(A)$ takes a unique value for every square matrix A. $\square$