MA341 – Matrix Analysis and Positivity
2023 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Thursday, September 7* in class, or previously in office hours)

Question 1 (*The (de)compression trick*)**.**

(1) Fix real scalars a, b, c and integers $m, n > 0$. Show that the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive semidefinite (psd), if and only if the matrix $\begin{pmatrix} a & a & b \\ a & a & b \\ b & b & c \end{pmatrix}$ is psd,

or more generally, if and only if the block matrix $\begin{pmatrix} a\mathbf{1}_{n\times n} & b\mathbf{1}_{n\times m} \\ b\mathbf{1}_{m\times n} & c\mathbf{1}_{m\times m} \end{pmatrix}$ is psd.

Here, $\mathbf{1}_{m\times n}$ denotes the $m \times n$ matrix of all ones.

(2) (Merely) State the generalization of this result to arbitrary real symmetric matrices being positive semidefinite or not.

Question 2 (*The correlation trick*)**.** Recall that a positive semidefinite matrix is a *correlation matrix* if all its diagonal entries are 1.

(1) Prove that for every positive definite matrix A, there exists a unique positive definite diagonal matrix D and correlation matrix C such that $A = DCD$.

(2) Fix a dimension $n \geq 1$. Does the procedure in the previous part recover all $n \times n$ correlation matrices? Prove or find a counterexample.

(3) Prove that A and C have the same *pattern of zeros* and the same rank. By the former, we mean that if $a_{jk} = 0$ for some j, k then $c_{jk} = 0$ as well.

Question 3. If the columns of an $m \times n$ real matrix A are linearly independent, verify that its Moore–Penrose inverse is $A^\dagger = (A^T A)^{-1} A^T$.

Question 4. Suppose $n \geq 1$ is an integer and $C_{n\times n}$ is a correlation matrix, i.e. C is positive semidefinite with all diagonal entries 1.

(1) Show that $n\mathrm{Id} - C$ is positive semidefinite.

(2) Show with an example that the coefficient n in the preceding question is sharp (i.e., cannot be reduced).

(3) More generally, show that if $A \in \mathbb{P}_n$ and D is the diagonal matrix with (j,k)-entry $\delta_{j,k} a_{jj}$, then $nD - A$ is positive semidefinite.

Question 5. Another construction of new positive definite matrices from older ones: W. Pusz and S. L. Woronowicz, *Functional calculus for sesquilinear forms and the purification map*, Rep. Math. Phys. 8 (1975), 159–170.

(1) Verify that the *geometric mean* of two positive definite (real) $n \times n$ matrices, given by
$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$$
is positive definite.

(2) Verify that $A \# B$ is the unique positive definite solution X to the *Riccati equation* $XA^{-1}X = B$. (Hint: First do this for $A = \mathrm{Id}$.)

(3) Consequently, show that $A \# B = B \# A$, and
$$(C^{-1}AC) \# (C^{-1}BC) = C^{-1}(A \# B)C$$
for positive definite A, B and unitary C.

(4) When A, B commute, show that $A \# B = (AB)^{1/2}$.