MA341 – Matrix Analysis and Positivity
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 3 (*due by 5pm on Monday, September 16 in Instructor's office*)

Question 1. Suppose $n \geq 1$ is an integer and $C_{n \times n}$ is a correlation matrix, i.e. $C \in \mathbb{P}_n$ with all diagonal entries 1.

(1) Show that $n\mathrm{Id} - C$ is positive semidefinite.
(2) Show with an example that the coefficient n in the preceding question is sharp (i.e., cannot be reduced).
(3) More generally, show that if $A \in \mathbb{P}_n$ and D is the diagonal matrix with (j,k)-entry $\delta_{j,k} a_{jj}$, then $nD - A$ is positive semidefinite.

Question 2. Suppose $A \in \mathbb{P}_n$ has all entries either 1 or -1. Prove that A has rank-one, i.e. there exists a vector u such that $A = uu^T$. (Hint: Let u be the first column of A; then by considering the principal minor with the $1, j, k$ columns, show that the (j,k)-entry of A equals $u_j u_k$.)

Question 3 (Hershkowitz, Neumann, Schneider)**.** Show the same result (as in the preceding question) for a complex Hermitian positive semidefinite matrix A whose entries all lie on the unit circle, i.e., have modulus one. (Note that A is positive semidefinite if $u^* A u \geq 0$ for all vectors $u \in \mathbb{C}^n$, with u^* the transpose of the entrywise conjugate of u.) The same hint as above applies with u_k replaced by $\overline{u_k}$.

Question 4. Given an integer $n \geq 1$, let $\mathcal{S}_n$ denote the set of truncated moment matrices of measures $\mu \geq 0$ on $[0, \infty)$. (In other words, the matrices $(s_{j+k}(\mu))_{j,k=0}^{n-1}$.) With the convention $0^0 := 1$, classify the entrywise powers that preserve total non-negativity on this set.

Question 5.

(1) Using the Kronecker product, prove the 'nonsingular' version of the Schur product theorem: show that if A, B are positive definite, then so is $A \circ B$.
(2) Show the same result using just the spectral theorem.

Question 6. Recall Tanvi Jain's 2017 result (which I mentioned in class): If $x_1, \dots, x_n$ are pairwise distinct positive real numbers, and $A := (1 + x_j x_k)_{j,k=1}^n$ for $n \geq 2$, then $A^{\circ\alpha}$ is positive semidefinite if and only if $\alpha \in \mathbb{Z}^{\geq 0} \cup [n-2, \infty)$.

(1) Suppose p_j, q_j for $1 \leq j \leq n$ are positive real numbers such that the ratios p_j/q_j are pairwise distinct. Define $A := (p_j p_k + q_j q_k)_{j,k=1}^n$. Then $A^{\circ\alpha}$ is positive semidefinite if and only if $\alpha \in \mathbb{Z}^{\geq 0} \cup [n-2, \infty)$.

(2) We used Jain's 2017 result above to show that if $\alpha \notin \mathbb{Z}^{\geq 0} \cup [n-2, \infty)$, then there exists a Hankel moment matrix 'counterexample' A such that $A^{\circ\alpha} \notin \mathbb{P}_n$.

Now use the previous part to show the *Toeplitz* analogue of the same result. Namely, if $\alpha \notin \mathbb{Z}^{\geq 0} \cup [n-2, \infty)$, then produce a Toeplitz 'counterexample' A such that $A^{\circ\alpha} \notin \mathbb{P}_n$. This proves the classification of the entrywise powers preserving positivity on Toeplitz $n \times n$ matrices.