MA341 – Matrix Analysis and Positivity
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Friday, August 23* in TA's office hours, or previously in class)

Question 1 (*Graph Laplacians*)**.** Suppose G is a weighted graph on nodes $1, \dots, n$. In other words, attach a non-negative real weight $w_{ij} = w_{ji}$ to each pair of nodes $\{i, j\}$ with $i \neq j$ (where $w_{ij} = 0$ denotes a lack of an edge). Now define the *graph Laplacian* of G to be the $n \times n$ matrix L_G with (i, j) entry $-w_{ij}$ for $i \neq j$, and (i, i) entry $\sum_{j \neq i} w_{ij}$.

Show that L_G is always positive semi-definite.

Question 2 (*The (de)compression trick*)**.**

(1) Fix real scalars a, b, c and integers $m, n > 0$. Then the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive semidefinite, if and only if the block matrix $\begin{pmatrix} a\mathbf{1}_{n\times n} & b\mathbf{1}_{n\times m} \\ b\mathbf{1}_{m\times n} & c\mathbf{1}_{m\times m} \end{pmatrix}$ is positive semidefinite. Here, $\mathbf{1}_{m\times n}$ denotes the $m \times n$ matrix of all ones.

(2) Extend – and prove – the previous statement to arbitrary positive semidefinite matrices.

Question 3 (*The correlation trick*)**.** Recall that a positive semidefinite matrix is a *correlation matrix* if all its diagonal entries are 1.

(1) Prove that for every positive definite matrix A, there exists a unique positive definite diagonal matrix D and correlation matrix C such that $A = DCD$.

(2) Fix a dimension $n \geq 1$. Does the procedure in the previous part recover all $n \times n$ correlation matrices? Prove or find a counterexample.

(3) Prove that A and C have the same *pattern of zeros* and the same rank. By the former, we mean that if $a_{jk} = 0$ for some j, k then $c_{jk} = 0$ as well.

Question 4. Here is an example of positive semidefiniteness arising from a combinatorial problem. Given integers $n, r > 0$ and an alphabet with r letters $\{1, \dots, r\}$, consider the set $S_{(n)}$ of all words/strings of length n. Also fix non-negative *weights*

$w_1, \ldots, w_r$. Now given two words $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{s}' = (s'_1, \ldots, s'_n)$ of length n, define their *interaction* to be

$$f(\mathbf{s}, \mathbf{s}') := \sum_{i=1}^{r} w_i \cdot \#\{1 \leq j \leq n : s_j = s'_j = i\}.$$

Prove that the $S_{(n)} \times S_{(n)}$ matrix with $(\mathbf{s}, \mathbf{s}')$-entry $f(\mathbf{s}, \mathbf{s}')$ is positive semidefinite. (Hint: Consider the refinement in which we use weights w_{ij} for the two strings both having letter i in position j.)

Question 5. Another construction of new positive definite matrices from older ones: W. Pusz and S. L. Woronowicz, *Functional calculus for sesquilinear forms and the purification map*, Rep. Math. Phys. 8 (1975), 159–170.

(1) Verify that the *geometric mean* of two positive definite (real) $n \times n$ matrices, given by
$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$
is positive definite.

(2) When A, B commute, show that $A\#B = (AB)^{1/2}$.