MA221 – Analysis I : Real Analysis
2017 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 6 (*due by Friday, Nov 17,* in class or TA's office hours)

Question 1.

(i) The Dirichlet function $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Show that the function f is not Riemann integrable.

(ii) Let $f : [0,1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is Riemann integrable on $[0,1]$ and find $\int_0^1 f$.

(iii) Find $\int_0^1 x^2$ from first principles, i.e., without using the Fundamental Theorem of Calculus or anti-derivatives.

Question 2. Prove that the Riemann integral is linear, that is, for any pair of Riemann integrable functions $f, g : [0,1] \to \mathbb{R}$, you have to show that (i) $\int_0^1 cf = c\int_0^1 f$, $c \in \mathbb{R}$, and $\int_0^1 f + \int_0^1 g = \int_0^1 (f+g)$.
[Part (ii) was done in class; solve part (i) in full detail, and from first principles.]

Question 3. Prove that a monotonic function $f : [0,1] \to \mathbb{R}$ is Riemann integrable.

Question 4. Prove or disprove the statement: Every continuous function $f : [a,b] \to \mathbb{R}$ is the derivative of some other function g defined on $[a,b]$.

Question 5. Assume $f : [a,b] \to \mathbb{R}$ is integrable. Show that $|f|$ is integrable and that $\left|\int_a^b f\right| \le \int_a^b |f|$.
[This will not be graded.]

Question 6. Let $f : [0,1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

Show that:

(i) f is not continuous at 0,
(ii) f is Riemann integrable on $[0,1]$, and
(iii) f has an anti-derivative.