MA221 – Analysis I : Real Analysis
2017 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 4 (*due by Friday, October 20*, in class or TA's office hours)

Question 1. Rudin Chapter 3 Problems 6, 9, 16.

Question 2. Rudin Chapter 8 Problem 6(b), 10.

The next set of problems constructs a rather exotic metric space. Namely, it continues beyond a previous set of homework questions, which showed that the set of all norms on $\mathbb{R}^k$ are 'similar' (i.e., gave rise to the same open sets=topology). Our goal below is to show that (informally speaking,) *the space of these metrics itself forms a metric space*!

Question 3. We will say that two norms $N, N' : \mathbb{R}^k \to \mathbb{R}$ are *equivalent*, written $N \sim N'$, if there exists a scalar $\alpha > 0$ such that $N'(\mathbf{x}) = \alpha \cdot N(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^k$. Prove that $\sim$ is an equivalence relation on the set of all norms $: \mathbb{R}^k \to \mathbb{R}$.

We now prove that the set $\mathcal{S}$ of *equivalence classes* of norms forms a metric space. The next question proves what is needed to define the distance between two such norms.

Question 4. Suppose N_1, N_2 are two norms on $\mathbb{R}^k$.

(a) Prove that (the boundary of) the 'unit N_1-ball'

$$B_1 := \{\mathbf{x} \in \mathbb{R}^k : N_1(\mathbf{x}) = 1\}$$

is compact.

Hint: Using the 'similarity' of N_1 and the usual 'Euclidean norm' $\|\mathbf{x}\|_2 := (x_1^2 + \cdots + x_k^2)^{1/2}$ which was proved in HW3, show that B_1 is closed and bounded in $(\mathbb{R}^k, \|\cdot\|_2)$. But from HW3, compact sets in $\|\cdot\|_2$ are compact in any norm on $\mathbb{R}^k$.

(b) Prove that $N_2 : (\mathbb{R}^k, N_1) \to \mathbb{R}$ is continuous, where N_1 induces the metric on $\mathbb{R}^k$.

(c) Using the previous two parts, prove that there exist real numbers $0 < m \leqslant M$ such that

$$N_2/N_1 : \mathbb{R}^k \setminus \{\mathbf{0}\} \to \mathbb{R}$$

maps inside $[m, M]$, and the extreme values are attained.

(d) Finally, prove that if $N_1' \sim N_1$ and $N_2' \sim N_2$ are any other equivalent norms (as in the previous question), then there exists $c > 0$ such that

$$N_2'/N_1' : \mathbb{R}^k \setminus \{\mathbf{0}\} \to \mathbb{R}$$

maps inside $[cm, cM]$, and the extreme values are attained.

Question 5. Now we can define the distance between two points in $\mathcal{S}$. Given two equivalence classes of norms in $\mathcal{S}$, choose any two representative norms N_1, N_2 from these classes, and define

$$d_{\mathcal{S}}(N_1, N_2) := \log(M/m),$$

where $0 < m \leqslant M$ are as in Question 4(c) above.

(a) If $N_1 \sim N_1'$ and $N_2 \sim N_2'$ are similar norms (i.e., in the same equivalence classes), then verify that

$$d_{\mathcal{S}}(N_1', N_2') = d_{\mathcal{S}}(N_1, N_2).$$

Hence $d_{\mathcal{S}}$ is a well-defined function on $\mathcal{S} \times \mathcal{S}$.

(b) Prove that $d_{\mathcal{S}}$ is a metric on $\mathcal{S}$. (Note: given the previous part, in this part you can work with pairs of 'actual' norms instead of equivalence classes of norms.)

(c) Suppose $k = 1$. What is the metric space $\mathcal{S}$ of (equivalence classes of) norms on $\mathbb{R}^1$?

Next, let us compute the distances in this exotic space for general $\mathbb{R}^k$ (let us call it $\mathcal{S}_k$) between some 'standard' norms.

Question 6. Fix an integer $k > 0$. For every real scalar $p \in [1, \infty)$, define the *p-norm* to be:

$$\|\mathbf{x}\|_p := (|x_1|^p + \cdots + |x_k|^p)^{1/p}.$$

Note that $\|\cdot\|_2$ is the usual norm / Euclidean distance in $\mathbb{R}^k$.

Our goal here is to calculate the distance in the metric space $\mathcal{S}_k$, between the p-norm and the q-norm for any $1 \leqslant p < q < \infty$.

(a) As a special case, prove directly that for all vectors $\mathbf{x} \in \mathbb{R}^k$, we have:

$$\|\mathbf{x}\|_2 \leqslant \|\mathbf{x}\|_1 \leqslant \sqrt{k}\|\mathbf{x}\|_2,$$

and both inequalities are sharp – i.e., equality can be attained in both of them. Hence, what is the distance between (the equivalence classes of) $\|\cdot\|_1$ and $\|\cdot\|_2$ in $\mathcal{S}_k$?

(b) The previous part involved a fundamental inequality. For the general case, we will require another fundamental inequality, by *Hölder*. The inequality (in our special case of interest) says that for all $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$,

$$\left(\frac{1}{k}(|x_1|^p + \cdots + |x_k|^p)\right)^{1/p} \leqslant \left(\frac{1}{k}(|x_1|^q + \cdots + |x_k|^q)\right)^{1/q}.$$

(Do not prove this, just assume it.) Using Hölder's inequality above, obtain one inequality that compares the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ in $\mathcal{S}_k$.

(c) For the 'other' inequality, we claim that $\|\mathbf{x}\|_p \geqslant \|\mathbf{x}\|_q$ for all $\mathbf{x}$, if $1 \leqslant p \leqslant q < \infty$. Clearly this holds for $\mathbf{x} = \mathbf{0}$; else it suffices to assume $\|\mathbf{x}\|_p = 1$. (Why?) Now prove the inequality.

(d) Use the previous two parts to compute the distance between (the equivalence classes of) the p-norm and the q-norm in $\mathcal{S}_k$. Note: first you will need to check – as in part (a) – that there exist *nonzero* vectors in $\mathbb{R}^k$ at which the two inequalities above are attained.

(e) There is another standard norm, which we saw in HW3: $\|\mathbf{x}\|_\infty := \max(|x_1|, \ldots, |x_k|)$. Prove that for all $p \in [1, \infty)$, we have

$$\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_p \leqslant k^{1/p} \|\mathbf{x}\|_\infty,$$

whence $d_{\mathcal{S}_k}(\| \cdot \|_p, \| \cdot \|_\infty) = \frac{\log k}{p}$.

Finally, one can ask how the set of these (equivalence classes of) p-norms *looks like* as a metric subspace of $\mathcal{S}_k$.

Question 7. A map of metric spaces $f : (X, d_X) \to (Y, d_Y)$ is called an *isometry* if f is 'distance-preserving':

$$d_Y(f(x), f(x')) = d_X(x, x'), \ \forall x, x' \in X.$$

(a) Prove that every isometry is injective, i.e., one-to-one – as well as continuous.
(b) As an example, the next few parts classify all the isometries from the normed space $\mathbb{R}$ to itself. Indeed, given such an isometry f, define $a := f(0)$. Then $f(1) = a + 1$ or $a - 1$, say $a + \epsilon$ for some $\epsilon = \pm 1$. Now successively compute $f(2), f(3), \ldots$ as well as $f(-2), f(-3), \ldots$.
(c) Next, compute $f(1/2)$, hence $f(n/2)$ for all integers n, as in (b).
(d) Compute $f(1/4)$, hence $f(n/4)$ for all integers n, as in (b).
(e) In general, guess $f(n/2^k)$ for all integers $k > 0$ and n.
(f) Finally, prove using (a),(e) that $f(x) = xf(1) + (1 - x)f(0)$ for all $x \in \mathbb{R}$. In other words, f must be linear. Conversely, verify that every linear map : $\mathbb{R} \to \mathbb{R}$ with slope ± 1 is an isometry.

Question 8. Finally, given an integer $k > 0$, let $\mathcal{S}'_k$ denote the subset of equivalence classes of norms $\{\| \cdot \|_p : 1 \leqslant p \leqslant \infty\}$ on $\mathbb{R}^k$. Let the map

$$f : \mathcal{S}'_k \to [0, \log k]$$

be given by: $f(\| \cdot \|_p) := \frac{\log k}{p}$ if $p < \infty$, and $f(\| \cdot \|_\infty) := 0$. Prove that f is an isometry. This means that the subset of (equivalence classes of) norms $\| \cdot \|_p$ looks like the interval $[0, \log k]$ equipped with the usual metric in $\mathbb{R}$.