MA221 – Analysis I : Real Analysis
2017 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 3 (*due by Thursday, September 28*, in class or earlier in the week)

Question 1. Given a space X, two metrics d, d' on X are said to be *similar* if there are *uniform* constants $0 < m \leqslant M$ such that

$$m \cdot d(x,y) \leqslant d'(x,y) \leq M \cdot d(x,y), \qquad \forall x, y \in X. \tag{1}$$

(1) Now **prove that** if d, d' are similar metrics on a space X, then they induce the same 'topology', i.e., the collection of open subsets of X

$$\mathcal{O}_d := \{U \subset \mathbb{R}^k : U \text{ is open with respect to the metric } d\} \tag{2}$$

equals the correspondingly defined collection $\mathcal{O}_{d'}$.

(2) Prove that a subset $K \subset X$ is compact in the d-metric, if and only if it is compact in the d'-metric.

Question 2. Recall that a *norm* on $\mathbb{R}^k$ is a function $N : \mathbb{R}^k \to \mathbb{R}$ satisfying:

- $N(\mathbf{x}) \geqslant 0 \ \forall \mathbf{x} \in \mathbb{R}^k$, with equality if and only if $\mathbf{x} = \mathbf{0}$;
- For all scalars $a \in \mathbb{R}$ and vectors $\mathbf{x} \in \mathbb{R}^k$, we have: $N(a \cdot \mathbf{x}) = |a| \cdot N(\mathbf{x})$; and
- $N(\mathbf{x} + \mathbf{y}) \leqslant N(\mathbf{x}) + N(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$.

We now claim that norms are the same as 'special' metrics:

(1) Prove that every norm N gives rise to a metric d_N which is
  - 'translation-invariant', i.e.: $d_N(\mathbf{x}+\mathbf{z}, \mathbf{y}+\mathbf{z}) = d_N(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$;
  - and 'scaling-equivariant', i.e.: $d_N(a \cdot \mathbf{x}, a \cdot \mathbf{y}) = |a| d_N(\mathbf{x}, \mathbf{y})$ for all $a \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$.

(2) Conversely, prove that every translation-invariant and scaling-equivariant metric d gives rise to a norm N_d.

Next, we study a very specific norm on $\mathbb{R}^k$:

Question 3. Suppose $N_\infty(\mathbf{x}) := \max_{1 \leqslant j \leqslant k} |x_j|$, for $\mathbf{x} \in \mathbb{R}^k$.

(1) Check that N_∞ is a norm on $\mathbb{R}^k$; it is called the *sup-norm*.

(2) What is the unit ball in N_∞? Namely, what is the set $B_\infty(\mathbf{0}, 1) := \{\mathbf{x} \in \mathbb{R}^k : N_\infty(\mathbf{x}) = 1\}$?

(3) Let $N_2(\mathbf{x}) := (x_1^2 + \cdots + x_k^2)^{1/2}$ denote the usual Euclidean norm. Prove that $N_\infty(\mathbf{x})$ is similar to $N_2(\mathbf{x})$, i.e., there exist universal constants $0 < m \leqslant M$ such that

$$m \cdot N_\infty(\mathbf{x}) \leqslant N_2(\mathbf{x}) \leqslant M \cdot N_\infty(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbb{R}^k.$$

(This is similar to something we worked out in class, about the convergence of a tuple of sequences.)

(4) Prove that the above unit N_∞-ball is compact in the metric space $(\mathbb{R}^k, N_\infty)$.

Question 4. A famous problem is to show that all norms N on $\mathbb{R}^k$ are similar (and in particular have the same topology). In terms of the previous problems, it means that for any two norms $N, N' : \mathbb{R}^k \to \mathbb{R}$, then one can find uniform constants $0 < m \leqslant M$ such that

$$m \cdot N(\mathbf{x}) \leqslant N'(\mathbf{x}) \leqslant M \cdot N(\mathbf{x}) \ \forall \mathbf{x} \in \mathbb{R}^k.$$

In a future homework, we will in fact prove a *stronger* statement than this fact, but for now our goal is to show this assertion (using the three questions above).

(1) Start by proving that 'similarity' is an equivalence relation on the space of all norms.

(2) Let $\mathbf{e}_j$ denote the unit vector with 1 in the jth coordinate, and 0 everywhere else. Then every vector $\mathbf{x} = (x_1, \ldots, x_k) = x_1\mathbf{e}_1 + \cdots + x_k\mathbf{e}_k$.

Now given any norm N, find a constant M such that $N(\mathbf{x}) \leqslant M \cdot N_\infty(\mathbf{x})$ for all $\mathbf{x}$.

(3) Prove that the inequality

$$N(\mathbf{x}) \leqslant M \cdot N_\infty(\mathbf{x}) \ \forall \mathbf{x} \in \mathbb{R}^k$$

implies that the map $N : (\mathbb{R}^k, N_\infty) \to \mathbb{R}$ is continuous.

(4) Evaluating the map N on the unit N_∞-ball

$$N : B_\infty(\mathbf{0}, 1) \to \mathbb{R},$$

and using the above results (and results from class), prove that there exists $m > 0$ such that $m \cdot N_\infty(\mathbf{x}) \leqslant N(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^k$. Together with Question 4.3 above, this proves N is similar to N_∞ for all N, and hence all norms are similar by Question 4.1.

Question 5. Prove the *Sandwich Theorem*: If $a_n \leqslant b_n \leqslant c_n$ are three real sequences, and $a_n \to L, c_n \to L$ for some $L \in \mathbb{R}$, then $b_n \to L$ as well.

Question 6. Rudin Chapter 3 Problem 23.

Question 7. Rudin Chapter 3 Problem 24.