MA219 – Linear Algebra
2023 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 9 (*due by Monday, November 27* by 5:30pm in TA's office, or previously in class)

Question 1. Suppose $A \in \mathbb{F}^{m\times n}$ for $m, n \geq 1$, and P, Q are square, invertible matrices over $\mathbb{F}$ (for an arbitrary field $\mathbb{F}$) such that PAQ is defined. Show that PAQ and A have the same rank.

Question 2. (Henceforth we work over $\mathbb{R}$ or $\mathbb{C}$.) Suppose a vector v_0 in an inner product space V is such that v_0 is orthogonal to every vector in V. Show that $v_0 = \mathbf{0}_V$.

Question 3. Verify the following *polarization identity* in a real inner product space V:

$$(x, y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right), \qquad \forall x, y \in V.$$

Question 4. Show the *triangle inequality* in a complex (or real) inner product space V: $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$, with equality if and only if $y = 0$ or x is a non-negative scalar multiple of y.

Question 5. Let $V = \mathbb{R}^3$ and

$$\mathbf{w}_1 = (\pi, 0, 0)^T, \quad \mathbf{w}_2 = (e, \pi, 0)^T, \quad \mathbf{w}_3 = (1, 1, 1)^T$$

(or forget the transposes and work without them). Apply the Gram–Schmidt algorithm to compute an orthogonal triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with the desired properties.

Question 6. Suppose $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^3$. Let $W \subset V$ be the subspace

$$W := \{(x, y, z)^T \in V : x + 2y + 3z = 0\}.$$

(1) Write down an orthogonal basis for W and one for $W^\perp$.
(2) Using this basis, compute $P_W(v)$, the projection onto W of $v = (1, 1, 1)^T$.
(3) Compute $P_W(v)$ differently, as $v - P_{W^\perp}(v)$.
(4) Suppose (w_1, w_2) form an orthonormal basis of W, and w_3 of $W^\perp$. Let $\mathcal{B} = (w_1, w_2, w_3)$. Compute $[P_W]_\mathcal{B}$. (In particular, this should tell you the eigenvalues of P_W and their algebraic (= geometric) multiplicities.)

Question 7. Show that if $A, B \in \mathbb{C}^{n\times n}$ are unitary matrices, then so are $AB, A^{-1}, A^T, \overline{A}$.

Question 8. We will show later that real symmetric matrices are diagonalizable, with all eigenvalues real – and this holds more generally for all complex Hermitian matrices ($A^* = A$). Similarly, you may have seen that if A is skew-Hermitian ($A^* = -A$), then A is diagonalizable with purely imaginary eigenvalues.

More generally now, suppose $z \in \mathbb{C}$ is a complex number, and suppose $A^* = zA$ for some matrix $A \in \mathbb{C}^{n\times n}$ and scalar $z \in \mathbb{C}$.

(1) If $|z| \neq 1$, show that $A = \mathbf{0}_{n\times n}$.
(2) Now suppose $|z| = 1$. Describe all diagonal matrices D with this property.
(3) Again suppose $|z| = 1$. Prove that every matrix A such that $A^* = zA$ is of the form UDU^*, where $U \in \mathbb{C}^{n\times n}$ is unitary and D is as in the previous part.