MA219 – Linear Algebra
2023 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 4 (*due by Thursday, September 14* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. We have seen that (by convention,) the zero vector space over any field has exactly one basis: the empty set. Now:

(1) Classify all nonzero vector spaces over all fields, which also have exactly one unordered basis – i.e., exactly one basis up to permuting its basis elements.
(2) Classify all nonzero vector spaces over all fields, which have exactly two unordered bases.

Question 2. Let V be finite-dimensional over $\mathbb{F}$, with an ordered basis $\mathcal{B}$. Find the coordinate-matrices of the linear transformations $\mathbf{0}, \mathrm{id}_V$ with respect to $\mathcal{B}, \mathcal{B}$.

Question 3. Suppose $\mathbb{F}$ is a field, and $T : \mathbb{F}^2 \to \mathbb{F}^2$ is the linear operator $T(x_1, x_2) := (x_2, -x_1)$, where $(x_1, x_2)^T = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ is with respect to the standard ordered basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$.

(1) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}}$?
(2) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}'}$, where $\mathcal{B}' = (\mathbf{e}_1 + \mathbf{e}_2, -\mathbf{e}_1)$?
(3) What is the transition matrix of $\mathcal{B}'$ into $\mathcal{B}$? Meaning, find the matrix P such that $[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}$ for all $v \in \mathbb{F}^2$.
(4) Suppose $\mathbb{F}$ has characteristic not 2 (so $2 = 1 + 1$ in $\mathbb{F}$). What is the coordinate vector of $(2, -1)^T$ in the standard basis, when written out in the basis $\mathcal{B}'$?

Question 4 (If you want, try this one after Tuesday's class – or see the videos online.)**.** The *direct product* of a family $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces is their Cartesian product, denoted

$$\prod_{i \in I} V_i = \times_{i \in I} V_i,$$

with a typical element $(v_i)_{i\in I}$. Also fix the *projection* maps

$$\pi_{i_0} : \prod_{i\in I} V_i \to V_{i_0}, \qquad (v_i)_{i\in I} \mapsto v_{i_0}.$$

(1) Verify that each π_{i_0} is a surjective $\mathbb{F}$-linear map.
(2) Write out the (complete) proof that this product satisfies the following "universal property":

Given any $\mathbb{F}$-vector space Z, and $\mathbb{F}$-linear maps $\varphi_i : Z \to V_i$ for all $i \in I$, there exists a unique $\mathbb{F}$-linear map $\varphi : Z \to \prod_{i\in I} V_i$ such that $\varphi_i = \pi_i \circ \varphi$ for all $i \in I$.

In other words, the Cartesian product proves the existence of an object that satisfies this universal property. (By class, every other "candidate" is isomorphic to this one.)