MA219 – Linear Algebra
2023 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 3 (*due by Thursday, September 7* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. (a) Show that three vectors in $\mathbb{R}^2$ are linearly dependent.
(b) Find three vectors in $\mathbb{R}^2$ such that any two are linearly independent.

Question 2. Suppose B is a domain set, and $Fun_0(B, \mathbb{F})$ is the set of functions $f : B \to \mathbb{F}$ such that $f(x) = 0$ for all but finitely many $x \in B$. (As we said in an earlier homework set, every vector space is akin to this, by assuming B to consist of the basis, and the value of f on B to be the coefficients such that f corresponds to that specific linear combination.)

The question here is to write down a basis of this vector space $Fun_0(B, \mathbb{F})$ over the ground field $\mathbb{F}$ (and to show that it is a basis).

Question 3. Suppose V, W are $\mathbb{F}$-vector spaces, and $T : V \to W$ is an $\mathbb{F}$-linear transformation.

(1) Show that $T(\mathbf{0}_V) = \mathbf{0}_W$ and that $T(-v) = -T(v)$ for all $v \in V$.
(2) Suppose T is a bijection of sets. Prove that the inverse map T^{-1} is also a linear transformation.

Question 4. Suppose V is an $\mathbb{F}$-vector space, with ordered basis $\mathcal{B} = (v_1, \ldots, v_n)$. Prove that the map $\eta : V \to \mathbb{F}^n$, sending a vector $v = c_1v_1 + \cdots + c_nv_n$ to the column vector $[v]_{\mathcal{B}} = (c_1, \ldots, c_n)^T$, is a vector space isomorphism.

Question 5. Suppose V, W are $\mathbb{F}$-vector spaces. Show that $Lin_{\mathbb{F}}(V, W)$, the space of $\mathbb{F}$-linear maps : $V \to W$, is a vector subspace of $Fun(V, W)$. (You are allowed to assume that the latter is an $\mathbb{F}$-vector space.)

Question 6. Suppose $A, B \in \mathbb{F}^{m\times n}$ for some integers $m, n \geq 1$. Prove that the following are equivalent:

(1) $A = B$.
(2) $Av = Bv$ for all vectors $v \in \mathbb{F}^n$.
(3) $A\mathbf{e}_j = B\mathbf{e}_j$ for all $1 \leq j \leq n$.

Question 7. Suppose $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$, and $\theta \in \mathbb{R}$. Suppose $T : V \to V$ is the linear transformation that rotates a vector counterclockwise by θ (radians). Compute the matrix of T with respect to the standard basis of V.

Question 8. Suppose $\mathbb{F}$ is a finite field of size $q \geq 2$, and V is an $\mathbb{F}$-vector space.

(1) If V is an $\mathbb{F}$-vector space, show that V is not the union of q-many proper subspaces.

(In particular, $\mathbb{R}^n$ is not the union of finitely many proper subspaces.)

*(Hint, for one possible approach: Suppose V **is** the union of q proper subspaces – let $2 \leq m \leq q$ be the smallest number of subspaces needed to cover V, say $W_1, \ldots, W_m \subset V$. Then there exist $w_i \in W_i$ such that $w_i \notin W_j$ for all $j \neq i$. Now consider certain $(q+1)$-many linear combinations of w_1, w_2.)*

(2) Now show that if we instead had $n \geq q+1$ (in fact $n = q+1$), then V can be a union of n proper subspaces (as long as V has dimension at least 2). To do so, first show that $\mathbb{F}^2$ is a union of $q+1$ proper subspaces.

(3) Now suppose $V \neq 0$ is an arbitrary $\mathbb{F}$-vector space of dimension at least 2 (and possibly infinite), and B is a basis of V. (Assume B exists.) Show that V is a union of $q+1$ proper subspaces.