MA219 – Linear Algebra
2023 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Thursday, August 31* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Solve the following systems of linear equations. Use row operations to obtain the RREF in all cases.

(1) $x + 2y + 3z = 0, \quad x + y + z = 1, \quad -x + z = 1$, over a field of characteristic 0.

(2) $x + 4y + 5z = 1, \quad y - z = 3, \quad x + z = 5$, over a field of characteristic 7.

Question 2. Suppose $\mathbb{F}$ is a field, and $n \geq 1$ an integer. For integers $1 \leq i, j \leq n$, define the $n \times n$ matrix E_{ij} as having all entries zero, except 1 in the (i, j)-entry.

Now find the span of the following sets – give (with some justification – maybe via the explicit description) the "conceptual description" (see e.g. towards the end of Lecture L06 in the videos).

(1) The matrices E_{ii} for $1 \leq i \leq n$.

(2) The matrices E_{ij} for $i < j$.

(3) The polynomials $x^2 - x, x^3 - x^2, \dots$ and the polynomial x, with $\mathbb{F} = \mathbb{R}$.

Question 3. For each of the following, explain whether or not the specified subset (of the corresponding vector space) is a subspace.

(1) The subset of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying: $f(1) - f(2) + 2f(3) = 0$.

(2) The subset of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying: $f(2) = f(3) + 1$.

(3) The subset of solutions to $Ax = b$ for some vector $b \neq 0$. Here $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ for some integers $m, n \geq 1$.

Question 4. Suppose V is a vector space over $\mathbb{F}$, and $L \subset V$ is a linearly independent subset. If $v \in V$ is not in the span of L, show that $L \cup \{v\}$ is also linearly independent.

Question 5. Let $\mathbb{F} = \mathbb{Q}$.

(1) Show that $\mathbb{R}$ is not a finite-dimensional $\mathbb{Q}$-vector space.
(2) Suppose V is a *countable*-dimensional $\mathbb{Q}$-vector space, i.e. a vector space with a countably infinite basis $v_1, v_2, \ldots, v_n, \ldots$ Show that V is the union of its finite-dimensional subspaces V_n spanned by $v_1, \ldots, v_n$.
(3) Show that $\mathbb{R}$ is not a countable-dimensional $\mathbb{Q}$-vector space.

Question 6. Suppose S is a linearly independent subset of a vector space W (over a field $\mathbb{F}$). Consider a chain of linearly independent subsets in W:

$$S = S_0 \subset S_1 \subset S_2 \subset \cdots$$

Prove that $\bigcup_{i \geq 0} S_i$ is also a linearly independent subset. (In a special case, this is the 'upper bound' of a 'chain' that is used in proving that every vector space has a basis, via Zorn's Lemma.)