MA219 – Linear Algebra
2023 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Thursday, August 17* in TA's office hours, or previously in class)

Question 1. The goal here is to show that the set of complex numbers

$$\mathbb{C} := \{a + bi = a + b\sqrt{-1} : a, b \in \mathbb{R}\}$$

under the operations

$$(a+bi)+(c+di) := (a+c)+(b+d)i, \quad (a+bi)\cdot(c+di) := (ac-bd)+(ad+bc)i,$$
$$0 := 0+0i, \quad 1 := 1+0i,$$
$$-(a+bi) := (-a)+(-b)i, \quad (a+bi)^{-1} := \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

is a field. (You are allowed to use that $\mathbb{R}$ is a field.)

I felt you should do some of these verifications at least once in your life – and it is not likely you will get to do this in any other course – so here, **prove that**:

(1) Multiplication is associative.
(2) $z \cdot z^{-1} = 1$ for all nonzero complex z.
(3) The distributive law holds.

Question 2. Suppose $\mathbb{F}$ is a field, with $a, b \in \mathbb{F}$. Prove the following statements.

(1) The elements a^{-1} (for $a \neq 0$) and $-a$ are unique in $\mathbb{F}$.
(2) $-a = (-1) \cdot a$.
(3) $(-1)^2 = 1$.
(4) $ab = 0$ in $\mathbb{F}$, if and only if $a = 0$ or $b = 0$.

Question 3. Show that if a field $\mathbb{F}$ contains a subfield $\mathbb{E}$, then $\mathbb{F}$ is a vector space over $\mathbb{E}$.

Question 4. Prove that a subset W of an $\mathbb{F}$-vector space V is a subspace if and only if $cw + w' \in W$ for all $w, w' \in W$ and $c \in \mathbb{F}$.

Question 5. Suppose V is a vector space over a field $\mathbb{F}$, and $c \in \mathbb{F}, v \in V$. Prove that $c \cdot \mathbf{0} = \mathbf{0} = 0 \cdot v$, where 0 is the zero in $\mathbb{F}$ and $\mathbf{0}$ is the zero in V.

Question 6. In class, we saw that the set of functions from any set to a field is a vector space. Now let me mention why **every** vector space is a subspace of such a space of functions.

Let B be a nonempty set, and $\mathbb{F}$ a field. Define $Fun_0(B, \mathbb{F})$ to be the set of all functions $f : B \to \mathbb{F}$ such that $f(b) = 0$ for all but finitely many $b \in B$. Show that $Fun_0(B, \mathbb{F})$ is an $\mathbb{F}$-vector space.

Note: You are allowed to use the fact mentioned in class, that the set of *all* functions $Fun(B, \mathbb{F})$ is an $\mathbb{F}$-vector space, which might help bypass a lot of the routine verifications for the subset Fun_0. (Similar to part (2) of the next question.)

(Later, we will say that every $\mathbb{F}$-vector space V has a "basis" B, and then the space V can in fact be identified precisely with our space here: $Fun_0(B, \mathbb{F})$.)

Question 7. Prove that:

(1) $(AB)^T = B^T A^T$ for all $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times p}$ – by checking entry by entry.

(2) $(B+C)A = BA+CA$ for $B, C \in \mathbb{F}^{m\times n}, A \in \mathbb{F}^{n\times p}$ – *without* using any entry-by-entry calculations, but using the previous part and that $A(B+C) = AB+AC$ (shown in class) whenever defined.

Question 8. The *trace* of a square matrix $A = (a_{ij})_{i,j=1}^n$ is the sum of its diagonal entries: $a_{11} + a_{22} + \cdots + a_{nn}$. Given integers $m, n \geq 1$ and matrices $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}$, prove that AB and BA have the same trace, even if they have different sizes.

Question 9. Suppose $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times p}$ are invertible. Prove that $(A^T)^{-1} = (A^{-1})^T$ and $(AB)^{-1} = B^{-1}A^{-1}$. (Prove only from one side, e.g. $M \cdot M^{-1} = \text{Id}$, not from both sides.)