MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 8 (*due by 9:30 am on Friday October 14* in TA's office, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose $V_1, \dots, V_n, W$ are $\mathbb{F}$-vector spaces, and

$$T_1, \dots, T_k : V_1 \times \cdots \times V_n \to W$$

are multilinear maps. Show that so is $\sum_{i=1}^k c_i T_i$, for any choice of scalars $c_i \in \mathbb{F}$.

(For your homework, it will suffice to check the linearity in the nth argument.)

Question 2. Using results from class about how the determinant changes under elementary row operations (or other results about the determinant), compute the determinants of the matrices $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Question 3. Suppose $A \in \mathbb{F}^{n \times n}$ is block *lower* triangular – with blocks M_{ij} where $M_{ij} = 0$ for $i < j$, and with square diagonal entries M_{ii}. Using results from class, compute $\det A$.

Question 4. Given a square matrix $A \in \mathbb{F}^{n \times n}$, define its *adjugate* matrix $adj(A) \in \mathbb{F}^{n \times n}$ to have (i, j) entry $(-1)^{i+j} \det A_{j|i}$, where $A_{j|i} \in \mathbb{F}^{(n-1)\times(n-1)}$ is the matrix obtained by removing the jth row and ith column of A. Prove the following properties for any matrix $A \in \mathbb{F}^{n \times n}$, say with $n \geq 2$:

(1) $adj(A) \cdot A = A \cdot adj(A) = (\det A)\mathrm{Id}_n$.
(2) If A is singular then $adj(A)$ is also singular.
(3) $\det(adj A) = (\det A)^{n-1}$.
(4) $adj(A^T) = adj(A)^T$.

Question 5. A *Vandermonde* matrix is a matrix of the form

$$M_{n\times n} = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

where $n \geq 1$ is an integer, and $a_1, \ldots, a_n \in \mathbb{F}$ are scalars.

Prove (e.g. by induction on n) that if $n \geq 2$, then $\det M = \prod_{1\leq i<j\leq n}(a_j - a_i)$.