MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 7 (*due by Thursday, October 6* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose $\mathbb{F}$-vector spaces V, W, X have ordered bases $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_m)$, and $(x_1, \ldots, x_p)$ for positive integers n, m, p respectively. Write down a basis of the vector space of bilinear maps : $V \times W \to X$, and prove that it is a basis.

Question 2. Suppose a $\mathbb{F}$-vector space V has an ordered basis $(v_1, \ldots, v_n)$. For $1 \leq i_0 \leq n$, define the *dual functionals* $\varphi_{i_0} : V \to \mathbb{F}$ via:

$$\varphi_{v_{i_0}} \left(\sum_{i=1}^{n} c_i v_i \right) := c_{i_0}.$$

We discussed in class that $\varphi_{i_0} \in V^*$ (so you can assume this). Now show, these vectors form a basis of V^*. (This is called the *dual basis*, as discussed in class.)

Question 3. Recall the direct product and direct sum (or coproduct) of a set $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces, constructed in class (and studied in the preceding homework set). The goal of this exercise is to show that

$$\left(\bigoplus_{i \in I} V_i \right)^* \cong \prod_{i \in I} V_i^*,$$

by going the 'reverse' way:

(1) Given $\mathbf{\Phi} = (\varphi_i)_{i \in I}$, with each $\varphi_i \in V_i^*$, first show that $(\varphi_i)_{i \in I}$ yields a linear map from $\bigoplus_{i \in I} V_i$ to $\mathbb{F}$. Let us call this map $T(\mathbf{\Phi})$.
(2) Show that the assignment $T : \mathbf{\Phi} \mapsto T(\mathbf{\Phi})$ is a linear map, from $\prod_{i \in I} V_i^*$ to $\left(\bigoplus_{i \in I} V_i\right)^*$.
(3) Show that T is one-to-one and onto.

Question 4. (*Interpolation.*) Let $V = \mathbb{F}[x]$, the space of polynomials. Also say $a_1, \ldots, a_n \in \mathbb{F}$ are pairwise distinct scalars.

Verify for yourselves that for every scalar $a \in \mathbb{F}$, the evaluation map

$$E_a : p(x) \mapsto p(a)$$

is a linear map from all functions to $\mathbb{F}$, hence a linear map from the subspace of polynomials V to $\mathbb{F}$. In other words,

$$E_a \in V^* = (\mathbb{F}[x])^*, \qquad \forall a \in \mathbb{F}.$$

The goal in this exercise is to prove that these evaluation maps are linearly independent. In fact, we will prove something stronger: the maps $E_{a_1}, \ldots, E_{a_n}$ for pairwise distinct $a_i \in \mathbb{F}$, are the dual basis to a specific set of polynomials of degree at most $n-1$.

(1) Define $p_i(x) := \displaystyle\prod_{1 \leq j \leq n,\ j \neq i} \frac{x - a_j}{a_i - a_j}$. Verify that $E_{a_j}(p_i) = p_i(a_j) = 0$ for all $j \neq i$, and $E_{a_i}(p_i) = p_i(a_i) = 1$.

(2) Show that $p_1, \ldots, p_n$ are linearly independent in V.

(3) Show that $p_1, \ldots, p_n$ form a basis of V_{n-1}, the space of polynomials in $\mathbb{F}[x]$ of degree at most $n-1$. (In particular, the E_{a_j} form the dual basis.)

(4) Given arbitrary scalars $c_1, \ldots, c_n \in \mathbb{F}$, find a polynomial $p(x) \in V_{n-1}$ of degree at most $n-1$ (the '*interpolation*') such that $p(a_i) = c_i$ for all i.