MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 6 (*not for submission*)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. The *direct product* of a family $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces is their Cartesian product, denoted

$$\prod_{i\in I} V_i = \times_{i\in I} V_i,$$

with a typical element $(v_i)_{i\in I}$. Also fix the *projection* maps

$$\pi_{i_0} : \prod_{i\in I} V_i \to V_{i_0}, \qquad (v_i)_{i\in I} \mapsto v_{i_0}.$$

(1) Verify that each π_{i_0} is a surjective $\mathbb{F}$-linear map.
(2) Write out the (complete) proof that this product satisfies the following 'universal property':

Given any $\mathbb{F}$-vector space Z, and $\mathbb{F}$-linear maps $\varphi_i : Z \to V_i$ for all $i \in I$, there exists a unique $\mathbb{F}$-linear map $\varphi : Z \to \prod_{i\in I} V_i$ such that $\varphi_i = \pi_i \circ \varphi$ for all $i \in I$.

In other words, the Cartesian product proves the existence of an object that satisfies this universal property. (By class, every other 'candidate' is isomorphic to this one.)

Question 2. The *coproduct*, or *direct sum*, of a family $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces is the subset of their Cartesian product given by:

$$\coprod_{i\in I} V_i = \bigoplus_{i\in I} V_i := \{(v_i)_{i\in I} : v_i = 0 \text{ for all but finitely many } i \in I\}.$$

Also define the inclusion maps

$$\eta_{i_0} : V_{i_0} \to \bigoplus_{i\in I} V_i,$$

which sends a vector $v \in V_{i_0}$ to $(v_i)_{i\in I}$ with $v_i = v$ for $i = i_0$, and $\mathbf{0}_{V_i}$ otherwise.

(1) Verify that each η_{i_0} is an injective $\mathbb{F}$-linear map.

(2) Write out the (complete) proof that this product satisfies the following 'universal property':

Given any $\mathbb{F}$-vector space Z, and $\mathbb{F}$-linear maps $\varphi_i : V_i \to Z$ for all $i \in I$, there exists a unique $\mathbb{F}$-linear map $\varphi : \bigoplus_{i\in I} V_i \to Z$ such that $\varphi_i = \varphi \circ \eta_i$ for all $i \in I$.

In other words, the direct sum proves the existence of an object that satisfies this universal property. (By class, every other 'candidate' is isomorphic to this one.)

(3) Verify that if the index set I is finite, then the product and coproduct 'candidates' discussed above are in fact equal.

Question 3. (*Quotient spaces.*) Suppose V is an $\mathbb{F}$-vector space, and $W \subset V$ a subspace. Define a relation $v \sim v'$ on V if $v - v' \in W$. Now define the quotient space V/W to be the set of distinct equivalence classes $[v]$, under the relation:

$$a[v] + b[v'] := [av + bv'], \qquad a, b \in \mathbb{F},\ v, v' \in V.$$

(1) Verify that $\sim$ above is an equivalence relation on V.
(2) Check that the addition defined above (setting $a = b = 1$) is associative.
(3) Check that $\mathbf{0}_{V/W} := [w]$ for $w \in W$ is an (or 'the') additive identity in V/W.
(4) Assuming that V/W is a vector space under the above operations (which it is!), check that the map $\pi : V \to V/W$ sending $v \in V$ to $[v]$ is a surjective $\mathbb{F}$-linear map.
(5) Prove that the quotient space satisfies the following 'universal property':

Given any $\mathbb{F}$-vector space Z, and a $\mathbb{F}$-linear map $\varphi : V \to Z$ which maps W to $\mathbf{0}_Z$, there exists a unique $\mathbb{F}$-linear map $\overline{\varphi} : V/W \to Z$ such that $\varphi = \overline{\varphi} \circ \pi$ (with π as in the previous part).