MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 5 (*due by Thursday, September 15* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose V, W are $\mathbb{F}$-vector spaces, and $T : V \to W$ is an $\mathbb{F}$-linear transformation.

(1) Show that $T(\mathbf{0}_V) = \mathbf{0}_W$ and that $T(-v) = -T(v)$ for all $v \in V$.
(2) Suppose T is a bijection of sets. Prove that the inverse map T^{-1} is also a linear transformation.

Question 2. Suppose V is an $\mathbb{F}$-vector space, with ordered basis $\mathcal{B} = (v_1, \ldots, v_n)$. Prove that the map $\eta : V \to \mathbb{F}^n$, sending a vector $v = c_1 v_1 + \cdots + c_n v_n$ to the column vector $[v]_\mathcal{B} = (c_1, \ldots, c_n)^T$, is a vector space isomorphism.

Question 3. Suppose V, W are $\mathbb{F}$-vector spaces. Show that $Lin_\mathbb{F}(V, W)$, the space of $\mathbb{F}$-linear maps : $V \to W$, is a vector space.

Question 4. Suppose $A, B \in \mathbb{F}^{m \times n}$ for some integers $m, n \geq 1$. Prove that the following are equivalent:

(1) $A = B$.
(2) $Av = Bv$ for all vectors $v \in \mathbb{F}^n$.
(3) $A\mathbf{e}_j = B\mathbf{e}_j$ for all $1 \leq j \leq n$.

Question 5. Suppose $\mathbb{F}$ is a field, and $T : \mathbb{F}^2 \to \mathbb{F}^2$ is the linear operator $T(x_1, x_2) := (x_2, -x_1)$, where $(x_1, x_2)^T = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ is with respect to the standard ordered basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$.

(1) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}}$?
(2) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}'}$, where $\mathcal{B}' = (\mathbf{e}_1 + \mathbf{e}_2, -\mathbf{e}_1)$?
(3) What is the transition matrix of $\mathcal{B}'$ into $\mathcal{B}$? Meaning, find the matrix P such that $[v]_\mathcal{B} = P[v]_{\mathcal{B}'}$ for all $v \in \mathbb{F}^2$.
(4) Suppose $\mathbb{F}$ has characteristic not 2 (so $2 = 1+1$ in $\mathbb{F}$). What is the coordinate vector of $(2, -1)^T$ in the standard basis, when written out in the basis $\mathcal{B}'$?

Question 6. Suppose $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$, and $\theta \in \mathbb{R}$. Suppose $T : V \to V$ is the linear transformation that rotates a vector counterclockwise by θ (radians). Compute the matrix of T with respect to the standard basis of V.

Question 7. We have seen that (by convention,) the zero vector space over any field has exactly one basis: the empty set. Now:

(1) Classify all nonzero vector spaces over all fields, which also have exactly one unordered basis – i.e., exactly one basis up to permuting its basis elements.

(2) Classify all nonzero vector spaces over all fields, which have exactly two unordered bases.