MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 4 (*due by Thursday, September 8* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. (a) Show that three vectors in $\mathbb{R}^2$ are linearly dependent.
(b) Find three vectors in $\mathbb{R}^2$ such that any two are linearly independent.

Question 2. Suppose V is a vector space over $\mathbb{F}$, and $L \subset V$ is a linearly independent subset. If $v \in V$ is not in the span of L, show that $L \cup \{v\}$ is also linearly independent.

Question 3. Suppose D is a domain set, and $Fun(D, \mathbb{F})$ is the set of functions $f : D \to \mathbb{F}$ such that $f(x) = 0$ for all but finitely many $x \in D$.

(1) Show that $Fun(D, \mathbb{F})$ is a vector space under pointwise addition and scalar multiplication.
(2) Find a basis of this vector space (over the ground field $\mathbb{F}$).

Question 4. Let $\mathbb{F} = \mathbb{Q}$ and $V = \mathbb{R}$, a $\mathbb{Q}$-vector space.

(1) Show that $\mathbb{R}$ is not a finite-dimensional $\mathbb{Q}$-vector space.
(2) Suppose V is a *countable*-dimensional $\mathbb{Q}$-vector space, i.e. a vector space with a countably infinite basis $v_1, v_2, \ldots, v_n, \ldots$ Show that V is the union of its finite-dimensional subspaces V_n spanned by $v_1, \ldots, v_n$.
(3) Show that $\mathbb{R}$ is not a countable-dimensional $\mathbb{Q}$-vector space.

Question 5. Suppose $\mathbb{F}$ is a finite field of size $q \geq 2$, and V is an $\mathbb{F}$-vector space. You showed in HW3 Q6 that V is not a union of $n \leq q$ proper subspaces. The goal is now to show that if we instead had $n \geq q + 1$ (in fact $n = q + 1$), then this is not so.

(1) First show that $\mathbb{F}^2$ is a union of $q + 1$ proper subspaces.
(2) Now suppose $V \neq 0$ is an arbitrary $\mathbb{F}$-vector space of dimension at least 2 (and possibly infinite), and B is a basis of V. (Assume B exists.) Show that V is a union of $q + 1$ proper subspaces.

Question 6. Suppose S is a linearly independent subset of a vector space W (over a field $\mathbb{F}$). Consider a chain of linearly independent subsets in W:

$$S = S_0 \subset S_1 \subset S_2 \subset \cdots$$

Prove that $\bigcup_{i \geq 0} S_i$ is also a linearly independent subset. (In a special case, this is the 'upper bound' of a 'chain' that is used in proving that every vector space has a basis, via Zorn's Lemma.)