MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 3 (*due by Thursday, August 25* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Let $n \geq 1$ be an integer, and let $A \in \mathbb{F}^{n \times n}$ be an upper triangular matrix. Thus, $a_{ij} = 0$ for all $i > j$. Prove that A is invertible if and only if the diagonal entries of A are all non-zero.

Question 2. Suppose $\mathbb{F}$ is a field, and $n \geq 1$ an integer. For integers $1 \leq i, j \leq n$, define the $n \times n$ matrix E_{ij} as having all entries zero, except 1 in the (i, j)-entry.

Now find the span of the following sets – give (with some justification) the "conceptual description", as I said towards the end of Lecture L06.

(1) The matrices E_{ii} for $1 \leq i \leq n$.
(2) The matrices E_{ij} for $i < j$.
(3) The polynomials $x^2 - x, x^3 - x^2, \dots$ and the polynomial x, with $\mathbb{F} = \mathbb{R}$.

Question 3. Prove that the space of polynomials $\mathbb{F}[x]$ is not the span of a finite set.

Question 4. For each of the following, explain whether or not the specified subset (of the corresponding vector space) is a subspace.

(1) The subset of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying: $f(1) - f(2) + 2f(3) = 0$.
(2) The subset of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying: $f(2) = f(3) + 1$.
(3) The subset of solutions to $Ax = b$ for some vector $b \neq 0$. Here $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ for some integers $m, n \geq 1$.

Question 5.

(1) Suppose S_1, S_2 are subsets of an $\mathbb{F}$-vector space. Prove that S_1, S_2 have the same spans if and only if each set is contained in the span of the other.
(2) The *row space* of a matrix is the span of its rows. If $A, B \in \mathbb{F}^{m \times n}$ are row-equivalent, prove that their row spaces are equal.

Question 6. Suppose $q \geq 1$ is an integer, and $\mathbb{F}$ is a (finite or infinite) field of size at least q. If V is an $\mathbb{F}$-vector space, show that V is not the union of q-many proper subspaces.

(In particular, $\mathbb{R}^n$ is not the union of finitely many proper subspaces.)

(Hint, for one possible approach: Suppose V ***is*** *the union of q proper subspaces – let $2 \leq m \leq q$ be the smallest number of subspaces needed to cover V, say $W_1, \ldots, W_m \subset V$. Then there exist $w_i \in W_i$ such that $w_i \notin W_j$ for all $j \neq i$. Now consider certain $(q+1)$-many linear combinations of w_1, w_2.)*