MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Thursday, August 18* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$ for integers $m, n, p \geq 1$.

(1) Show that $(AB)^T = B^T A^T$.
(2) Suppose C is also a matrix over $\mathbb{F}$ such that BC is defined. Prove that $A(BC) = (AB)C$.
(3) Suppose $C \in \mathbb{F}^{n\times p}$ instead. Show that $A(B + C) = AB + AC$.
(4) Suppose $P, Q \in \mathbb{F}^{n\times n}$ are both upper triangular (recall, this means $a_{ij} = b_{ij} = 0$ for $i > j$). Prove that $P + Q, PQ$ are also upper triangular, with diagonal entries $p_{ii} + q_{ii}$ and $p_{ii}q_{ii}$ respectively.

Question 2. Suppose $A \in \mathbb{F}^{m\times n}$. Show the following statements *without* computing individual entries. You can use that $\mathbb{F}^{r\times s}$ forms a vector space for all integers $r, s \geq 1$.

(1) $A \cdot \mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}$.
(2) $(P + Q)A = PA + QA$ for $P, Q \in \mathbb{F}^{p\times m}$.
(3) $\mathbf{0}_{p\times m} \cdot A = \mathbf{0}_{p\times n}$.
(4) Suppose $P, Q \in \mathbb{F}^{n\times n}$ are both lower triangular (recall, this means $a_{ij} = b_{ij} = 0$ for $i < j$). Prove that $P + Q, PQ$ are also lower triangular, with diagonal entries $p_{ii} + q_{ii}$ and $p_{ii}q_{ii}$ respectively.
(5) Suppose $P, Q \in \mathbb{F}^{n\times n}$ are both diagonal (recall, this means $a_{ij} = b_{ij} = 0$ for $i \neq j$). Prove that $P + Q, PQ$ are also diagonal, with diagonal entries $p_{ii} + q_{ii}$ and $p_{ii}q_{ii}$ respectively. (Once again, while this is trivial to verify by hand, I am asking you to prove this *without* computing individual entries.)

Question 3. Suppose $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$ for integers $m, n, p \geq 1$. Also suppose A, B are *invertible*. In other words, (by results from class) there exist unique matrices

$$A^{-1} \in \mathbb{F}^{n\times m}, \quad B^{-1} \in \mathbb{F}^{p\times n}$$

such that

$$AA^{-1}, \quad A^{-1}A, \quad BB^{-1}, \quad B^{-1}B$$

are identity matrices of suitable orders. Prove the following – basically, just verify from *only one side* that $A \cdot A^{-1} = \mathrm{Id}$. (You need not verify that $A^{-1} \cdot A = \mathrm{Id}$.)

(1) AB is invertible, with inverse $B^{-1}A^{-1}$.
(2) cA is invertible for a scalar $0 \neq c \in \mathbb{F}$, with inverse $c^{-1}A^{-1}$.
(3) If A is square, and $k \geq 1$ is an integer, then A^k is invertible, with inverse $(A^{-1})^k$.
(4) What is the inverse of A^{-1}? Give reasons.

Question 4. Suppose $A \in \mathbb{F}^{m\times n}$ as above. Show that A is invertible if and only if A^T is invertible, and if this happens then $(A^T)^{-1} = (A^{-1})^T$. (Once again, verify the product is the identity from only one side, not from both sides.)

Question 5. (Elementary matrices.) In each part, verify that the product is the identity from only one side, not from both sides – but learn how to write this down for an $m \times m$ matrix for *general* $m \geq 1$.

(1) Let $E_1^{(i,c)}$ denote the elementary row operation where the ith row is rescaled by $0 \neq c \in \mathbb{F}$. Show that
$$E_1^{(i,c)}(\mathrm{Id}_m)^{-1} = E_1^{(i,c^{-1})}(\mathrm{Id}_m).$$
(2) Let $E_2^{(i,j)}$ denote the elementary row operation where the ith and jth rows are interchanged. (So $i \neq j$.) Show that
$$E_2^{(i,j)}(\mathrm{Id}_m)^{-1} = E_2^{(i,j)}(\mathrm{Id}_m).$$
(3) Let $E_3^{(i,c,j)}$ denote the elementary row operation where c times the ith row is added to the jth row. (So $i \neq j$.) Show that
$$E_3^{(i,c,j)}(\mathrm{Id}_m)^{-1} = E_3^{(i,-c,j)}(\mathrm{Id}_m).$$

Question 6. Solve the following systems of linear equations (over real numbers, or over any field of characteristic zero). Use row operations to obtain the RREF in all cases.

(1) $x + 2y + 3z = 0, \quad x + y + z = 1, \quad -x + z = 1.$
(2) $x + 2y + 3z = 3, \quad x + y + z = 1, \quad -x + z = 1.$
(3) $x + 4y + 5z = 1, \quad y - z = 3, \quad x + z = 5.$ (For this part, assume the field has characteristic 7.)