# MA219 – Linear Algebra
# 2021 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

**Homework Set 12** (*due by Thursday, November 17* in TA's office hours, or previously in class)

**Question 1.** Suppose $A \in \mathbb{F}^{m\times n}$ for $m, n \geq 1$, and $P, Q$ are square, invertible matrices over $\mathbb{F}$ (for an arbitrary field $\mathbb{F}$) such that $PAQ$ is defined. Show that $PAQ$ and $A$ have the same rank.

**Question 2.** (Henceforth we work over $\mathbb{R}$ or $\mathbb{C}$.) Suppose a vector $v_0$ in an inner product space $V$ is such that $v_0$ is orthogonal to every vector in $V$. Show that $v_0 = \mathbf{0}_V$.

**Question 3.** Verify the following *polarization identity* in a real inner product space $V$:

$$(x, y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right), \qquad \forall x, y \in V.$$

**Question 4.** Show the *triangle inequality* in a complex (or real) inner product space $V$: $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$, with equality if and only if $y = 0$ or $x$ is a non-negative scalar multiple of $y$.

**Question 5.** Let $V = \mathbb{R}^3$ and

$$\mathbf{w}_1 = (\pi, 0, 0)^T, \quad \mathbf{w}_2 = (e, \pi, 0)^T, \quad \mathbf{w}_3 = (1, 1, 1)^T$$

(or forget the transposes and work without them). Apply the Gram–Schmidt algorithm to compute an orthogonal triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with the desired properties.