

MA219 – Linear Algebra 2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 11 (*due by Friday, November 11 by 5pm in TA's office, else in TA's office hours on Thursday, or previously in class*)

Throughout this homework (and this course), \mathbb{F} denotes an arbitrary field.

Question 1. Suppose $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. Compute $A^3 + A^2 + A$, without multiplying 3×3 matrices. (Hint: Compute the characteristic polynomial of A .)

Question 2. Suppose $\lambda \in \mathbb{F} = \mathbb{R}$ and $J = J(3, \lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ is a Jordan block.

- (1) Write down a formula for J^k for any integer $k \geq 1$, and prove it.
- (2) More generally, if f is a polynomial with real coefficients, prove that

$$f(J) = \begin{pmatrix} f(\lambda) & f'(\lambda) & f''(\lambda)/2! \\ 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & f(\lambda) \end{pmatrix}.$$

- (3) Write down (but don't prove) a formula for $f(J)$, where f is an arbitrary polynomial with real coefficients, and $J = J(n, \lambda)$ for arbitrary $n \geq 1$.

Question 3. Suppose \mathbb{F} is any field, $\lambda \in \mathbb{F}$ is any scalar, and $n \geq 1$ is any integer. Let $J = J(n, \lambda)$ be a Jordan block.

- (1) Compute the algebraic and geometric multiplicities of all eigenvalues of J .
- (2) Show that the minimal and characteristic polynomials of J agree.
- (3) Compute the k th power of $J(n, 0)$, for all integers $k \geq 1$.

Question 4. Suppose a real matrix A can be written in Jordan canonical form, with Jordan blocks

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \quad (4), \quad (1), \quad (0), \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Compute the following, with some reasoning.

- (1) The characteristic polynomial of A .
- (2) The minimal polynomial of A .
- (3) The algebraic and geometric multiplicities of all eigenvalues of A .
- (4) The rank of A .

Question 5. This question shows that every *complex* square matrix is conjugate to its transpose. (The same holds true over every field, but this is harder.)

- (1) Show that a Jordan block matrix over any field, say $J = J(n, \lambda) \in \mathbb{F}^{n \times n}$, is conjugate to its transpose: $J^T = PJP$, where $P = P^{-1} = P^T$ is the matrix with 1s along the *anti-diagonal*. In other words, $P_{ij} = 1$ if $j = n + 1 - i$, and 0 otherwise.
- (2) Now suppose $A \in \mathbb{C}^{n \times n}$. Show that $A^T = QAQ^{-1}$ for some $Q \in \mathbb{C}^{n \times n}$ invertible.