MA219 – Linear Algebra
2022 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Thursday, August 11* in TA's office hours, or previously in class)

Question 1. The goal here is to show that the set of complex numbers

$$\mathbb{C} := \{a + bi = a + b\sqrt{-1} : a, b \in \mathbb{R}\}$$

under the operations

$$(a+bi)+(c+di) := (a+c)+(b+d)i, \quad (a+bi)\cdot(c+di) := (ac-bd)+(ad+bc)i,$$

$$0 := 0+0i, \quad 1 := 1+0i,$$

$$-(a+bi) := (-a)+(-b)i, \quad (a+bi)^{-1} := \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

is a field. (You are allowed to use that $\mathbb{R}$ is a field.)

I felt you should do some of these verifications at least once in your life – and it is not likely you will get to do this in any other course – so here, **prove that**:

(1) Multiplication is associative.
(2) $z \cdot z^{-1} = 1$ for all nonzero complex z.
(3) The distributive law holds.

Question 2. Suppose $\mathbb{F}$ is a field, with $a, b \in \mathbb{F}$. Prove the following statements.

(1) The elements $1, -a, a^{-1}$ are unique (for the last, we need $a \neq 0$).
(2) $0 \cdot a = 0$.
(3) $-a = (-1) \cdot a$.
(4) $(-1)^2 = 1$.
(5) $ab = 0$ in $\mathbb{F}$, if and only if $a = 0$ or $b = 0$.

Question 3. Show that if a field $\mathbb{F}$ contains a subfield $\mathbb{E}$, then $\mathbb{F}$ is a vector space over $\mathbb{E}$.

Question 4. Suppose $\mathbb{F}$ is a field with finitely many elements, say n. Prove that $1+1+\cdots+1$ (n times) equals 0 in $\mathbb{F}$. (Hint: Note that $\mathbb{F}$ is a group using $(0, +, -)$.)

Question 5. Suppose V is a vector space over a field $\mathbb{F}$, and $c \in \mathbb{F}, v \in V$. Prove that $c \cdot \mathbf{0} = \mathbf{0} = 0 \cdot v$, where 0 is the zero in $\mathbb{F}$ and $\mathbf{0}$ is the zero in V.

Question 6. The *trace* of a square matrix $A = (a_{ij})_{i,j=1}^n$ is the sum of its diagonal entries: $a_{11} + a_{22} + \cdots + a_{nn}$. Given integers $m, n \geq 1$ and matrices $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}$, prove that AB and BA have the same trace, even if they have different sizes.