MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 9 (*due by Friday, November 15* in TA's office hours, or previously in class)

Question 1. Suppose F is a field, and $0 \neq p(x) \in F[x]$ a nonzero polynomial of degree $d > 0$. Show that $\{1, \dots, x^{d-1}\}$ is an F-basis of the quotient ring $F[x]/(p(x))$.

Question 2. Suppose R is an integral domain, and M an R-module.

(1) Show that the set M_{tor} of all torsion elements is an R-submodule of M.
(2) Show that M/M_{tor} is torsion-free.
(3) Show that if M is free, then M is torsion-free.

Question 3. We now show that the 'converses' to the previous results fail in general, even over PIDs. Indeed, suppose $R = \mathbb{Z}$ and $M = \mathbb{Q}$.

(1) Show that M is torsion-free but not free.
(2) Show that M is not finitely generated over R.

Question 4. Let F be an algebraically closed field. Show that every square matrix in $F^{n \times n}$ is conjugate to its transpose. (Hint: First do so for a single $m \times m$ Jordan block-matrix J, by showing that $BJB = J^T$. Here, $B = B^{-1}$ is the matrix whose *anti-diagonal* has all entries 1, and all other entries 0.)

Question 5. Suppose R is a (unital commutative) ring and M an R-module. Given an element $p \in R$, define

$$M_p := \{m \in M : p^k m = 0 \text{ for some } k \geq 1\}.$$

(1) Prove that M_p is a submodule of M.
(2) Now suppose R is a PID, M is a finitely generated R-module, and p is prime in R. Prove that $M = M_p$ if and only if the annihilator of M is an ideal of the form (p^r) for some integer $r \geq 0$.