MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 8 (*due by Friday, November 8* in TA's office hours, or previously in class)

Question 1. Suppose R is a unique factorization domain (UFD).

(1) Show that R is a monoid under the operation of gcd. Classify the UFDs that are groups under gcd.

(2) Show that R is a monoid under the operation of lcm. Classify the UFDs that are groups under lcm. Classify the UFDs R such that $R \setminus \{0\}$ is a group under lcm.

Question 2. Suppose I is an ideal in a (unital commutative, as always) ring R. Then show that $I \cdot R[x]$ is an ideal in $R[x]$, and that the quotient ring is isomorphic to $(R/I)[x]$.

Question 3. Show that the polynomial $x^6 + 48x^5 - 24$ is irreducible over $\mathbb{Q}[x]$.

Question 4. Suppose $n \geq 3$ and $m_1, \dots, m_n \geq 1$ are integers. Show that the polynomial $\sum_{j=1}^n x_j^{m_j}$ is irreducible over a field F, if and only if the characteristic of F does not divide *all* m_j.

Question 5. Show without using Eisenstein's criterion the polynomial

$$p(x) := x(x+1)\cdots(x+n-1) - 1$$

is irreducible over $\mathbb{Q}[x]$, for all $n \geq 1$.

(Hint: Suppose $p = fg$; evaluate at $x = 0, -1, \dots, -(n-1)$ and then show that $f + g$ has degree $< n$ but at least n roots.)

Question 6. If $(G, +)$ is an abelian group, show that $\mathrm{Hom}_{gp}(G, G) = \mathrm{Hom}_{\mathbb{Z}}(G, G)$ is a ring under pointwise addition, and composition. (So it is unital, but not necessarily commutative.)

Question 7. Suppose $R_1, \ldots, R_n$ are unital commutative rings with $1 \neq 0$ for each $1 \leq j \leq n$. Prove that any ideal of a direct product of rings $\times_{j=1}^{n} R_j$ is of the form $\times_{j=1}^{n} I_j$, where each I_j is an ideal in R_j.

Question 8. Given a ring R and an abelian group $(M, +)$, show that M is an R-module if and only if M is a representation of R.

Question 9. Let F be a field and $R = F[x]$. Show that an R-module is the same as an F-vector space V together with a linear transformation $T : V \to V$.

Question 10. Show that if M is an R-module, then $\operatorname{Hom}_R(R, M) \cong M$ as R-modules.