MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 7 (*due by Friday, October 11* in TA's office hours, or previously in class)

Question 1 (Initial and terminal objects)**.**

(1) (*One ring to rule them all.*) Find a unital commutative ring R such that given any other such ring S, there is a ring morphism : $R \to S$.
(2) Show that in fact this morphism is unique, and the ring R is unique up to isomorphism. (In particular, think why R cannot be the zero ring.) This ring is called the *initial object* in the 'category of (unital commutative) rings'.
(3) Similarly, there is a 'terminal object': find a ring R such that given any other ring S, there is a ring morphism : $S \to R$. (Hint: this ring looks like a ring!)
(4) Verify once again that both the morphism and the ring R in the preceding part are unique.

Question 2. Prove: if $x \in R$ is nilpotent, then $a \in R^\times$ if and only if $a + x \in R^\times$.

Question 3. Suppose R is a ring and $p(x) = p_0 + p_1 x + \cdots + p_d x^d \in R[x]$.

(1) Show that p is a unit in $R[x]$ if and only if $p_0 \in R^\times$ and $p_1, \ldots, p_d$ are nilpotent.
(2) p is nilpotent if and only if $p_0, \ldots, p_d$ are nilpotent.

Question 4. If a non-unit $p \in R$ is prime, show that the ideal $(p) = Rp$ is prime.

Question 5. Prove that a finite integral domain of size ≥ 2 is a field.

Question 6 (Some questions on characteristic)**.**

(1) Show that the characteristic of an integral domain is a prime integer.
(2) Suppose R is an integral domain of characteristic $p \geq 2$. Prove that the Frobenius map $a \mapsto a^p$ is a morphism of rings : $R \to R$.
(3) Show that every prime subfield of a field is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$.
(4) Show that every finite field has order p^n for some $n \geq 1$, where $p \geq 2$ is the order of its prime subfield.
(5) Give an example of an infinite field, of positive characteristic.