MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 6 (*due by Friday, October 4* in TA's office hours, or previously in class)

Question 1. Similar to groups, the **direct product** of a family of rings $\{R_i : i \in I\}$ is defined to be the Cartesian product $R := \times_{i\in I} R_i$, under coordinatewise addition and multiplication.

There is a different notion: given a family of rings $\{R_i : i \in I\}$, their **product** is any ring R equipped with morphisms of rings $\pi_i : R \to R_i$ satisfying the following *universal property:*

Given a ring S and morphisms of rings $\varphi_i : S \to R_i$ for all $i \in I$, there exists a unique morphism of rings $\varphi : S \to R$ such that $\varphi_i = \pi_i \circ \varphi$ for all $i \in I$.

(1) Prove that any two rings which are products of the same family, are isomorphic to each other (by a 'unique' isomorphism).
(2) Prove that the direct product of rings $R = \times_{i\in I} R_i$ is a product, when equipped with the projection morphisms $\pi_{i_0} : R \twoheadrightarrow R_{i_0}$ sending $(r_i)_{i\in I}$ to r_{i_0}.

Question 2. We now construct products in other settings.

(1) (**For** submission.) Repeat the previous question for sets, where a morphism of sets is just a set-theoretic map. In other words, show that the Cartesian product of sets satisfies a universal property for sets, exactly corresponding to the one above (replacing 'ring(s)' by 'set(s)').
(2) (**Not** for submission.) Verify for yourself that the same properties hold for the (direct) products of semigroups, or monoids, or rings without unity, or vector spaces over $\mathbb{R}$. (You will see this in more detail in a later course – as an example of the *product* of a family of objects in any *category*.)
(3) (**For** submission.) Akin to Homework 5 Question 2: Show that the *disjoint union* of sets is the coproduct of sets – i.e., satisfies the universal property mentioned in HW5 Q2.
(4) (**Not** for submission.) Show that the direct sum of vector spaces over $\mathbb{R}$ satisfies the universal property, for vector spaces over $\mathbb{R}$. Also show that for **commutative** monoids / semigroups, the direct sum satisfies the same universal property. (Remember to put the word 'commutative' everywhere in this property!)

(5) (**Not** for submission.) Akin to Homework 5 Question 2: what is the *coproduct* of two unital commutative rings? It cannot be their product because the 'inclusion' morphisms $R \to R \times S$ would take $r \mapsto (r, 0)$, and this does not take 1 to $(1, 1)$, which is the multiplicative identity in $R \times S$.

In fact, this coproduct is called the *tensor product*, and we will hopefully see it later in this course.

Question 3. Suppose R is a (not necessarily unital/commutative) ring, and X is any set.

(1) Prove that the function space $Fun(X, R)$ is a ring, under pointwise addition and multiplication.
(2) Prove that $Fun(X, R)$ is unital/commutative if and only if R is unital/commutative.

Question 4. Let $Z(R)$ be the center of a (unital, possibly non-commutative ring).

(1) Show that $Z(R)$ is a subring of R (i.e., a subset containing 1 and closed under addition and multiplication).
(2) Show that the center of $M_n(\mathbb{R})$ is precisely the set of scalar matries $c \cdot \mathrm{Id}_{n\times n}$ for $c \in \mathbb{R}$.

Question 5. Suppose R is a commutative ring. Show that the set of nilpotent elements is an additive group that is closed under multiplication as well.

Question 6. Suppose R is a not necessarily commutative ring. Show that the sum of two zerodivisors need not be a zerodivisor (think of 2×2 real matrices). What if R is a commutative ring?

Question 7. Show that if R is an integral domain, then so is $R[x]$.

Question 8. Show that a morphism of groups is one-to-one if and only if its kernel is the trivial subgroup.