MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 5 (*due by 5pm on Monday, September 16 in Instructor's office*)

Question 1. Recall that we defined the **direct product** of a family of groups $\{G_i : i \in I\}$ to be the Cartesian product $G := \times_{i\in I} G_i$, under coordinatewise composition / identity / inverse.

There is a different notion: given a family of groups $\{G_i : i \in I\}$, their **product** is any group G equipped with group homomorphisms $\pi_i : G \to G_i$ satisfying the following *universal property:*

Given a group H and group homomorphisms $\varphi_i : H \to G_i$ for all $i \in I$, there exists a unique group homomorphism $\varphi : H \to G$ such that $\varphi_i = \pi_i \circ \varphi$ for all $i \in I$.

(1) Prove that any two groups which are products of the same family, are isomorphic to each other (by a 'unique' isomorphism).
(2) Prove that the direct product of groups $G = \times_{i\in I} G_i$ is a product, when equipped with the projection maps $\pi_{i_0} : G \twoheadrightarrow G_{i_0}$ sending $(g_i)_{i\in I}$ to g_{i_0}.

Question 2. A parallel notion is the **direct sum** of a family of *abelian* groups $\{G_i : i \in I\}$. Namely, define $\oplus_{i\in I} G_i$ to be the subgroup of the Cartesian product $\times_{i\in I} G_i$, consisting of all tuples $(g_i)_{i\in I}$ where at most finitely many components g_i are nontrivial, and the rest are simply the identity elements.

There is a different notion: given a family of abelian groups $\{G_i : i \in I\}$, their **coproduct** is any abelian group G equipped with group homomorphisms $\eta_i : G_i \to G$ satisfying the following *universal property:*

Given an abelian group H and group homomorphisms $\varphi_i : G_i \to H$ for all $i \in I$, there exists a unique group homomorphism $\varphi : G \to H$ such that $\varphi_i = \varphi \circ \eta_i$ for all $i \in I$.

(1) If G_i are all abelian groups, prove that $\oplus_{i\in I} G_i$ is also an abelian group.
(2) Prove that any two groups which are coproducts of the same family, are isomorphic to each other (by a 'unique' isomorphism).
(3) Prove that the direct sum of groups $G = \oplus_{i\in I} G_i$ is a coproduct, when equipped with the inclusion maps $\eta_{i_0} : G_{i_0} \hookrightarrow G$ sending g_{i_0} to $(g_i)_{i\in I}$, where $g_i = g_{i_0}$ for $i = i_0$, and the identity element otherwise.

Question 3. Let $X = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$, and $G = \text{Aut}(X) = (\mathbb{Z}/n\mathbb{Z})^\times$ consist of the coprime integers modulo n under multiplication. (We have seen this in class, and that each automorphism $m \mod n$ acts on $\mathbb{Z}/n\mathbb{Z}$ by left-multiplication by m.)

(1) For each $0 \leq m \leq n-1$, show that the G-orbit of $m+n\mathbb{Z}$ includes $gcd(m,n)+n\mathbb{Z}$.

(2) Now write down the class equation of X. (Hint: $\phi(d)$ denotes the number of integers in $[1,d]$ that are coprime to d.)

Question 4. Consider the following operation on the set of pairs of (row vector, square matrix) $S = \mathbb{R}^n \times \mathbb{R}^{n\times n}$:

$$(u^T, A)\cdot(v^T, B) := (v^T + u^T B, AB), \qquad u, v \in \mathbb{R}^n,\ A, B \in \mathbb{R}^{n\times n}.$$

(1) Show that S is a monoid under this operation. What is the identity?

(2) Let $G_0 \subset S$ consist of those pairs (u, A) such that A is invertible. Show that G_0 is a group. What is the inverse of a general element in G_0?

Question 5. Consider the action of $GL_{n+1}(\mathbb{R})$ on $\mathbb{R}^{n+1}$ for $n \geq 0$.

(1) Show that there are exactly two orbits.

(2) Show that the stabilizer of any nonzero vector $v \in \mathbb{R}^{n+1}$ is isomorphic to the group G_0 from the previous question. (Hint: first let $v = \mathbf{e}_1 = (1, 0, \ldots, 0)^T$.)

It's fine if you work with $\mathbb{R}^n$ as was originally stated, but then you need to use the G_0 from the previous question, with n now replaced by $n-1$.

Question 6. Let $p \geq 2$ be a prime, and suppose G is a finite group, each of whose elements has order p^n for some $n \geq 0$. Prove that G is a p-group.

Question 7. Let G a group of order 385. Find a prime $q \geq 2$, such that for all other primes $p \geq 2,\ p \neq q$, every p-Sylow subgroup of G is normal in G.

Question 8. Let $p \geq q \geq 2$ be prime numbers, and G a group of order pq. By Lagrange's theorem, every element $g \in G$ has order dividing pq. Compute the number of elements of each order, for every possible such group G.

Question 9. Suppose there are six beads, two each red, blue, and green. Show that there are precisely 11 ways in which they can be arranged in a circular necklace. (Here we treat rotations and reflections of one configuration as the same configuration.)