MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 4 (*due by Friday, September 6* in TA's office hours, or previously in class)

Question 1. If $H \leq G$ are groups, with $[G : H] = 2$, show that H is normal in G.

Question 2. Show that a group cannot be the union of two proper subgroups. Give an example of a group that is the union of three proper subgroups.

Question 3. Suppose a group has exactly 36 elements of order 7. How many subgroups does it contain of order 7?

Question 4. Suppose G is an abelian group. Show that the elements of G of finite order form a subgroup.

Question 5. Suppose N_1, N_2 are normal subgroups of G, such that G/N_1 and G/N_2 are abelian groups. Show that $G/(N_1 \cap N_2)$ is also abelian. (Hint: Use $[G, G]$.)

Question 6. **(a)** Suppose G is a p-group for some prime $p \geq 2$, and G acts on a finite nonempty set X whose size is not divisible by p. Show that X has a point fixed by every element of G.
(b) Use this to deduce (an alternate proof of the fact) that the center of G is nontrivial.

Question 7. Suppose $p \geq 2$ is a prime, and a p-group G acts on a finite set X. Let X^G denote the elements of X that are fixed by all $g \in G$. Show that p divides $|X| - |X^G|$.

Question 8. Suppose G acts transitively on sets X, Y, and for some $x_0 \in X, y_0 \in Y$, their stabilizer subgroups in G are conjugate. Prove that $X \cong Y$ as G-sets.

Question 9. Suppose G is a nontrivial p-group for some prime $p \geq 2$, and $H \leq G$ is a subgroup of index p. Show that H is normal in G. (Hint: G acts on the coset space G/H of size p, so there is a homomorphism $\varphi : G \to S_p$. Since $|S_p| = p!$ is not divisible by p^2, show that $\ker(\varphi) = H$.)