MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 3 (*due by Friday, August 30* in TA's office hours, or previously in class)

Question 1. Prove that all subgroups and quotients of a cyclic group are cyclic.

Question 2. If $G/Z(G)$ is cyclic (where $Z(G)$ is the center), then show that $G/Z(G)$ is trivial.

Question 3. Suppose S is a (nonempty) subset of a group G. Is the centralizer of S a subgroup of the normalizer of S? How about a normal subgroup?

Question 4. Suppose $\varphi : G \to H$ is a homomorphism of groups, and $g \in G$ has finite order.
(a) Show that the order of $\varphi(g)$ divides the order of g.
(b) If φ is an isomorphism, show that g and $\varphi(g)$ have the same order.

Question 5. Compute the automorphism group of the Klein-4 group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Question 6. Define the *commutator subgroup* of a group G to be the group $[G, G]$ generated by the set

$$\{ghg^{-1}h^{-1} : g, h \in G\}.$$

(a) Show that $[G, G]$ is a normal subgroup of G, and that it is trivial if and only if G is abelian.
(b) Suppose $\phi : G \to H$ is a homomorphism of groups. Show that $\phi([G, G]) \subset [H, H]$. In particular, if H is abelian then $[G, G] \subset \ker(\phi)$.

Question 7. Suppose G is the group of all upper triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with real entries and such that a, d are nonzero. (Under the operation of matrix multiplication.) Show that $[G, G]$ is the subgroup of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.