MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Friday, August 23* in TA's office hours, or previously in class)

Today we saw that if $\varphi : G \twoheadrightarrow H$ is a surjective group homomorphism, then $G/\ker(\varphi) \cong H$.

Question 1. (a) Prove the *fundamental homomorphism theorem:* Suppose $\varphi : G \to H$ is a group homomorphism. Let K be a normal subgroup of G such that $\varphi(K) = e_H$, and let $\pi : G \twoheadrightarrow G/K$ be the unique group homomorphism sending g to gK. Then there exists a unique group homomorphism

$$\overline{\varphi} : G/K \to H$$

such that $\varphi = \overline{\varphi} \circ \pi$.

We say that the map $\varphi : G \to H$ *factors through* $\overline{\varphi}$.

(b) Show that if φ is surjective, then so is $\overline{\varphi}$.

Question 2. Suppose $0 < m \leq n$ are integers such that m divides n. Show that there exists a unique surjective group homomorphism : $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ sending 1 to 1. Determine the kernel of this map.

Question 3. (Discussed at the end of class.) Suppose G is a subgroup of $(\mathbb{Z}, +)$ which is not trivial. Then G contains a positive element, hence a smallest positive element, say $n > 0$. Prove that $G = n\mathbb{Z}$.

Question 4. Suppose G is a group.

(1) Show that for any nonempty set X, the set of functions $Fun(X, G)$ from X to G is a group, where one defines:

$$(f_1 \cdot f_2)(x) := f_1(x) \cdot f_2(x), \qquad \forall x \in X \text{ and } f_1, f_2 : X \to G.$$

Show that this group is abelian if and only if G is abelian.

(2) If X is also a group, and G is abelian, then show that the set of group homomorphisms
$$\mathrm{Hom}(X, G)$$
is a (normal) subgroup of $Fun(X, G)$.

Question 5. Let G be a group. We will study the space of group homomorphisms $\mathrm{Hom}(\mathbb{Z}, G)$.

(1) Show that $\mathrm{Hom}(\mathbb{Z}, G)$ is in bijection with G. (Hint: $\mathbb{Z}$ is cyclic.)
(2) Show that if G is abelian, then $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ as abelian groups.

Question 6. If G is a group such that $g^2 = e$ for all elements $g \in G$, then show that G is abelian.