MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 10 (*due by Monday, November 25* in TA's office hours, or previously in class)

Question 1. Show that if $m, n \geq 1$ are coprime integers, then $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$.

Question 2. Given a prime $p \geq 2$, describe – with reasoning – the *group* of automorphisms of $\mathbb{Q}(\sqrt{p})$. (You can assume / need not show that $\sqrt{p}$ is irrational.)

Question 3.

(1) Suppose R is an integral domain and $p \in R[x]$ is a nonzero polynomial. Show that p has at most $\deg(p)$ roots in R.

(2) Just for fun, verify the following 'Lagrange interpolation-type' identity:

$$\frac{7}{7-3} \cdot \frac{5}{5-3} + \frac{7}{7-5} \cdot \frac{3}{3-5} + \frac{5}{5-7} \cdot \frac{3}{3-7} = 1.$$

Do *not* submit this for credit.

(3) Here is the general statement – which you do have to prove. If $a_1, \ldots, a_k$ are distinct elements in any field $\mathbb{F}$, show that

$$\sum_{i=1}^{k} \prod_{j=1,\ j\neq i}^{k} \frac{a_j}{a_j - a_i} = 1.$$

(Hint: Replace each numerator a_j by $a_j - X$.)

Question 4.

(1) Find the remainder when $1^{1728} + 2^{1728} + \cdots + 2016^{1728}$ is divided by 2017. (Hint: 2017 is a prime.)

(2) More generally, suppose $n \geq 1$, and G is a finite subgroup of the units in a field $\mathbb{F}^{\times}$. Show that in $\mathbb{F}$, one has:

$$\sum_{g \in G} g^n = \begin{cases} |G|, & \text{if } n \equiv 0 \mod |G|, \\ 0, & \text{otherwise.} \end{cases}$$

Question 5. Suppose $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ form a tower of fields. Show that their union $\bigcup_{n \geq 0} K_n$ is a field.

(More questions to follow...)

Question 6. Suppose $L \supset K$ are fields, $l \in L$ is algebraic over K, and $a \in K[l]$. Prove that the degree $[K[a] : K]$ divides $[K[l] : K]$, hence is bounded above by it.

Question 7. If $K \subset L$ are fields, with K countable, then show that $K(l)$ is countable for every $l \in L$.

Question 8. Suppose K is a field. Show that the rational functions $1/(T-k),\ k \in K$ are K-linearly independent in the field $K(T)$.

Question 9. Write down the unique factorization in $\mathbb{F}[x]$ of $x^{14} - 4x^7 + 4$, where $\mathbb{F}$ has 49 elements.

Question 10. Fix a prime $p \geq 2$, and let K be any algebraically closed field of characteristic p. (This exists by Artin's theorem.) We are now going to construct the algebraic closure of *every* finite field (of characteristic p).

(1) Prove for each $n \geq 1$ that K has a unique finite subfield K_n of size $p^{n!}$.
(2) Show that $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ in K.
(3) Prove that the union K_∞ (also called the "direct limit") of the K_n is a countable field.
(4) Now fix $q = p^n$. Show that $\mathbb{F}_q$ is contained inside some K_N, and hence inside K_∞.
(5) Prove that K_∞ is an algebraic extension of $\mathbb{F}_q$.
(6) Now show that every irreducible $f \in \mathbb{F}_q[T]$ has a root in some K_N (not necessarily the same N as earlier).
(7) Conclude that K_∞ is an algebraic closure of $\mathbb{F}_q$ (in particular, of $\mathbb{F}_{p^1}$ as well).

Question 11. (Not for credit; do not submit.) Show that every element of a finite field can be written as a sum of at most two squares in it.