MA212 – Algebra I
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Friday, August 16* in TA's office hours, or previously in class)

Question 1. Let G be the semigroup of all nonzero complex numbers under multiplication:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i,$$

where a, b, c, d are real.

(1) Show that G is in fact a group.
(2) Show that the circle $S^1 := \{a + bi : a, b \in \mathbb{R}, a^2 + b^2 = 1\}$ is a subgroup of G.

Question 2. Let G be a group. Show that the intersection of an arbitrary collection of subgroups of G is a subgroup. What if 'intersection' is replaced by 'union'? Give a proof or a counterexample.

Question 3. Let H be a subgroup of a group G (both possibly infinite). Find a bijectiive map that takes a left-coset of H to a right coset, and vice versa. (Hint: gH does not go to Hg – as we could do if H is normal in G.)

Now one can define $[G : H]$, even for infinite groups G, to be the size of the left or right coset spaces G/H, $H\backslash G$.

Question 4. Let $H \geq G$ be groups and let $g \in G$.

(1) Show that H and gHg^{-1} are in bijection.
(2) Show that the coset spaces G/H and G/gHg^{-1} are in bijection.

Question 5. Let $n \geq 1$ be an integer, and $GL_n(\mathbb{R})$ the group of invertible $n \times n$ real matrices under multiplication. Given $A \in GL_n(\mathbb{R})$ and a vector $u \in \mathbb{R}^n$, define the map

$$\tau_{A,u} : \mathbb{R}^n \to \mathbb{R}^n, \qquad \tau_{A,u}(v) := Av + u.$$

Show that the set of all $\tau_{A,u}$ is a group under composition, with explicit formulas for the product of two general elements in this group, as well as for $\tau_{A,u}^{-1}$.

Question 6. If G is a finite group with an even number of elements, show that there exists $g \neq e$ in G, such that $g^2 = e$.

Question 7. Suppose G is a group, and H is a subgroup of order 2019 in G – and the only subgroup of order 2019. Prove that H is normal in G.

Question 8. Suppose M, N are normal subgroups of a group G.

(1) Show that $MN := \{m \cdot n : m \in M, n \in N\}$ is also a normal subgroup of G.
(2) Suppose $M \cap N = (e)$. Show that elements of M, N commute:

$$mn = nm, \qquad \forall m \in M,\ n \in N.$$