MA341 – Matrix Analysis and Positivity

(*2019 Autumn Semester*)

Place & Time: Math. Building, Lecture Hall LH-3, Tue Thu 3:30 – 5 pm (Credits 3:0)

Instructor: Apoorva Khare **Office:** Math Dept, X26
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Teaching Assistant: Prateek Kumar Vishwakarma **Office:** Math Dept, R24
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Course website: http://www.math.iisc.ac.in/∼khare/teaching.html#2019-341

Course goals: This course explores matrix positivity and operations that preserve it. These involve fundamental questions that have been extensively studied in the mathematics literature over the past century, and are still being studied owing to modern applications to high-dimensional covariance estimation. The course will bring together techniques from different areas: analysis, linear algebra, combinatorics, and symmetric functions.

Pre-requisites: A course in linear algebra, and a course in calculus/real analysis.

List of topics (not in sequential order, and time permitting):

(1) *The cone of positive semidefinite matrices. Totally positive/non-negative matrices.* Examples of PSD and TP/TN matrices (Gram, Hankel, Toeplitz, Vandermonde, $\mathbb{P}_G$). Matrix identities (Cauchy–Binet, Andréief). Generalized Rayleigh quotients and spectral radius.

(2) *Positivity preservers.* Schur product theorem. Pólya–Szegö observation. Schoenberg's theorem. Positive definite functions to correlation matrices. Rudin's (stronger) theorem. Herz, Christensen–Ressel.

(3) *Fixed-dimension problem.* Introduction and modern motivations. H.L. Vasudeva's theorem and simplifications. Horn/Loewner's theorem and simplifications.

(4) *Proof of Schoenberg's theorem.* Characterization of (Hankel total) positivity preservers in the dimension-free setting.

(5) *Analytic/polynomial preservers – I.* Which coefficients can be negative? Bounded and unbounded domains: Horn-type necessary conditions.

(6) *Schur polynomials.* Two definitions and properties. Specialization over fields and for real powers. First-order approximation.

(7) *Analytic/polynomial preservers – II.* Sign patterns: The Horn-type necessary conditions are best possible. Sharp quantitative bound. Extension principle I: dimension increase.

(8) *Entrywise maps preserving total positivity.* Extension principle II: Hankel TN matrices. Variants for all TP matrices and for symmetric TP matrices. Matrix completion problems.

(9) *Entrywise powers preserving positivity.* Application of Extension principle I. Low-rank counterexamples. Tanvi Jain's result.

(10) *Characterizations for functions preserving $\mathbb{P}_G$.* Extension principle III: pendant edges. The case of trees. Chordal graphs and their properties. Functions and powers preserving $\mathbb{P}_G$ for G chordal. Non-chordal graphs.

Grading & Exam Schedule (Credits 3:0)

Homework	25%	due on Fridays in TA office hours, or earlier in class
Mid-sem	25%	Saturday, September 21 (in class), 11 am to 12:30 pm
Final	50%	TBA

Suggested books:

(1) R. Bhatia, *Matrix analysis*, vol. 169 of Graduate Texts in Mathematics, Springer, 1997.
(2) R. Bhatia, *Positive definite matrices*, Princeton Series in Applied Mathematics, 2007.
(3) R.A. Horn and C.R. Johnson, *Matrix analysis*, Cambridge University Press, 1990.
(4) R.A. Horn and C.R. Johnson, *Topics in matrix analysis*, Cambridge University Press, 1991.
(5) S. Karlin, *Total positivity*, Stanford University Press, 1968.
(6) A. Khare, *Matrix analysis and positivity preservers*, Course notes, 2019.

Homework:

Homework will be assigned weekly – posted on the course webpage – and is due by Friday of the following week in the TA office hours – or earlier in the week in class. Each student should hand in their independently written solutions, written in their own words.

Miscellaneous

(1) **Exams:** There will be no backup mid-sem, so please be on campus and available for the mid-sem.

(2) **Emails:** Please write MA341 in the subject heading of all email correspondence with the instructor/TA. This is in general effective in filtering / weeding out spam email.

(3) **Homework:** The TA is responsible for the homework component of the class. We are unable to accept late homework. Homework extensions only delay the grading of solutions.

Please submit HW on time, in the TA office hours on Friday or in class. Do not place your homework under the office doors of the TA/instructor. In the past, homework placed under office doors has sometimes been lost or misplaced.

(4) **Office Hours:** Please note that it is difficult to discuss mathematics questions by email due to the lack of an appropriate interface. You are encouraged to ask your technical questions during office hours. As the TA has responsibilities other than this course, please go to him only during office hours (unless prior arrangements have been made).

(5) **Ethics:** Read the information on the IISc student ethics page.

(6) In the past poor time management and planning have led to homework being handed in late etc... To be fair to the TA and your fellow students, please do not unnecessarily burden the teaching team with unreasonable requests. Please be responsible and plan ahead if you have to be away from campus during the semester.