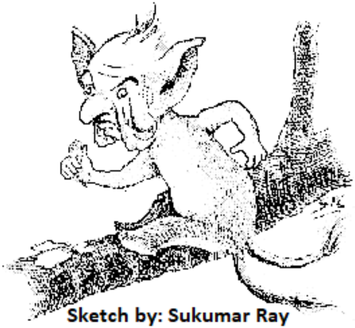
How to tell a tale of two tails?



Parthanil Roy (Indian Statistical Institute)

Collaborators: Ayan Bhattacharya, Rajat Subhra Hazra, Krishanu Maulik, Zbigniew Palmowski, Souvik Ray and Philippe Soulier

What is this talk about?

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Branching Random Walks

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Key Question: How do these two tails affect BRW?

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We will try to answer the question in a special case.

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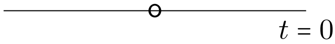
- This is a toy model: (a) no interaction between the particles, (b) particles can cross each other without colliding.
- **Question:** What is the long run configuration of the positions of particles?
- This model was introduced by Hammerseley (1974), Kingman (1975) and Biggins (1976).

The dynamics

- Start with a single particle, *Root*, at the origin at time $t = 0$.



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- Start with a single particle, *Root*, at the origin at time $t = 0$.
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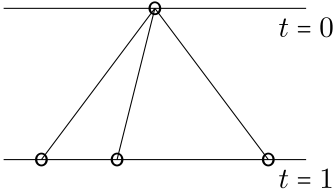

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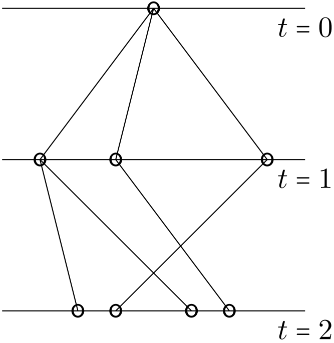

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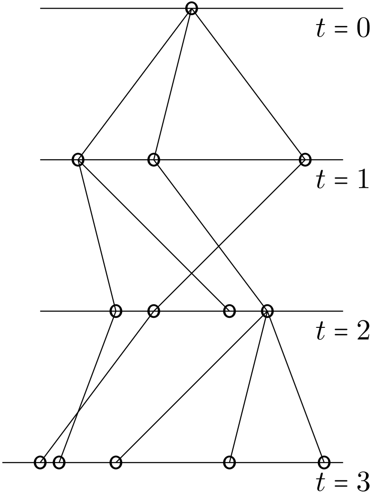

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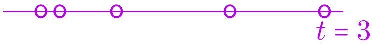
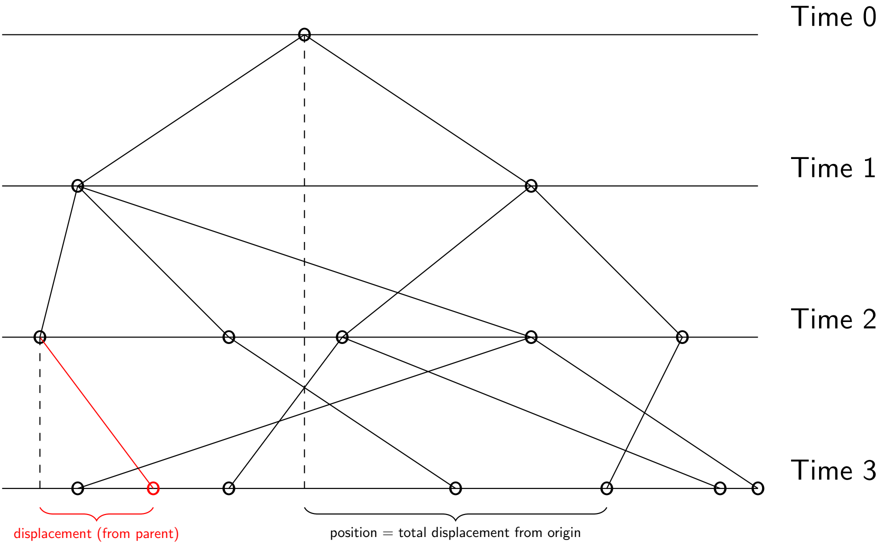
- This dynamics goes on.
- We get a BRW. We only see a snapshot of particles at each time $t \in \mathbb{N}$.



Tree representation



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- In our rather simplified model, we only allow the particles to move along a single ray.
- More complicated models can also be considered where particles move in a plane or in a box.
- For the purpose of this talk, we shall restrict ourselves to the simple model and study long run configuration of the positions of particles.

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- Gaussian multiplicative chaos: Rhodes and Vargas (2014),
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See Shi (2015): Google "Zhan Shi + branching random walk notes"


QUESTIONS?
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Important question

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How fast does the system of particles grow as time goes to infinity and what is its scaling limit?

A more precise question

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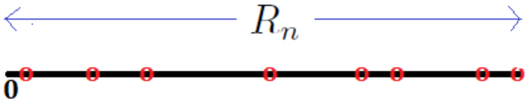

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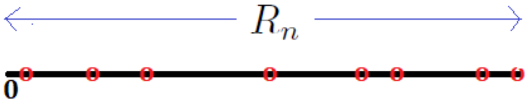

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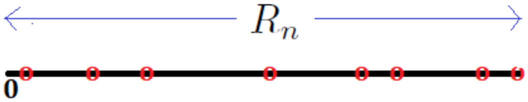

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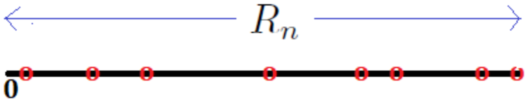

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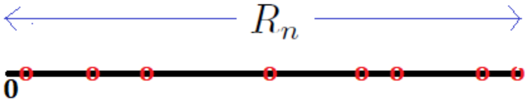

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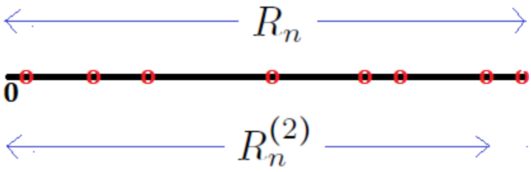
$R_n^{(2)}$:= position of the 2^{nd} rightmost particle at time n.



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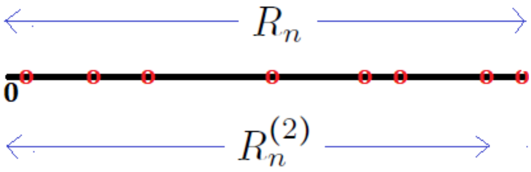

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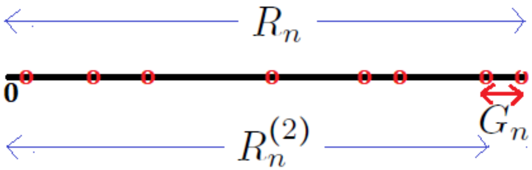

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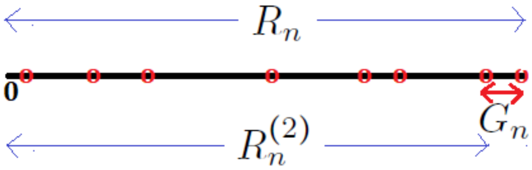

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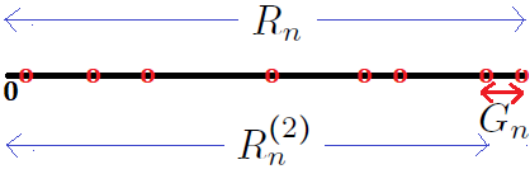

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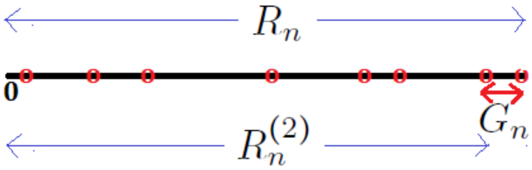

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Question: Weak limts of $\frac{R_n^{(2)}}{2^{n/\alpha}}$, $\frac{G_n}{2^{n/\alpha}}$, etc?

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We shall see a glimpse of **Method 2** in this talk. The full generality of this method will need technicalities like weak convergence on an appropriate Polish space, tightness, vague metric, etc. and we shall happily sweep all technicalities under the carpet.


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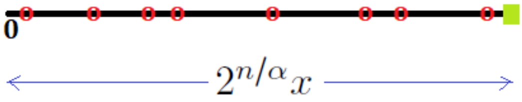

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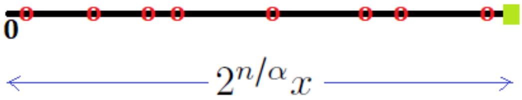

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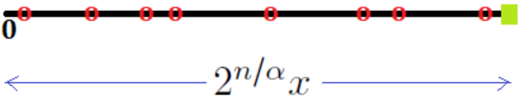

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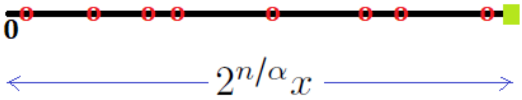

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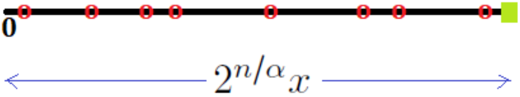

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$$\begin{aligned}\mathbb{P}(2^{-n/\alpha}R_n^{(2)} \le x) = \mathbb{P}(R_n^{(2)} \le 2^{n/\alpha}x) &= \mathbb{P}(N_{n,x} = 0) + \mathbb{P}(N_{n,x} = 1) \\ &\longrightarrow \mathbb{P}(N_{\infty,x} = 0) + \mathbb{P}(N_{\infty,x} = 1)\end{aligned}$$

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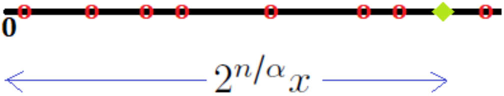

The main result

Theorem (Bhattacharya, Hazra and R. (2017))

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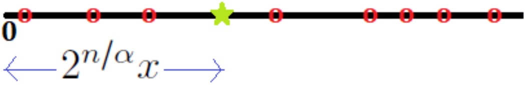

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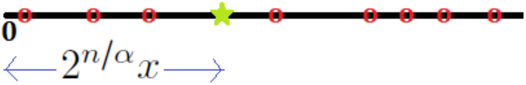

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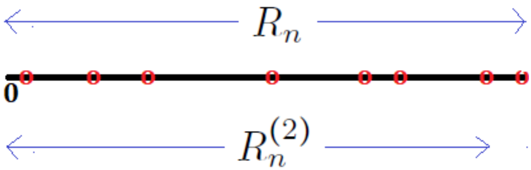
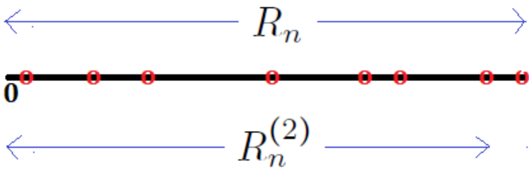
when $P_x = 2$, then $N_{\infty,x} = \sum_{i=1}^{2} 2^{G_i} = 2^{G_1} + 2^{G_2}$, and so on.

Exercise problems





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For all $x > 0$,

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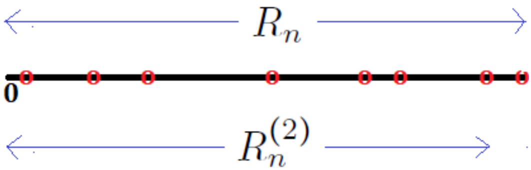

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(Bhattacharya, Hazra and R. (2017)).


QUESTIONS?
COMMENTS?
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Tree representation revisted

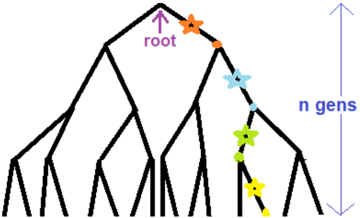

Figure: In this case, $n = 4$

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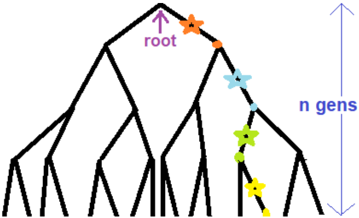

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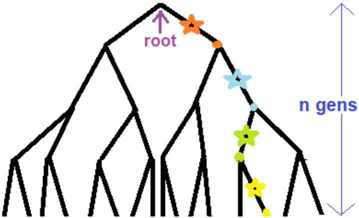

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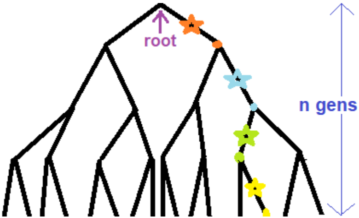

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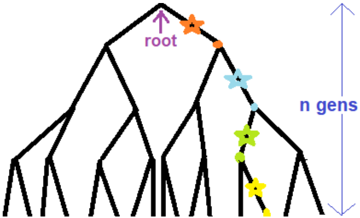

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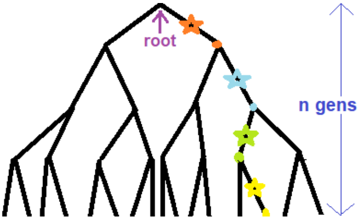

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Easy Exercise: $L_n \xrightarrow{d} 2^G$, where $G \sim Geo(1/2)$ ($\mathbb{N} \cup \{0\}$-valued).

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- Statistical inference for branching random walks.

References to our work

4. **Branching random walk with infinite progeny mean: a tail of two tails** (jointly with Souvik Ray, Rajat Subhra Hazra and Philippe Soulier): *arXiv:1909.08948*.

3. **Extremes of multi-type branching random walks: Heaviest tail wins** (jointly with Ayan Bhattacharya, Krishanu Maulik and Zbigniew Palmowski): *Advances in Applied Probability* (2019) 51(2): 514-540.

2. **Branching random walks, stable point processes and regular variation** (jointly with Ayan Bhattacharya and Rajat Subhra Hazra): *Stochastic Processes and their Applications* (2018) 128(1): 182-210.

1. **Point process convergence for branching random walks with regularly varying steps** (jointly with Ayan Bhattacharya and Rajat Subhra Hazra). *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques* (2017) 53(2): 802-818.

We are grateful to Remco van der Hofstad for useful discussions.

Thank you very much for your attention.

Please take care and stay safe.