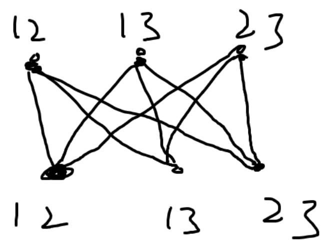
Polynomial method for /list/ colorings.

Def $G=(V,E)$ - finite graph,
C - set of colors, map $f: V \to C$ - proper coloring, if $f(v) \neq f(u)$ for all $uv \in E$.

DEF Each vertex $v \in V$ has a list $C(v)$ of appropriate colors.
MAP $\{v \to C(v) \mid v \in V\}$ proper list coloring, if $f(v) \neq f(u)$ for $uv \in E$.
$f(v) \in$

DEF $\chi(G) = \min |C|$ s.t. proper coloring exists
$ch(G) = \min\{k :$ whenever $|C(v)| \geq k$, a list coloring exists$\}$.
for all $v \in V$

Obvious $ch(G) \geq \chi(G)$. **Ex** $G = K_{3,3}$



LCC for LINE Graphs $ch = \chi$.

Polynomials Assume that all $C(v) \subseteq \mathbb{R}$.

Graph polynomial $P_G(\{x_v \mid v \in V\}) = \pm \prod_{\substack{e \in E \\ \shortparallel \\ uv}} (x_u - x_v)$.

Proper list coloring $\Leftrightarrow \{x_v \in C(v)\} : P_G \neq 0$.

Combinatorial Nullstellensatz / N. Alon, 90's /

$[xy](x+y)^2 = 2$

$F(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ / K - a field /

$\deg f = d_1 + \dots + d_n$, $[x_1^{d_1} \dots x_n^{d_n}] f \neq 0$

then for all set $A_1, \dots, A_n \subseteq K$, $|A_i| = d_i + 1$,

$\exists\, a_1 \in A_1, \dots, a_n \in A_n$: $f(a_1, \dots, a_n) \neq 0$.

Cor. If $\sum_{v \in V} |C(v)| - 1 = |E|$, and $[\prod_v x_v^{|C(v)|-1}] P_G \neq 0$

$\Downarrow$

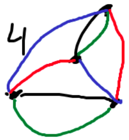
Proper list coloring exists.

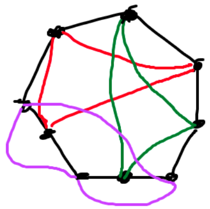
Alon, Tarsi

Theorems

1) Alon, Tarsi : ch(planar Bipartite graph) ≤ 3.

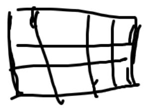
2) Ellingham and Goddyn; G - planar (multi)-graph, regular of degree r,

$r=4$  $\exists$ r-coloring of edges. Then ch(edges of G) = r.

3) Erdös cycle+triangles graph.  3n vertices add n triangles with disjoint vertices

$\chi(G)=3$.

Fleischner and Stiebitz. Even ch(G) = 3.

4) Toroidal $n \times m$ grid 

(Li, Shao, Gorder, P.) If nm is even, then ch(G) ≤ 3.

COEFFICIENT FORMULA.

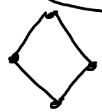
$f(x_1,\dots,x_n)\in K[x_1,\dots,x_n]$, $\deg f \le d_1+\dots+d_n$.

$A_1,A_2,\dots,A_n \subseteq K$, $|A_i| = d_i+1$.

$$[x_1^{d_1}\cdots x_n^{d_n}]\, f = \sum_{\substack{a_1\in A_1\\ a_2\in A_2\\ \vdots \\ a_n\in A_n}} \frac{f(a_1,a_2,\dots,a_n)}{\prod_{i=1}^{n}\prod_{b\in A_i\setminus a_i}(a_i-b)}.$$

(U. Schauz
M. Lason
R. Karasev & P)
2008-2012.
K. G. Jacobi

$\boxed{\text{for any } U\subseteq V,\ |E(U)|\le 2\cdot|U|}$



WANT: ORIENTATION of a planar bipartite graph with indegrees in one part ≤ 2

Th4 $A_i=\{1,\omega,\omega^2\}$, $\omega=e^{2\pi i/3}$.

COEF $= \mathrm{tr}(A^n)$, A – anti-hermitian

n-even $\Rightarrow \mathrm{tr}=\sum \lambda_i^n$.

Th1 Planar bipartite $\pm\prod_{uv=\text{edge}}(x_u+x_v)$,

$x_u \longrightarrow x_v$ $\qquad$ $x_v \longrightarrow -x_v$