The Lovász Local Lemma and its Consequences: Part 1

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Introduction: Probabilistic Methods

Consider the following motivating example:

Existence of Bipartite Subgraph

Every graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ contains a bipartite subgraph with at least $|\mathcal{E}|/2$ edges.

Introduction: Probabilistic Methods

In probabilistic methods, we essentially try to prove the existence or non-existence of a mathematical object satisying a given property in a (finite) collection of objects. The main idea is that if such an object does exist, then there is a positive probability that a randomly chosen object from the collection fulfills the property.

Introduction: Probabilistic Methods

Here is yet another example, possibly of the first known use of probabilistic method:

Hamiltonian Paths in Tournaments (Szele, 1943)

There exists a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

Introduction: Probabilistic Methods

Consider the following problem of providing lower bounds to $R(k,k)$ in Ramsey Theory, Here is a result due to Paul Erdös:

Lower Bound on $R(k,k)$ (Erdös, 1947)

If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. So, there exists a 2-coloring of the edges of $\mathcal{K}_n$ without a monochromatic $\mathcal{K}_k$.

Upon optimizing n, we conclude:

$$R(k,k) > \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{\frac{k}{2}}$$

Introduction: Probabilistic Methods

Can we do better? Yet another coloring argument gives the following bound:

Lower Bound on $R(k, k)$ (Alteration Method)

$R(k, k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$, for any k, n.

Again upon optimizing n, we obtain:

$$R(k, k) > \left(\frac{1}{e} + o(1) \right) k 2^{\frac{k}{2}}$$

Motivation: 2-Colorability of Hypergraphs

We begin with the following definitions:

Definition: k-Uniform Hypergraph

A *k-uniform hypergraph* given by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph for which every $e \in \mathcal{E}$ is a k-tuple of vertices $\in \mathcal{V}$.

Definition: 2-Colorability of a Hypergraph

A *2-coloring* of a hypergraph $\mathcal{G}$ is a mapping from $\mathcal{V}$ to a binary set of colors such that no edge is monochromatic. $\mathcal{G}$ is said to be 2-colorable if such a coloring of $\mathcal{V}$ exists.

Motivation: 2-Colorability of Hypergraphs

Suppose we 2-color the vertices of $\mathcal{G}$ uniformly at random, independently of each other. Then under what conditions can we guarantee the existence of a coloring assignment such that there are no monochromatic edges in $\mathcal{G}$?
We observe that for, $|\mathcal{E}| < 2^{k-1}$, such a 2-coloring will surely exist.
Our goal is to come up with much better sufficient conditions that do not rely on the cardinality of $\mathcal{E}$.

Framework: Limited Dependence

Suppose we have n bad events $E_1, \dots, E_n \subset \Omega$ (finite) that we wish to "avoid", i.e., we want:

$$\mathbb{P}(\bigcap_{i=1}^{n} E_i^c) > 0.$$

What are some sufficient conditions under which we shall be able to ensure this?

- $\sum_{i=1}^{n} \mathbb{P}(E_i) < 1$.
- $E_1, \dots, E_n$ are independent, where $\mathbb{P}(E_i) \in [0, 1) \forall$ $i = 1, \dots, n$.

Such sufficient conditions are rarely met in practice. We thus try to see whether we can work under a framework of *restricted* dependence between $E_1, \dots, E_n$,

The Symmetric Lovász Local Lemma

Going ahead, we shall need the notion of "*mutual independence*". We make it precise with the following definition:

Definition: Mutual Independence

Let $A_0, A_1, \ldots, A_n \subset \Omega$ denote events. We say that A_0 is mutually independent of $\{A_1, \ldots, A_n\}$ if A_0 is independent of every event of the form $\bigcap_{i=1}^{n} B_i$ where each $B_i = A_i$ or A_i^c, $i = 1, \ldots, n$.

The Symmetric Lovász Local Lemma

We are now well-equipped to state the symmetric version of the Local Lemma:

Theorem: Symmetric Lovász Local Lemma

Let $E_1, \dots, E_n \subset \Omega$ be events, with $\mathbb{P}(E_i) \leq p$, $\forall\ i = 1, \dots, n$. Suppose that each E_i is mutually independent from a set of all other E_j except for at most d many of them. If $ep(d+1) \leq 1$, then there is a positive probability that none of the E_i's occur.

Application: 2-Colorability of Hypergraphs

Going back to the 2-colorability problem of a k-uniform hypergraph $\mathcal{G}$, we can now make the following claim:

Claim

A k-uniform hypergraph is 2-colorable if every edge intersects at most $\left(\frac{2^{k-1}}{e} - 1\right)$ other edges.

As a result, we have the following corollary:

Corollary

For $k \geq 9$, every k-uniform k-regular hypergraph is 2-colorable. (*Note:* In a k-regular hypergraph given by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, each vertex in $\mathcal{V}$ belongs to exactly k many edges in $\mathcal{E}$.)

Application: Lower Bound on Ramsey Numbers

The best known lower bound (till date) on $R(k,k)$ was obtained by Joel Spencer with the help of the Symmetric Lovász Local Lemma:

Lower Bound on $R(k,k)$ (Spencer, 1977)

If $e\left(\binom{k}{2}\binom{n-2}{k-2}+1\right)2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

Upon optimizing n, we conclude:

$$R(k,k) > \left(\frac{\sqrt{2}}{e} + o(1)\right) k 2^{\frac{k}{2}}$$

The Asymmetric Lovász Local Lemma

Here is a more general version of the Local Lemma which is rather unwieldy in applications because its conditions are in general very difficult to verify.

Theorem: Asymmetric Lovász Local Lemma

Let $E_1, \dots, E_n \subset \Omega$ be events. For each $i \in [n]$, let $N(i) \subset [n]$ be such that E_i is mutually independent of $\{E_j : j \notin \{i\} \cup N(i)\}$. If $\exists$ real numbers $x_1, \dots, x_n \in [0, 1)$ that satisfy:

$$\mathbb{P}(E_i) \leq x_i \prod_{j \in N(i)} (1 - x_j),$$

$\forall\ i \in [n]$, then with probability at least $\prod_{i=1}^{n}(1 - x_i)$, none of the events E_i occur.

The Asymmetric Lovász Local Lemma

Asymmetric LLL includes the Symmetric LLL as a special case. More formally:

Corollary

Let $E_1, \ldots, E_n$ are such that the conditions of the Symmetric LLL are satisfied. Then setting $x_i = \frac{1}{d+1}$ $\forall$ $i = 1, \ldots, n$ in the previous theorem gives us:

$$P(\bigcap_{i=1}^{n} E_i^c) \geq \left(\frac{d}{d+1}\right)^n > 0.$$

Weakening LLL Conditions

- While proving the Asymmetric LLL, we exploit the following fact:
$$\mathbb{P}(E_i \cap \bigcap_{j \in S \setminus N^*(i)} E_j^c) = \mathbb{P}(E_i)\mathbb{P}(\bigcap_{j \in S \setminus N^*(i)} E_j^c),$$
for any $S \subseteq [n]$, where $N^*(i) = \{i\} \cup N(i)$.
- However, for the proof, all we need is $\leq$. This paves way for considering *"negative dependencies"* in the events $E_1, \ldots, E_n$, giving rise to the Lopsided Local Lemma.

Weakening LLL Conditions

Theorem: Lopsided Local Lemma (Erdös and Spencer, 1991)

Suppose, in addition to the conditions of the Asymmetric Local Lemma, for every E_i, $i = 1, \ldots, n$, let $N(i)$ be such that:

$$\mathbb{P}(E_i \cap \bigcup_{j \in T} E_j) \geq \mathbb{P}(E_i)\mathbb{P}(\bigcup_{j \in T} E_j),$$

for every subset T of $[n] \setminus N^*(i)$. Then the conclusions of the Asymmetric LLL continue to hold good, i.e.,

$$P(\bigcap_{i=1}^{n} E_i^c) \geq \prod_{i=1}^{n}(1 - x_i) > 0.$$

Application: Latin Transversals

We begin with the following definition:

Definition: Latin Transversal

Given an $n \times n$ array, a Latin transversal is a set of n distinct entries with one in every row and column.

Application: Latin Transversals

An application of the Lopsided LLL yields the following interesting result:

Latin Transversals (Erdös and Spencer, 1991)

Every $n \times n$ array where every entry appears at most $\frac{n}{4e}$ times has a Latin transversal.

Looking Beyond LLL

Theorem: Symmetric Shearer's Lemma (Shearer)

Let $E_1, \ldots, E_n \subset \Omega$ be events such that each E_i is mutually independent from a set of all other E_j except for at most d many of them. If $\mathbb{P}(E_i) \leq \frac{(d-1)^{d-1}}{d^d}$, $\forall\ i = 1, \ldots, n$, then there is a positive probability that none of the E_i's occur.

Looking Beyond LLL

Theorem: Cluster Expansion Lemma (Bissacot et. al., 2011)

Let $E_1, ..., E_n \subset \Omega$ be events. For each $i \in [n]$, let $N(i) \subset [n]$ be such that E_i is mutually independent of $\{E_j : j \notin \{i\} \cup N(i)\}$. Let $\mathcal{I}$ denote the power set of (indices of) mutually independent events. If $\exists$ real numbers $y_1, \ldots, y_n > 0$ that satisfy:

$$\mathbb{P}(E_i) \leq \frac{y_i}{\sum_{I \subseteq N^*(i), I \in \mathcal{I}} \prod_{j \in I} y_j},$$

$\forall\ i \in [n]$, then there is a positive probability that none of the events E_i occur.