Random walk on networks
Or: reversible Markov chains, revisited

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Setting

By "Markov chain" we will mean a discrete time Markov chain on a finite or countably infinite state space, that is to say a sequence $(X_t)_{t\geq 0}$ of random variables indexed by the non-negative integers $0, 1, ...$ and taking values in a countable or finite state space $\mathcal{X}$ such that for any $t \geq 0$ and any $A_0, ..., A_t \subseteq \mathcal{X}$ with $\mathbf{P}\{X_t \in A_t, ..., X_0 \in A_0\} \neq 0$ and any $y \in \mathcal{X}$ one has

$$\mathbf{P}\{X_{t+1} = y | X_t \in A_t, ..., X_0 \in A_0\} = \mathbf{P}\{X_{t+1} = y | X_t \in A_t\}.$$

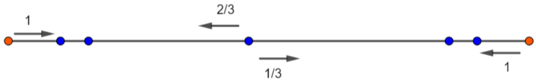
Reversible Markov chains

A Markov chain $(X_t)_{t \geq 0}$ or more generally a stochastic matrix P on a state space $\mathcal{X}$ is said to be reversible if there is a non-zero, finite measure π on $\mathcal{X}$ satisfying $\pi(x)P(x,y) = \pi(y)P(y,x)$ for every $x, y \in \mathcal{X}$.

Note that, if $\mathcal{X}$ is finite, then one can normalize π to a probability measure on $\mathcal{X}$.

Examples and non-examples

- Simple, symmetric random walk on graphs are reversible.
- Random walk on the line graph



- The random walk on a $n - cycle$ which visits the anticlockwise neighbour with probability $2/3$ and the clockwise neighbour with probability $1/3$ is however *not* reversible.

Harmonic functions

We call a function $h : \mathcal{X} \to \mathbf{R}$ to be harmonic at a vertex x if

$$h(x) = \sum_{y \in \mathcal{X}} h(y) P(x, y).$$

Harmonic extension principle

Let (X_t) be an irreducible Markov chain on a state space $\mathcal{X}$. Let $B \subseteq \mathcal{X}$ and suppose that $h_B : B \to \mathbf{R}$ is a function defined on B. Then the function $x \mapsto \mathbf{E}_x[h_B(X_{\tau_B})]$ is the *unique* extension of h_B to $\mathcal{X}$ which is harmonic on $\mathcal{X} \setminus B$.

Random walk on networks

A network is an edge-weighted graph (weights $\in [0, \infty)$).

A simple random walk on a network visits at each step a neighbour of its present location where the probabilities of transitions to the various neighbours are proportional to the weights on the edge joining these vertices to its present location.

Some terminologies

It will be helpful to use the vocabulary of electrical networks.

- The weights on the edges of a network will be called conductances.
- Reciprocal of conductance will be called resistance.

Here one draws the analogy that higher the conductance of an edge, more is the probability of a transition along that edge.

Also for any node a of a network, we shall write $c(a)$ to denote the sum of all the conductances on the edges incident to the node a.

The transition probabilities

Consider the Markov chain on the vertices of a network G with the transition matrix $P(x, y) = \frac{c(x,y)}{c(x)}$.

- Note: this is reversible with respect to the measure π where $\pi(x) = \frac{c(x)}{\sum_y c(y)}$.
- More interestingly one has that: *any reversible Markov chain can be modeled as a random walk on a network*.

Voltage and current flows

Given a network $(G, \{c(e)\})$ we distinguish two vertices a, z and call them the source and the sink respectively.

A voltage is a function defined on $\mathcal{X}$ which is harmonic on $\mathcal{X} \setminus \{a, z\}$.
Thanks to the harmonic extension principle, a voltage is uniquely determined by its values at the source and the sink.

An oriented edge is an *ordered* pair (x, y) of vertices of a network.
A flow is a function θ on oriented edges which is *antisymmetric*, that is to say, $\theta(\vec{xy}) = -\theta(\vec{yx})$.

The divergence of a flow θ at a node x is the quantity $\text{div}_\theta(x)$ defined by

$$\text{div}_\theta(x) = \sum_{y:y\sim x} \theta(\vec{xy}).$$

Divergence conservation principle

For a flow in a network, divergences of all nodes add up to zero.

Proof: By a double-counting argument, we get that, if θ is a flow then one has

$$\sum_x \text{div}_\theta(x) = \sum_x \sum_{y:y\sim x} \theta(\vec{xy}) = \sum_{x,y:x\sim y} \theta(\vec{xy}) + \theta(\vec{yx}) = 0. \quad \square$$

A flow from source to sink is a flow θ satisfying

(i) *Kirchoff's node law*: $\text{div}_\theta(x) = 0$ for all $x \notin \{a, z\}$,

(ii) $\text{div}_\theta(a) \geq 0$.

Intuitively, these can be interpreted as saying that the source of a network is where the flow originates, and a flow cannot be created or cannot accumulate at any node other than the source and the sink.

- The strength of a flow θ from a to z is the quantity $||\theta|| := \text{div}_\theta(a)$. A unit flow is a flow having unit strength. Thanks to the divergence conservation principle, it follows that $||\theta|| = \text{div}_\theta(a) = -\text{div}_\theta(z)$.
- **Orientation:** conductances and resistances can be defined for oriented edges by setting $c(\{x, y\}) = c(\vec{xy})$, and $r(\{x, y\}) = r(\vec{xy})$.
  Note that however unlike flows these are not antisymmetric, rather these are symmetric $c(\vec{xy}) = c(\vec{yx})$ and $r(\vec{xy}) = r(\vec{yx})$.

- Given a voltage W, the current flow associated with W is a flow I given by

$$I(\vec{xy}) = c(\vec{xy})[W(x) - W(y)] = \frac{W(x) - W(y)}{r(\vec{xy})}.$$

- One easily verifies that I is antisymmetric, and thanks to harmonicity of a voltage at all nodes $\notin \{\text{source}, \text{ sink}\}$ one has at every node $x \notin \{\text{source}, \text{ sink}\}$

$$\sum_{y:y\sim x} I(\vec{xy}) = \sum_{y:y\sim x} c(\vec{xy})[W(x) - W(y)]$$

$$= \sum_{z:z\sim x} c(\vec{xz}) \cdot [W(x) - \sum_{y:y\sim x} W(y)P(x,y)] = 0$$

which verifies Kirchoff's law and does justice to the terminology.

- Several other laws of electrical circuits fall in place with this definition of current. For instance the famous *Ohm's law* in this context is just the statement $r(\vec{xy})I(\vec{xy}) = W(x) - W(y)$.

Node law / cycle law

For a finite network, if θ is a flow from a to z satisfying the *cycle law*

$$\sum_{i=1}^{m} r(\vec{e_i})\theta(\vec{e_i}) = 0, \text{ for every cycle } \vec{e_1}, \dots, \vec{e_m} \text{ in the network},$$

and if $||\theta|| = ||I||$, then $\theta = I$.

Effective resistance

The effective resistance is the quantity

$$\mathcal{R}(a \leftrightarrow z) := \frac{W(a) - W(z)}{||I||}.$$

- Why the name "effective resistance"?
  If we replace all the edges of a network by a single edge between the source and the sink, then the resistance that one needs to put on this edge for the same current to flow from the source to the sink is $\mathcal{R}(a \leftrightarrow z)$.

Hitting-time notations

In many cases, it may be of interest to study quantities of the form $\mathbf{P}_x\{\tau_a < t\}$, or $\mathbf{P}_x\{\tau_a < \tau_b\}$, etc., where τ_a is the first hitting time to a, that is to say that

$$\tau_a := \min\{t : X_t = a\};$$

to remove the germ of triviality when a chain starts at state a itself, we would write τ_a^+ to denote the first return time to a, that is to say that, if the chain starts at state a, then

$$\tau_a^+ := \min\{t \geq 1 : X_t = a\}.$$

Often it is also of relevance to consider hitting times to subsets of the state space, thus for example if $A \subseteq \mathcal{X}$, then

$$\tau_A := \min\{t : X_t \in A\}.$$

Effective resistance meets escape probabilities

Proposition

For any $a \neq z \in \mathcal{X}$ one has

$$\mathbf{P}_a\{\tau_z < \tau_a^+\} = \frac{1}{c(a)\mathcal{R}(a \leftrightarrow z)}$$

Proof: Let $B = \{a, z\}$ and $h_B := \mathbf{1}_{\{z\}}$. Then by the harmonic extension principle, the function

$$x \mapsto \mathbf{E}_x[h_B(X_{\tau_B})] = \mathbf{P}_x\{\tau_z < \tau_a\}$$

is the unique extension of h_B on $\mathcal{X}$ which is harmonic on $\mathcal{X} \setminus \{a, z\}$, with value 0 at a and 1 at z.

However, since the function

$$x \mapsto \frac{W(a) - W(x)}{W(a) - W(z)}$$

is also harmonic on $\mathcal{X} \setminus \{a, z\}$, so these two extensions must be equal. Writing $\mathbf{P}_a\{\tau_z < \tau_a^+\}$ as $\sum_{x \neq a,z} P(a, x)\mathbf{P}_x\{\tau_z < \tau_a\}$, expressing the transition probability $P(a, x)$ in terms of conductances on edges leaving a and rearranging the obtained equations proves the claim. $\square$

Thomson's principle

- For a flow θ we define its energy as $\mathcal{E}(\theta) := \sum_{e \in E} \theta(e)^2 r(e)$.

Thomson's principle

$$\mathcal{R}(a \leftrightarrow z) = \inf\{\mathcal{E}(\theta) : \theta \text{ is a unit flow from } a \text{ to } z\}.$$

Rayleigh's monotonicty principle

Rayleigh's monotonicity principle

If $\{r(e)\}$ and $\{r'(e)\}$ are two assignments of resistances to the edges of the same graph G and if $r(e) \leq r'(e)$ for all edges e then one has

$$\mathcal{R}(a \leftrightarrow z; r) \leq \mathcal{R}(a \leftrightarrow z; r').$$

Proof: Note that $\inf_{\theta:||\theta||=1} \sum_e r(e)\theta(e)^2 \leq \inf_{\theta:||\theta||=1} \sum_e r'(e)\theta(e)^2$ and apply Thomson's principle. $\square$

Nash-Williams inequality

- A subset Π of edges is said to be an edge-cutset separating a from z if every path from a to z includes an edge from Π.

Nash-Williams inequality

If $(\Pi)_k$ are disjoint edge-cutsets separating a from z then

$$\mathcal{R}(a \leftrightarrow z) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1} .$$

Proof of Nash-Williams inequality: We first prove the following preliminary bound:

- **Claim 1**: If θ is a flow from a to z and Π is a cutset separating a from z then one has

$$sum_{e\in\Pi}|\theta(e)| \geq ||\theta||.$$

- **Proof**: We give a double counting argument. Suppose $S = \{x : a \text{ and } x \text{ are connected in } G\setminus\Pi\}$. Thanks to node law which holds for all $x \in S, x \neq a$, one has

$$\sum_{x\in S}\sum_{y}\theta(\vec{xy}) = ||\theta||;$$

but also (thanks to antisymmetry) one has

$$\sum_{x\in S}\sum_{y}\theta(\vec{xy}) = \sum_{x\in S}\sum_{y\notin S}\theta(\vec{xy}) \leq \sum_{e\in\Pi}\theta(e),$$

which proves Claim 1. □

Now returning back to the proof of Nash-Williams inequality, let θ be a unit flow from a to z. Thanks to the Cauchy-Schwarz inequality, for all k one has

$$\sum_{e\in\Pi_k} c(e)\cdot\sum_{e\in\Pi_k} r(e)\theta(e)^2 \geq \left(\sum_{e\in\Pi_k}\sqrt{c(e)}\sqrt{r(e)}|\theta(e)|\right)^2$$

$$= \left(\sum_{e\in\Pi_k}|\theta(e)|\right)^2$$

and by Claim 1 the right hand side is $\geq ||\theta||^2 = 1$ (since θ is taken to be a unit flow). Thus $\sum_e r(e)\theta(e)^2 \geq \sum_k \left(\sum_{e\in\Pi_k} c(e)\right)^{-1}$; finally by applying Thomson's principle completes the proof. $\square$

Symmetric random knight-walk on a bi-infinite chessboard

- Suppose a chess knight performs a symmetric random walk on a bi-infinite chessboard.
- One readily sees that this random walk is irreducible.
- Is the walk recurrent or is it transient?[1]

[1] Thanks to irreducibility, this is a meaningful question to ask.

Proposition

The above random walk is recurrent.

Proof: We view the chessboard as a network where the cells are the nodes, and between two nodes we have an edge if and only if a chess knight can visit one of these cells from the other in 1 move. We put unit conductance on each edge of the network (because we want our knight to move uniformly at each step).
Let the knight begin at $(0, 0)$. We consider the following cutsets:

$B_0 = (0, 0)$;

B_1 be the 5×5 section of the board centered at $(0, 0)$;

and for all natural numbers n let B_{n+1} be formed from B_n by adding two rows to each side of B_n and two columns to each side of B_n and then removing B_n from this.

Define for each n, E_n to be the set of edges having one end point in B_{n-1} and the other in B_n. Then notice that these sets E_n are cutsets separating $(0, 0)$ from infinity, in the sense that if a knight has to reach somewhere in the band B_m then it must go through some edge in each of $E_1, ..., E_{m-1}$.
Also observe that by virtue of our construction these E_j are disjoint.
Finally notice that the sum of conductance of the edges in E_n is basically the number of edges in E_n which is at most the number of cells in B_{n-1} times 8, and the number of cells in B_{n-1} is bounded above by $C \cdot n \log n$, where C is a fixed constant not varying with n (this is because the number of cells in B_n is of the order of n).
Thus, the number of edges in E_n is at most $8Cn \log n$, whence by an application of Nash-Williams inequality the result gets proved.

□


Thank you
Merci
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