The critical exponent of a graph

Apoorva Khare
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Working example

Definition. A real symmetric matrix A is *positive (semidefinite)* if all eigenvalues of A are $\geqslant 0$. (Equivalently, $u^T Au \geqslant 0$ for all vectors u.)

Notation. Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive (semidefinite) matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

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Example: Consider the following correlation matrices in $\mathbb{P}_5$:

$$A = \begin{pmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0.6 & 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 0.4 & 0 \\ 0 & 0 & 0.4 & 1 & 0.3 \\ 0 & 0 & 0 & 0.3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}.$$

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Question: Raise each entry to the αth power for some $\alpha > 0$.
For which α are the resulting matrices positive?

Entrywise functions preserving positivity

More generally: For which functions $f : I \to \mathbb{R}$ is it true that

$$f[A] := (f(a_{jk})) \in \mathbb{P}_N \text{ for all } A \in \mathbb{P}_N(I)?$$

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Theorem (Schoenberg, *Duke Math. J.* 1942; Rudin, *Duke Math. J.* 1959)

Suppose $I = (-1, 1)$ and $f : I \to \mathbb{R}$. The following are equivalent:

1. $f[A] \in \mathbb{P}_N$ *for all* $A \in \mathbb{P}_N(I)$ *and all* N.
2. f *is analytic on* I *and has nonnegative Taylor coefficients.*
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Question: What about positivity preservers for *fixed* N? **Open** for $N \geqslant 3$.

Modern motivation from high-dimensional statistics

Graphical models: Connections between statistics and combinatorics.

Let $X_1, \dots, X_p$ be a collection of random variables.

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Classical estimator (sample covariance matrix):

$$S := \frac{1}{n-1} \sum_{j=1}^n (x_j - \overline{x})(x_j - \overline{x})^T.$$

In modern "large p, small n" problems, S is known to be a *poor* estimator of Σ: (a) low rank, (b) no graphical structure.

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$$\text{True } \Sigma = \begin{pmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0.95 & 0.18 & 0.02 \\ 0.18 & 0.96 & 0.47 \\ 0.02 & 0.47 & 0.98 \end{pmatrix}$$

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- **Question:** When does this procedure preserve positivity (psd)? (Critical for applications, since covariance matrices are psd.)

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- Proofs crucially involve Schur polynomials, Schur positivity, ...

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   Well-studied family in theory and applications:
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   Question: Which power functions applied entrywise preserve positivity on $\mathbb{P}_N$ for fixed N? (Subject of this talk.)

Powers preserving positivity

Theorem (FitzGerald and Horn, *J. Math. Anal. Appl.* 1977)

Let $N \geqslant 2$. *Then:*

1. $f(x) = x^\alpha$ *preserves positivity on* $\mathbb{P}_N((0,\infty))$ *if* $\alpha \geqslant N-2$.

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So for $A = \begin{pmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0.6 & 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 0.4 & 0 \\ 0 & 0 & 0.4 & 1 & 0.3 \\ 0 & 0 & 0 & 0.3 & 1 \end{pmatrix}$, all powers $\alpha \in \mathbb{N} \cup [3,\infty)$ work.

Can we do better?

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The proof of FitzGerald and Horn's result is easy, but ingenious.

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- Suppose $a_{NN} \neq 0$. Write $A = \begin{pmatrix} B & \xi \\ \xi^T & a_{NN} \end{pmatrix}, \quad \zeta := \frac{1}{\sqrt{a_{NN}}} \begin{pmatrix} \xi \\ a_{NN} \end{pmatrix}$.

  Note: $A - \zeta\zeta^T$ is psd, with last row and column zero.

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  Note: $A - \zeta\zeta^T$ is psd, with last row and column zero.
- By elementary calculus, for any $x, y \geqslant 0$,

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Refine the FitzGerald–Horn problem for matrices with zeros.

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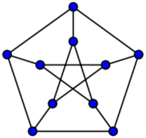

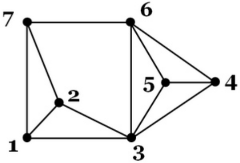

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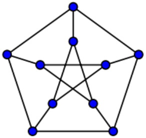
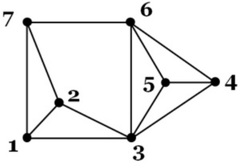


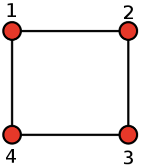


Given a graph $G = (V, E)$ with $V = \{1, \dots, n\}$, define

$$\mathbb{P}_G := \{A \in \mathbb{P}_n : a_{jk} = 0 \text{ if } (j,k) \notin E \text{ and } j \neq k\}.$$

Note: a_{jk} can be zero if $(j,k) \in E$.

Example:



$$\begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{pmatrix}$$

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Problem 1: Compute the set of powers preserving positivity for G:

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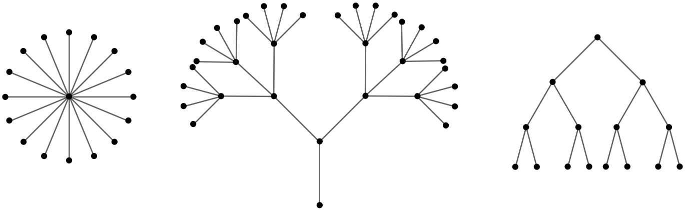

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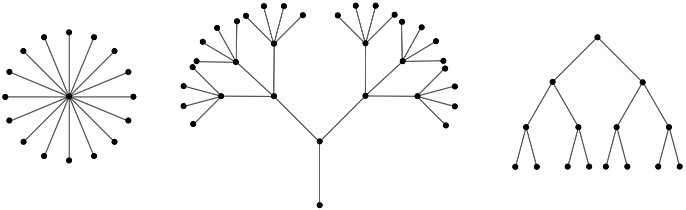

Theorem (Guillot, K., Rajaratnam, *Trans. AMS* 2016)

Let T be a tree with at least 3 vertices. Then $\mathcal{H}_T = [1,\infty)$.

Functions preserving positivity for trees

More generally, classify *all functions* preserving positivity for trees:

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- First known characterization for non-complete graphs.
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$(\mathbf{1}) \implies (\mathbf{2})$. Immediate.

$(\mathbf{2}) \implies (\mathbf{3})$. Suppose $f[-] : \mathbb{P}_{A_3} \to \mathbb{P}_{A_3}$.
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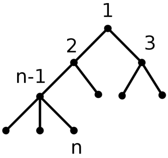
$(\mathbf{3}) \implies (\mathbf{1})$.
The proof uses induction on n, and Schur complements:



$$M := \left(\begin{array}{cccc|c} & & \mathbf{A}_{(n-1)\times(n-1)} & & \begin{matrix} 0 \\ \vdots \\ 0 \\ a_{n-1,n} \end{matrix} \\ \hline 0 & \cdots & 0 & a_{n-1,n} & a_{nn} \end{array}\right)$$

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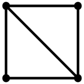
Answer: No. Counterexample: $G = K_4^{(1)}$ (K_4 minus a chord).



$K_4^{(1)}$

Clearly, the maximal clique is K_3. However, we can show that $\mathcal{H}_{K_4^{(1)}} = \{1\} \cup [2, \infty)$.

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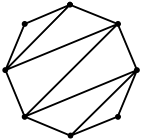

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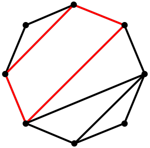

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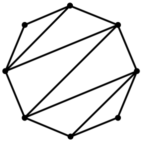

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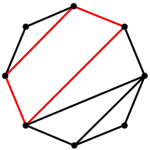

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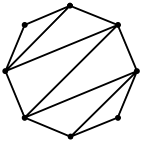

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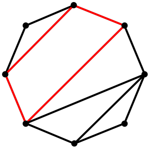

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- *Examples:* Trees, complete graphs, triangulation of any graph, Apollonian graphs, band graphs, split graphs, etc.

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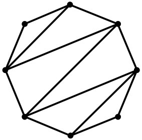

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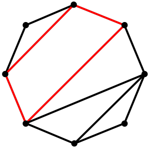

Not Chordal

- *Names:* Triangulated, decomposable, rigid circuit graphs...
- *Examples:* Trees, complete graphs, triangulation of any graph, Apollonian graphs, band graphs, split graphs, etc.
- Occur in many *applications*: positive definite completion problems, maximum likelihood estimation in graphical models, Gaussian elimination, etc.

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The following are equivalent:

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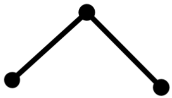

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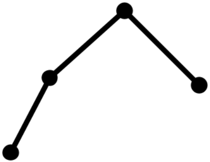

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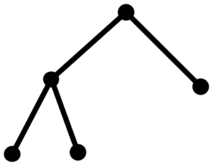

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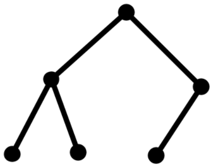

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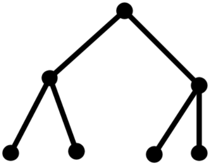

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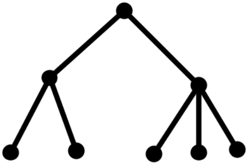

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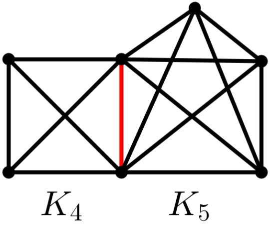

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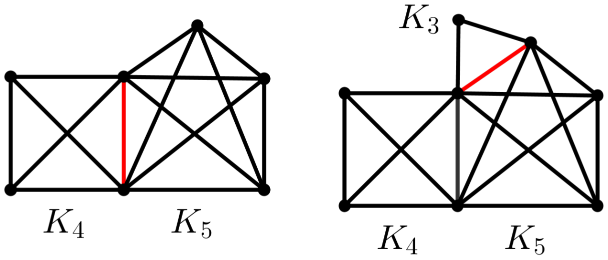

Powers preserving positivity for chordal graphs

Theorem (Guillot, K., Rajaratnam, *J. Combin. Theory Ser. A* 2016)

Let G be any chordal graph with at least 2 vertices and let $r = m_G$ be the largest integer such that either K_r or $K_r^{(1)}$ is an induced subgraph of G. Then

$$\mathcal{H}_G = \mathbb{N} \cup [m_G - 2, \infty).$$

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So for $B = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$, all powers $\geqslant 2 = d$ work.

Critical exponents of well-known chordal graphs

Example 2: Complete graph K_r or almost complete graph $K_r^{(1)}$:
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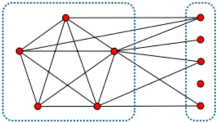

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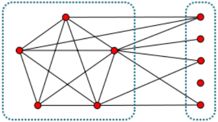

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Example 5: *Apollonian graphs* are obtained by recursively subdividing triangles.
I.e., maximal planar graphs.
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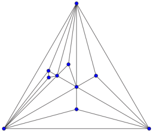

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$$f[A+B] - (f[A] + f[B]) \in \mathbb{P}_n \qquad \forall A, B.$$

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3. Induction and properties of chordal graphs (decomposition, ordering of cliques, etc.).

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Using the above ideas, we show more strongly, a sufficient condition for *general functions* preserving positivity on $\mathbb{P}_G$:

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Let G be a chordal graph with a perfect elimination ordering of its vertices $\{v_1, \dots, v_n\}$. For all $1 \leqslant k \leqslant n$, let G_k denote the induced subgraph formed by $\{v_1, \dots, v_k\}$.

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If $d = 1$ then $c = 2$ and G is a tree, and the converse is also true. [G.-K.-R., *Trans. AMS* 2016]

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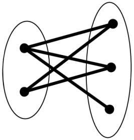
If G is a connected bipartite graph with $n \geqslant 3$ vertices, then $\mathcal{H}_G = [1, \infty)$.



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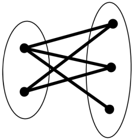

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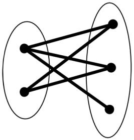

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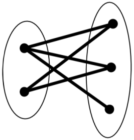

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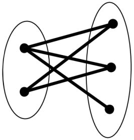

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Suppose $G_1, \dots, G_k$ are connected graphs with at least one edge each. Let G denote any coalescence of the G_i.
Then x^α preserves positivity on $\mathbb{P}_G$ if and only if $\alpha \geqslant 1$ and x^α preserves positivity on all $\mathbb{P}_{G_i}$.

Non-chordal graphs (cont.)

Theorem (Guillot, K., Rajaratnam, *J. Combin. Theory Ser. A* 2016)

For all $n \geqslant 3$, $\mathcal{H}_{C_n} = [1, \infty)$.

Remark: 1 is the biggest $r - 2$ such that K_r or $K_r^{(1)} \subset C_n$.

Question: For any graph G, is $\mathcal{H}_G = \mathbb{N} \cup [m_G - 2, \infty)$, where $r = m_G$ is the biggest integer such that K_r or $K_r^{(1)} \subset G$?

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Then x^α preserves positivity on $\mathbb{P}_G$ if and only if $\alpha \geqslant 1$ and x^α preserves positivity on all $\mathbb{P}_{G_i}$. In particular,

$$CE(G) = \max(1, CE(G_1), \dots, CE(G_k)).$$

Matrices containing negative entries

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Even and odd extensions of the power functions:

$$\phi_\alpha(x) := |x|^\alpha, \qquad \psi_\alpha(x) := \operatorname{sgn}(x)|x|^\alpha, \qquad \forall\, x \in \mathbb{R} \setminus \{0\},$$

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$$\mathcal{H}_G^\psi := \{\alpha \in \mathbb{R} : \psi_\alpha[A] \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G(\mathbb{R})\},$$
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- Our main result about chordal graphs **extends** to $\mathcal{H}_G^\psi$ and $\mathcal{H}_G^\phi$:

$$\mathcal{H}_G^\psi = (-1 + 2\mathbb{N}) \cup [r-2, \infty),$$
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- We have $\mathcal{H}_{C_n}^\psi = [1,\infty)$. However, $\mathcal{H}_{C_4}^\phi = [2,\infty)$.

Open problems

1. Does the same combinatorial rule for chordal/cycle graphs, also work for $\mathbb{P}_G$ for *all* G? Namely, for all G, is $\mathcal{H}_G = \mathcal{H}_G^{\psi} = \mathbb{N} \cup [m_G - 2, \infty)$, where $r = m_G$ is the biggest integer such that K_r or $K_r^{(1)} \subset G$?
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2. For which graphs G is $CE^{\psi}(G) \neq CE^{\phi}(G)$?
3. The critical exponent of a graph always appears to be an integer. Can this be proved directly (without computing the critical exponent explicitly)?
4. Connections to other (purely combinatorial) graph invariants?

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