THE NON-COMPACT NORMED SPACE OF NORMS ON A FINITE-DIMENSIONAL BANACH SPACE

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ABSTRACT. We discuss a new pseudometric on the space of all norms on a finite-dimensional vector space (or free module) $\mathbb{F}^k$, with $\mathbb{F}$ the real, complex, or quaternion numbers. This metric arises from the Lipschitz-equivalence of all norms on $\mathbb{F}^k$, and seems to be unexplored in the literature. We initiate the study of the associated quotient metric space, and show that it is complete, connected, and non-compact. In particular, the new topology is strictly coarser than that of the Banach–Mazur compactum. For example, for each $k \geqslant 2$ the metric subspace $\{\|\cdot\|_p : p \in [1,\infty]\}$ maps isometrically and monotonically to $[0, \log k]$ (or $[0,1]$ by scaling the norm), again unlike in the Banach–Mazur compactum.

Our analysis goes through embedding the above quotient space into a normed space, and reveals an implicit functorial construction of function spaces with diameter norms (as well as a variant of the distortion). In particular, we realize the above quotient space of norms as a normed space.

We next study the parallel setting of the – also hitherto unexplored – metric space $\mathcal{S}([n])$ of all metrics on a finite set of n elements, revealing the connection between log-distortion and diameter norms. In particular, we show that $\mathcal{S}([n])$ is also a normed space. We demonstrate embeddings of equivalence classes of finite metric spaces (parallel to the Gromov–Hausdorff setting), as well as of $\mathcal{S}([n-1])$, into $\mathcal{S}([n])$. We conclude by discussing extensions to norms on an arbitrary Banach space and to discrete metrics on any set, as well as some questions in both settings above.

CONTENTS

1. THE METRIC SPACE OF NORMS: DEFINITION AND MAIN RESULT

It is a folklore result that all norms on a finite-dimensional (real or complex) normed linear space are topologically equivalent – i.e., Lipschitz – with respect to one another. The space of norms has long been studied using the Banach–Mazur pseudometric. Our goal in this work is to explain a new, strictly coarser topology on the space of norms on $\mathbb{R}^k$ – the equivalence classes are now given by dilations – which leads us to a *non-compact* quotient metric space $\mathcal{S}_k(\mathbb{R})$. The Banach–Mazur continuum turns out to be a (compact) quotient of this space; we will see for instance that the two topologies agree on the sets of p-norms for $p \in [1,2]$ and $[2,\infty]$, but not for $p \in [1,\infty]$.

We then study the space $\mathcal{S}_k(\mathbb{R})$ by working in a broader context of function spaces with diameter norms. As we explain below, (a) this function space construction is functorial and applies in a special case to the setting of $\mathcal{S}_k(\mathbb{R})$; (b) we deduce that the metric on $\mathcal{S}_k(\mathbb{R})$ is in fact a norm; and

Date: October 31, 2018.

Key words and phrases. Norms, metric space, diameter seminorm, distortion, Banach–Mazur compactum, Gromov–Hausdorff distance.

(c) we interpret this new metric/norm through a variant of the distortion between metric spaces. (d) We also apply this functorial framework to deduce similar structural properties of the metric space of all metrics on each finite set (see Section 4), and of families of norms on an arbitrary Banach space. Hence the present paper, as we were surprisingly unable to find these results recorded in the literature.

We begin by setting notation. Fix an integer $k > 0$ and a Clifford algebra $\mathbb{F}$ over $\mathbb{R}$ that is a division ring (equivalently, $\mathbb{F}$ lacks zerodivisors), that is, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. We will denote $\dim_{\mathbb{R}} \mathbb{F}$ by d; also let $1, i$ (and j, k) denote the standard $\mathbb{R}$-basis elements in $\mathbb{C}$ (or $\mathbb{H}$). Recall the conjugation operation $\alpha \mapsto \alpha^*$ in $\mathbb{F}$, which is the unique $\mathbb{R}$-linear anti-involution that fixes 1 and acts as multiplication by -1 on $\{i, j, k\} \cap \mathbb{F}$. Now a *norm* on $\mathbb{F}^k$ is a function $N : \mathbb{F}^k \to \mathbb{R}$ satisfying the following properties for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}^k$ and $\alpha \in \mathbb{F}$:

(1) *Positivity:* $N(\mathbf{x}) \geqslant 0$, with equality if and only if $\mathbf{x} = 0$.
(2) *Homogeneity:* $N(\alpha \mathbf{x}) = |\alpha| N(\mathbf{x})$, where $|\alpha| := \sqrt{\alpha \alpha^*}$ will be termed the *absolute value* of $\alpha \in \mathbb{F}$ (to distinguish it from the norm). Recall $|\cdot|$ is multiplicative on $\mathbb{F}$.
(3) *Sub-additivity:* $N(\mathbf{x} + \mathbf{y}) \leqslant N(\mathbf{x}) + N(\mathbf{y})$.

Denote the space of all norms on $\mathbb{F}^k$ by $\mathcal{N}(\mathbb{F}^k)$. Here are some basic properties of this space, some of which are used below.

Lemma 1.1. *For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and an integer $k > 0$, the space $\mathcal{N}(\mathbb{F}^k)$ is closed under the following operations:*

- *Addition.*
- *Multiplication by $\mathbb{R}^{>0}$. (Thus, $\mathcal{N}(\mathbb{F}^k)$ is a convex cone.)*
- *Pointwise limits, as long as the limiting function is positive except at $\mathbf{0}$.*
- *Pre-composing by continuous additive maps $A : \mathbb{F}^k \to \mathbb{F}^k$ with trivial kernel – equivalently, real-linear maps $A \in GL_{dk}(\mathbb{R})$ under some identification of $\mathbb{F}^k$ with $\mathbb{R}^{dk}$. In other words,*

$$A \in GL_{dk}(\mathbb{R}),\ N \in \mathcal{N}(\mathbb{F}^k) \quad \implies \quad (\mathbf{x} \mapsto N(A\mathbf{x})) \in \mathcal{N}(\mathbb{F}^k).$$

Notice that there are also other ways to construct norms, e.g. adding norms on subspaces of $\mathbb{F}^k$ to a given norm in $\mathcal{N}(\mathbb{F}^k)$. See Equation (3.7) below for an example.

The next result is standard for $\mathbb{F} = \mathbb{R}$, and easily extends to $\mathbb{C}$ or $\mathbb{H}$.

Lemma 1.2. *All norms in $\mathcal{N}(\mathbb{F}^k)$ are Lipschitz-equivalent, i.e., for any two norms $N, N' \in \mathcal{N}(\mathbb{F}^k)$ there exist constants $0 < m \leqslant M$ such that*

$$m \cdot N(\mathbf{x}) \leqslant N'(\mathbf{x}) \leqslant M \cdot N(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbb{F}^k. \tag{1.3}$$

For completeness we include a proof-sketch, in a slightly more general setting that is relevant to the present work (below).

Proof. This follows from two observations – (i) $\mathbb{F}^k$ is a finite-dimensional vector space over $\mathbb{R}$, and (ii) $\mathcal{N}(\mathbb{F}^k) \subset \mathcal{N}(\mathbb{R}^{dk})$ under the $\mathbb{R}$-linear homeomorphism $\mathbb{F} \cong \mathbb{R}^d$, where $d = \dim_{\mathbb{R}} \mathbb{F}$. These observations reduce the situation to the well-known case of $\mathbb{F} = \mathbb{R}$, where we remind that the result again follows from two observations: (a) Every norm is bounded above by a positive multiple of the sup-norm; one obtains this by working on the boundary ∂C of the cube $C = [-1, 1]^d$. (b) Given a compact metric space (X, d) (such as $X = \partial C$), all continuous maps in $C(X, (0, \infty))$ are pairwise 'Lipschitz equivalent'. A more general statement is that given any set X, all set maps : $X \to (0, \infty)$ with image bounded away from 0 and ∞ are pairwise 'Lipschitz equivalent'. $\square$

The preceding result is well-known. A less well-known result (which we were unable to find in the literature) is the following construction, which was mentioned by V.G. Drinfeld in a lecture at the University of Chicago in the early 2000s:

Proposition 1.4. *Say that two norms N, N' on $\mathbb{F}^k$ are* equivalent, *written $N \sim N'$, if $N' \equiv \alpha N$ for some positive real number α. Then the space $\mathcal{S}_k(\mathbb{R}) := \mathcal{N}(\mathbb{R}^k)/\sim$ is a metric space, with metric*

$$d_{\mathcal{S}_k(\mathbb{R})}([N],[N']) := \log(M_{N,N'}/m_{N,N'}), \tag{1.5}$$

where $M = M_{N,N'}$, $m = m_{N,N'}$ denote the largest and smallest Lipschitz constants respectively, in Equation (1.3).

Once formulated, the result is shown in a straightforward manner. Informally, 'the space of metrics forms a metric space'. The reader may also recognize (1.5) as a variant of the *log-distortion* between metrics; see Section 4 for more on this.

Remark 1.6. We now record the connection between the space $\mathcal{S}_k(\mathbb{F})$ and the Banach–Mazur compactum (see e.g. [16]), where two k-dimensional Banach spaces U, V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ have distance

$$\log \inf\{\|T\| \cdot \|T^{-1}\| : T \in GL(U,V)\}.$$

Now if two norms are proportional, and thus represent the same point in $\mathcal{S}_k(\mathbb{R})$, then they also do the same in the Banach–Mazur compactum: just note that $\|T\| \cdot \|T^{-1}\| = 1$ for T the identity map on $\mathbb{R}^k$. It follows that the Banach–Mazur compactum is a quotient of $\mathcal{S}_k(\mathbb{R})$. One consequence of our main result (Theorem 1.8 below) is that the topology in $\mathcal{S}_k(\mathbb{R})$ is strictly coarser.

The space $\mathcal{S}_k(\mathbb{F})$ does not seem to be known to experts, nor is it defined or analyzed in the literature; we initiate its study in the present work. In light of the preceding remark, we hope that subsequent, continued analysis of $\mathcal{S}_k(\mathbb{F})$ will also yield additional information about the Banach–Mazur compactum.

We begin with an immediate consequence of the above observation that $\mathcal{N}(\mathbb{F}^k) \subset \mathcal{N}(\mathbb{R}^{dk})$: in a sense, it suffices to work with $\mathbb{F} = \mathbb{R}$ (as we do below):

Corollary 1.7. *The space $\mathcal{S}_k(\mathbb{F}) := \mathcal{N}(\mathbb{F}^k)/\sim$ is a closed metric subspace of $\mathcal{S}_{dk}(\mathbb{R})$, with common metric given by* (1.5).

Proof. We show that $\mathcal{S}_k(\mathbb{F})$ is closed in $\mathcal{S}_{dk}(\mathbb{R})$. Suppose $[N_l] \to [N]$ in $\mathcal{S}_{dk}(\mathbb{R})$, with $N_l \in \mathcal{N}(\mathbb{F}^k)\ \forall l$ and $N \in \mathcal{N}(\mathbb{R}^{dk})$. Without loss of generality, rescale the N_l and assume via (1.3) that

$$\frac{N_l}{N} : \mathbb{F}^k \setminus \{\mathbf{0}\} \to [1, M_l], \qquad \forall l > 0$$

with $M_l \to 1$ as $l \to \infty$. But then, $N_l \to N$ pointwise on $\mathbb{F}^k$. In particular, given $\alpha \in \mathbb{F}$ and nonzero $\mathbf{x} \in \mathbb{F}^k$,

$$\frac{N(\alpha\mathbf{x})}{N(\mathbf{x})} = \lim_{l\to\infty} \frac{N_l(\alpha\mathbf{x})}{N_l(\mathbf{x})} = |\alpha|,$$

and from this it follows that $N \in \mathcal{N}(\mathbb{F}^k)$ as desired. □

We now state the main result of the present work (with the caveat that this result is placed in a more general, functorial framework introduced in the following section). It implies as a consequence that $d_{\mathcal{S}_k(\mathbb{F})}$ is not just a metric, but also a norm:

Theorem 1.8. *For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the space $\mathcal{S}_k(\mathbb{F})$ is a complete, path-connected metric subspace of a real Banach space. It is a singleton set for $k = 1$, and unbounded for $k > 1$.*

(In less formal terms: 'the space of norms lies in a normed linear space.') A second consequence is that in dimensions two and higher, the space of equivalence classes of norms is not compact.

Acknowledgments. This work is partially supported by Ramanujan Fellowship SB/S2/RJN-121/2017 and MATRICS grant MTR/2017/000295 from SERB (Govt. of India), by grant F.510/25/CAS-II/2018(SAP-I) from UGC (Govt. of India), and by a Young Investigator Award from the Infosys Foundation. I thank Terence Tao for valuable discussions – especially about the final section – as well as Gautam Bharali, Javier Cabello Sánchez, Hariharan Narayanan, and M. Amin Sofi for several useful comments that helped improve previous versions of this manuscript. Part of this work was carried out during a visit to UCLA, to which I am thankful for its hospitality.

2. Diameter norms and an endofunctor

The goal of this section and the next is (to proceed toward) proving Theorem 1.8. While it is possible to provide a direct proof, our construction of a family of Banach spaces that each encompass $\mathcal{S}_k(\mathbb{F})$ (as asserted in Theorem 1.8) turns out to be part of a broader functorial setting – which we will use below in more than one setting. Thus we explain this setting in the present section, and complete the proof in Section 3.

The most primitive framework we consider is that of an abelian topological semigroup $(\mathscr{G}, +, d_{\mathscr{G}})$ with an associative, commutative binary operation $+ : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ and a translation-invariant metric $d_{\mathscr{G}}$, i.e.,

$$d_{\mathscr{G}}(x+z, y+z) = d_{\mathscr{G}}(x, y), \qquad \forall x, y, z \in \mathscr{G}. \tag{2.1}$$

Notice that in such a semigroup, one does not necessarily have inverses (i.e., 'negatives') or the identity element $e = 0_{\mathscr{G}}$. However, the following is easily shown (see e.g. [11] for details).

- the semigroup always has at most one idempotent $2e = e$;
- if such an e exists, it is the unique identity element in (the monoid) $\mathscr{G}$;
- if $\mathscr{G}$ does not contain an idempotent, one can formally attach such an idempotent e with metric $d_{\mathscr{G}}(e, z) := d_{\mathscr{G}}(z, 2z), d_{\mathscr{G}}(e, e) = 0$, and this creates the unique smallest monoid (with translation-invariant metric) containing $\mathscr{G}$.

Examples of such semigroups $\mathscr{G}$ abound in the literature, the most prominent being Banach spaces. However, there are several 'intermediate' classes of such abelian semigroups, including monoids (i.e. '$\mathbb{N} \cup \{0\}$-modules'), groups (i.e. $\mathbb{Z}$-modules), torsion-free divisible groups (i.e. $\mathbb{Q}$-modules), and normed linear spaces (i.e. $\mathbb{R}$-modules). More generally, one can consider metric R-modules, where $R \subset \mathbb{R}$ is a unital subring. As a further variant, one has the subclass of R-normed R-modules (see Definition 2.2).

Our first goal is to show that the seminorm defined in Theorem 2.3 below endows all $\mathscr{G}$-valued function spaces with the structure of a (pseudo-)metric. In fact, we show that this holds on a more structural level. For instance, it is clear that if $\mathscr{G}$ is a monoid, then so is the corresponding function space; and this holds for the finer structures mentioned above as well. What we also show is that the function space construction is also compatible with 'good' homomorphisms.

A systematic way to carry out this bookkeeping is that each of the above classes of semigroups is in fact a category, and the function space construction is a *covariant endofunctor* for each of these categories. This is now explained.

Definition 2.2. Let $R \subset \mathbb{R}$ denote any unital subring.

1. Let `Semi` denote the category of abelian topological semigroups $(\mathscr{G}, +, d_{\mathscr{G}})$ with translation-invariant metric $d_{\mathscr{G}}$, as above, and whose morphisms are semigroup homomorphisms that are Lipschitz.
2. Let `Mon` denote the full subcategory of `Semi`, whose objects are monoids.
3. Let R-`Mod` denote the subcategory of `Semi`, whose objects are R-modules such that multiplication by scalars in R are Lipschitz maps, and whose morphisms are Lipschitz R-module maps.
4. Let R-`NMod` denote the full subcategory of R-`Mod`, whose objects are *R-normed* R-modules $\mathscr{G}$. In other words, $d_{\mathscr{G}}(0, ra) = |r| d_{\mathscr{G}}(0, a)$ for all $r \in R$ and $a \in \mathscr{G}$.

(5) Let $\overline{R}$-Mod $\subset R$-Mod (note the unconventional notation) denote the full subcategory of R-Mod whose objects are complete metric spaces; and similarly define $\overline{R}$-NMod $\subset R$-NMod.

For each of these classes of objects, we now prove that the following function space construction is functorial:

Theorem 2.3. *Suppose $\mathscr{C}$ is one of the categories in Definition 2.2 (i.e.,* Semi, Mon, R-Mod, $\dots$, $\overline{R}$-NMod*). Given a set X and an object $M \in \mathscr{C}$, let $F_b(X, M)$ denote the set of bounded functions $f : X \to M$, and define*

$$d(f, g) \coloneqq \sup_{x,x' \in X} d_M(f(x) + g(x'), f(x') + g(x)). \tag{2.4}$$

For every set X that is not a singleton:

(1) *d is a pseudometric on $F_b(X, M)$.*
(2) *If one defines $f \sim g$ to mean $d(f, g) = 0$, then $\sim$ is an equivalence relation, and the quotient space assignment $M \mapsto F_b(X, M)/\sim$ is a covariant isometric endofunctor of the category $\mathscr{C}$.*

Here, we term a functor $F : \mathscr{C} \to \mathscr{C}'$ to be isometric *if: (a) all* Hom*-spaces in $\mathscr{C}, \mathscr{C}'$ are metric spaces, and (b) $F : \mathrm{Hom}(C_1, C_2) \to \mathrm{Hom}(F(C_1), F(C_2))$ is an isometry for all objects $C_1, C_2 \in \mathscr{C}$.*

Remark 2.5. As we show in Section 3, the connection to norms arises from the fact that the space $\mathcal{S}_k(\mathbb{F}) \subset \mathcal{S}_{dk}(\mathbb{R})$ embeds into $F_b(X, \mathbb{R})/\sim$ for some compact subset $X \subset \mathbb{F}^k$. Thus by Theorem 2.3 for $\mathscr{C} = \overline{\mathbb{R}}$-NMod, the metric on the space of norms arises as a norm in a Banach space.

For more categorical consequences and ramifications related to Theorem 2.3, we refer the reader to e.g. [11]. Also notice that if X is a singleton then $F_b(X, M) \simeq M$, whence $F_b(X, M)/\sim$ is the trivial semigroup (or Banach space). In this case the above result is true, except perhaps for the word 'isometric'.

Remark 2.6. For general abelian metric semigroups $M \in \mathscr{C}$ as in Theorem 2.3, an example of functions $f, g \in F_b(X, M)$ with distance zero is to choose and fix $m_0 \in M$, and take $g(x) \equiv m_0 + f(x)$ on all of X. If M is moreover a monoid and $f \equiv 0_M$, then these are the only examples.

In the further special case when M is an abelian metric group, the functions $g \equiv m_0 + f$ turn out to be the only examples of equivalent functions, for all $f \in F_b(X, M)$. Moreover, the (pseudo-)metric defined above has a more accessible interpretation as a *diameter seminorm*:

$$d(f, g) = N(f - g), \quad \text{where} \quad N(f) \coloneqq \sup_{x,x' \in X} d_M(f(x), f(x')) = \mathrm{diam}(\mathrm{im}(f)). \tag{2.7}$$

In particular, the equivalence relation $f \sim g$ amounts to $f - g$ being a constant function. For (abelian) monoids $(M, 0_M)$, essentially this last assertion also follows if one restricts to the set of bounded functions $f : X \to M$ satisfying: $0_M \in \mathrm{im}(f)$. More precisely, if $d(f, g) = 0$ for such functions, there exists $m_0 \in M$ such that $-m_0 \in M$ and $g \equiv m_0 + f$ on M.

Proof of Theorem 2.3. We begin by showing that (1) holds for all abelian semigroups M, and then turn to (2) for each successively smaller category. To show (1), we will show the triangle inequality; note this also proves the transitivity of $\sim$ and hence that $\sim$ is an equivalence relation on $F_b(X, M)$. Given $x, y \in X$,

$$\begin{aligned}
&d_M(f(x) + g(y), f(y) + g(x)) \\
&= d_M(f(x) + g(y) + h(y), f(y) + g(x) + h(y)) \\
&\leqslant d_M(f(x) + g(y) + h(y), f(y) + g(y) + h(x)) \\
&\qquad + d_M(f(y) + g(y) + h(x), f(y) + g(x) + h(y)) \\
&\leqslant d(f, h) + d(h, g).
\end{aligned} \tag{2.8}$$

As this inequality holds for all $x, y \in X$, the triangle inequality follows. This proves (1), and as a consequence, $F_b(X,M)/\sim$ is always a metric space under the metric (2.4). That this metric is translation-invariant (2.1) is straightforward.

We now claim that (2) holds, first for the category `Semi`. Indeed, one defines the (bounded) function $f+g$ pointwise for $f, g \in F_b(X,M)$. We claim that if $f \sim f'$ and $g \sim g'$ (i.e., they have distances zero between them) in $F_b(X,M)$, then $f + g \sim f' + g'$. This follows because

$$\begin{aligned} & d_M((f+g)(x) + (f'+g')(y), (f+g)(y) + (f'+g')(x)) \\ = \ & d_M([f(x)+f'(y)] + [g(x)+g'(y)], [f(y)+f'(x)] + [g(y)+g'(x)]) = 0, \end{aligned}$$

which in turn follows from the equalities: $f(x)+f'(y) = f(y)+f'(x)$ and $g(x)+g'(y) = g(y)+g'(x)$, for all $x, y \in X$.

Thus, $+$ is well-defined on $F_b(X,M)/\sim$. Similarly, one verifies that if $f_n \to f$ and $g_n \to g$ in $F_b(X,M)/\sim$, then $f_n + g_n \to f + g$. Next, given a semigroup morphism $\varphi : M \to N$, post-composing by φ defines a map $\varphi \circ -$ of semigroups : $F_b(X,M) \to F_b(X,N)$; and if $d(f,g) = 0$ in $F_b(X,M)$, then $d(\varphi \circ f, \varphi \circ g) = 0$ in $F_b(X,N)$. Thus φ induces a well-defined map

$$[\varphi] : F_b(X,M)/\sim \ \to F_b(X,N)/\sim . \tag{2.9}$$

Finally, we verify that the given functor induces isometries on Hom-spaces. The first sub-step is to claim that every Hom-space $\operatorname{Hom}(M,N)$ in `Semi` is itself a semigroup, with translation-invariant metric given by:

$$d(\eta, \varphi) := \sup_{m \neq m' \in M} \frac{d_N(\eta(m) + \varphi(m'), \varphi(m) + \eta(m'))}{d_M(m,m')}.$$

We only show that if $d(\eta,\varphi) = 0$ then $\eta = \varphi$; the remainder of the claim is straightforward (with the triangle inequality following similarly to (2.8)). Indeed, if $d(\eta,\varphi) = 0$, then:

$$d(\eta,\varphi) = 0 \quad \Longrightarrow \quad d_M(\eta(m) + \varphi(2m), \eta(2m) + \varphi(m)) = 0 \ \forall m \in M \quad \Longrightarrow \quad \eta \equiv \varphi.$$

This proves the above claim.

We now show that the map $\varphi \mapsto [\varphi]$ is an isometry

$$[-] : \operatorname{Hom}_{\mathscr{C}}(M,N) \to \operatorname{Hom}_{\mathscr{C}}(F_b(X,M)/\sim, F_b(X,N)/\sim).$$

Suppose $\eta, \varphi : M \to N$ are semigroup morphisms, and let $d(\eta,\varphi) = L$. If $L = 0$ then the preceding computation shows $\eta \equiv \varphi$ and hence $[\eta] = [\varphi]$. Otherwise, we compute from first principles:

$$\begin{aligned} d([\eta],[\varphi]) = & \sup_{[f] \neq [g] \in F_b(X,M)/\sim} \frac{d(\eta \circ [f] + \varphi \circ [g], \varphi \circ [f] + \eta \circ [g])}{d([f],[g])} \\ = & \sup_{[f] \neq [g] \in F_b(X,M)/\sim} \frac{1}{d([f],[g])} \sup_{x,x' \in X} d_N(\eta(m) + \varphi(m'), \varphi(m) + \eta(m')), \end{aligned}$$

where $m := f(x) + g(x')$ and $m' := f(x') + g(x)$. Now note that

$$d_N(\eta(m) + \varphi(m'), \varphi(m) + \eta(m')) \leqslant d(\eta,\varphi) d_M(f(x) + g(x'), f(x') + g(x)) \leqslant L \cdot d([f],[g])$$

for all $x, x' \in X$. It follows that $d([\eta],[\varphi]) \leqslant L = d(\eta,\varphi)$.

To show the reverse inequality, suppose $m_l \neq m'_l, l \in \mathbb{N}$ are sequences in M such that the sequences

$$d_l := \frac{d_N(\eta(m_l) + \varphi(m'_l), \varphi(m_l) + \eta(m'_l))}{d_M(m_l, m'_l)}$$

are non-decreasing to $L = d(\eta,\varphi)$ as $l \to \infty$. We now use that X is not a singleton, whence for a fixed element $x_1 \in X$, we consider

$$f_l|_X \equiv m_l, \qquad g_l|_{X \setminus x_1} \equiv m_l, \qquad g_l(x_1) := m'_l, \qquad l \in \mathbb{N}.$$

Clearly $d([f_l],[g_l]) = d_M(m_l, m_l')$, whence for any $x_2 \in X \setminus x_1$, we compute from above:

$$\begin{aligned}
& d([\eta],[\varphi]) \\
\geqslant & \sup_{l\in\mathbb{N}} \frac{d_N(\eta(f_l(x_1)+g_l(x_2))+\varphi(f_l(x_2)+g_l(x_1)), \eta(f_l(x_2)+g_l(x_1))+\varphi(f_l(x_1)+g_l(x_2)))}{d([f_l],[g_l])} \\
= & \sup_{l\in\mathbb{N}} \frac{d_N(\eta(m_l)+\varphi(m_l'), \varphi(m_l)+\eta(m_l'))}{d_M(m_l,m_l')} \\
= & \sup_{l\in\mathbb{N}} d_l = L = d(\eta,\varphi).
\end{aligned}$$

This proves the theorem for metric semigroups, i.e. for $\mathscr{C} = \texttt{Semi}$. We next impose the additional structure in each smaller subcategory one by one, and show the result for the remaining $\mathscr{C}$. Clearly, if M is a monoid, then so is $F_b(X,M)/\sim$, with identity $f \equiv 0_M$. Now the result is easily verified for $\mathscr{C} = \texttt{Mon}$. (Note that all morphisms are automatically monoid maps.)

Next suppose $\mathscr{C} = R\text{-}\texttt{Mod}$. One checks that if $f \sim g$ then $rf \sim rg$, where $rf \in F_b(X,M)$ is defined in the usual (pointwise) fashion. Also, multiplication by r is Lipschitz on $F_b(X,M)/\sim$ if it is so on M itself. Now the result is easily verified in this setting. Finally, if M is also R-normed, then one checks that so is $F_b(X,M)/\sim$.

This shows the result for all categories except for $\overline{R}\text{-}\texttt{Mod}, \overline{R}\text{-}\texttt{NMod}$. For these latter cases, it suffices to show the claim that if M is a complete abelian metric group then so is $F_b(X,M)/\sim$. We begin by isolating the main component of this argument into a standalone result (together with some related preliminaries).

Lemma 2.10. *Fix a set X and an abelian metric semigroup $(M,+,d_M)$.*

1. *$F_b(X,M)$ is a metric space under the sup-norm*
$$d_\infty(f,g) := \sup_{x\in X} d_M(f(x),g(x)).$$
Moreover, $F_b(X,M)$ is complete if and only if M is complete.
2. *The quotient map of metric spaces : $F_b(X,M) \to F_b(X,M)/\sim$ is Lipschitz of norm at most 2.*
3. *If M is moreover a group, there exists a section $\Phi_{x_0} : F_b(X,M)/\sim \ \to F_b(X,M)$ which is a sub-contraction:*
$$d_\infty(\Phi_{x_0}([f]),\Phi_{x_0}([g])) \leqslant d([f],[g]), \qquad \forall x_0 \in X,\ [f],[g] \in F_b(X,M), \tag{2.11}$$
and such that the image of Φ_{x_0} is precisely the set of functions vanishing at x_0.

Proof. (1) is well-known, and (2) is standard using:

$$\begin{aligned}
& d_M(f(x)+g(x'), f(x')+g(x)) \\
\leqslant & d_M(f(x)+g(x'), g(x)+g(x')) + d_M(g(x)+g(x'), g(x)+f(x')) \\
\leqslant & 2d_\infty(f,g), \qquad \forall f,g \in F_b(X,M),\ x,x' \in X.
\end{aligned}$$

To show (3), choose any representative f of $[f] \in F_b(X,M)/\sim$, recalling by Remark 2.6 that f is unique up to translation by an element of M. Now the 'Kuratowski' map $\Phi_{x_0}([f]) := f(x) - f(x_0)$ satisfies (2.11). (We also point out for completeness some related observations at the start of [4, Section 3].) ☐

Returning to the proof of the above claim, suppose $[f_n] \in F_b(X,M)/\sim$ is Cauchy, with M complete. For any fixed $x_0 \in X$, this implies by Lemma 2.10(3) that $\Phi_{x_0}([f_n])$ is Cauchy, whence it converges in the d_∞ metric to some bounded map f by Lemma 2.10(1). Hence, $[f_n] \to [f]$ by Lemma 2.10(2). ☐

We now refine the above categorical construction, when the domain X is additionally equipped with a topology:

Corollary 2.12. *Given a topological space X and an abelian metric semigroup $(M, +, d_M)$, define $C_b(X, M)$ to be the set of bounded continuous functions $: X \to M$. Now fix a unital subring $R \subset \mathbb{R}$.*

1. *If M is in fact an abelian group, then $C_b(X, M)/\sim$ is a closed subobject of $F_b(X, M)/\sim$.*
2. *With notation as in Theorem 2.3 (for any of the categories $\mathscr{C}$), $C_b(X, M)$ forms a pseudometric subspace of $F_b(X, M)$, whence $C_b(X, M)/\sim$ is a subobject of $F_b(X, M)/\sim$ in $\mathscr{C}$. Moreover, $M \mapsto C_b(X, M)/\sim$ is also a covariant endofunctor of $\mathscr{C}$, which is isometric if X is not a singleton.*

Note that the case of X compact Hausdorff and $M = \mathbb{R}$ was studied in greater detail in [4], and is part of a broader program to study isometries and linear isomorphisms of spaces of continuous functions. See e.g. [1, 4, 10], and the references therein.

Proof. To show (1), if $f_n : X \to M$ are continuous and $[f_n] \to [f]$ for some $f \in F_b(X, M)$, then $\Phi_{x_0}([f_n]) \to \Phi_{x_0}([f])$ uniformly by (2.11), whence $\Phi_{x_0}([f])$ is continuous and hence so is f.

For the categories whose objects are not all complete, the assertion (2) follows from Theorem 2.3 and the continuity of the R-module operations. For the categories $\mathscr{C} = \overline{R}$-Mod, $\overline{R}$-NMod, one further uses the previous part and that $F_b(X, M)/\sim$ is complete by Theorem 2.3. ☐

3. Distances between p-norms; proof of the main result

In this section we prove Theorem 1.8, deriving part of it from the functorial framework discussed in the previous section.

3.1. p-norm computations.

Part of the proof of Theorem 1.8 works with the p-norms on $\mathbb{F}^k$; thus, we begin by providing 'more standard' models for certain sets of such norms. Given $p \in [1, \infty)$, define for $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{F}^k$ its p-norm:

$$\|(x_1, \dots, x_k)\|_p := (|x_1|^p + \cdots + |x_k|^p)^{1/p}, \tag{3.1}$$

and also define $\|(x_1, \dots, x_k)\|_\infty := \max_j |x_j|$.

As is well-known in the Banach–Mazur framework for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ [6], if $p, q \in [1, \infty]$ and $2 - p, 2 - q$ have the same sign, then the norms $\| \cdot \|_p$ and $\| \cdot \|_q$ on $\mathbb{F}^k$ have Banach–Mazur distance $|1/p - 1/q| \cdot \log(k)$. However, this does not usually hold for $1 \leqslant p < 2 < q \leqslant \infty$ – for instance if $k = 2$ then $\| \cdot \|_1$ and $\| \cdot \|_\infty$ denote the same point in the Banach–Mazur compactum. In the present setting of $\mathcal{S}'_k(\mathbb{F})$, the p-norms share the above behavior for $p \in [1, 2]$ and $[2, \infty]$, but differ in the metric structure for $[1, \infty]$:

Proposition 3.2. *Let $\mathcal{S}'_k(\mathbb{F}) \subset \mathcal{S}_k(\mathbb{F})$ denote the equivalence classes of the norms $\{\| \cdot \|_p : 1 \leqslant p \leqslant \infty\}$. Then the map $f : \mathcal{S}'_k(\mathbb{F}) \to [0, \log k]$, given by $f(\| \cdot \|_p) := \frac{\log k}{p}$ for $p \in [1, \infty)$ and $f(\| \cdot \|_\infty) := 0$, is an isometric bijection.*

Thus the p-norms behave 'uniformly well': $\mathcal{S}'_k(\mathbb{R}) \cong \mathcal{S}'_k(\mathbb{C}) \cong \mathcal{S}'_k(\mathbb{H}) \cong [0, \log k]$.

Proof. For $1 \leqslant p < q < \infty$, Hölder's inequality implies $k^{-1/p} \|\mathbf{x}\|_p \leqslant k^{-1/q} \|\mathbf{x}\|_q$ for all $\mathbf{x} \in \mathbb{F}^k$, and equality is attained at the vectors with all equal coordinates. For the other way, we claim that $\|\mathbf{x}\|_p \geqslant \|\mathbf{x}\|_q$, with equality along the coordinate axes. Indeed, by rescaling one may assume $\|\mathbf{x}\|_p = 1$, whence $|x_j|^p \leqslant 1$ for all j. Thus $|x_j| \leqslant 1$, and it follows that

$$\|\mathbf{x}\|_q^q = \sum_j |x_j|^q \leqslant \sum_j |x_j|^p = 1,$$

whence $\|\mathbf{x}\|_q \leqslant 1 = \|\mathbf{x}\|_p$. From this it follows that $d_{\mathcal{S}_k(\mathbb{F})}(\| \cdot \|_p, \| \cdot \|_q) = \log k^{1/p - 1/q}$.

Finally, it is evident that $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_p \leqslant k^{1/p}\|\mathbf{x}\|_\infty$ for all $p \in [1,\infty)$ and $\mathbf{x} \in \mathbb{F}^k$, with equality attained in the same two cases as above. Thus, $d_{\mathcal{S}_k(\mathbb{F})}(\|\cdot\|_p, \|\cdot\|_\infty) = \log k^{1/p}$. This concludes the proof. □

Example 3.3. As another example, notice using Lemma 1.1 that given a non-negative measure μ supported on $[1,\infty)$, the function

$$N_\mu(\mathbf{x}) := \int_1^\infty \|\mathbf{x}\|_p \; d\mu(p)$$

is a norm, if convergent on $\mathbb{F}^k$. Now the same reasoning as in the above proof shows that for all such $\mu \geqslant 0$ with bounded support and positive mass,

$$d_{\mathcal{S}_k(\mathbb{F})}(N_\mu, \|\cdot\|_q) = \log \frac{\int_1^q k^{1/p} \; d\mu(p)}{k^{1/q}\int_1^q d\mu(p)} \leqslant \log(k)(1-q^{-1}), \tag{3.4}$$

where $\sup(\text{supp }\mu) < q < \infty$.

The above provide examples of subsets of $\mathcal{S}_k(\mathbb{F})$ with bounded diameter. However, this does not always happen, and we now mention such an example, which also serves to show the 'unboundedness' assertion in the main result.

Example 3.5. Given $p \in [1,\infty]$, $q \in [0,\infty)$, and an integer $1 \leqslant j \leqslant k$, define

$$N_{p,q,j}(\mathbf{x}) := \|\mathbf{x}\|_p + q|x_j|, \qquad \mathbf{x} \in \mathbb{F}^k \tag{3.6}$$

and consider the family of such norms for a fixed p:

$$\mathcal{S}_{k,p} := \{N_{p,q,j} : q \in [0,\infty), \; j \in [k]\}. \tag{3.7}$$

(one verifies easily that these are norms). We now claim that akin to Proposition 3.2 for the p-norms, the family $\mathcal{S}_{k,p}$ can also be realized as a more familiar metric subspace of a Banach space:

Proposition 3.8. *Suppose $k > 1$. The subset $\mathcal{S}_{k,p}$ defined in (3.7) isometrically embeds into $\mathbb{R}^k$ with the ℓ^1-norm, via $N_{p,q,j} \mapsto \log(1+q)\mathbf{e}_j$. The image of $\mathcal{S}_{k,p}$ is the union of the non-negative coordinate semi-axes.*

Proof. Notice that $N_{p,q,j}(\mathbf{x}) \leqslant (1+q)N_{p,q',j'}(\mathbf{x})$ for all $q,q' \geqslant 0$ and $j \neq j' \in \{1,\dots,k\}$, with equality attained at least for $\mathbf{x} = x\mathbf{e}_j$ (here, $\mathbf{e}_1,\dots,\mathbf{e}_n$ comprise the standard basis of $\mathbb{F}^k$). It follows that

$$d_{\mathcal{S}_k(\mathbb{F})}(N_{p,q,j}, N_{p,q',j'}) = \log(1+q) + \log(1+q').$$

One next shows that for a fixed $j \in \{1,\dots,k\}$,

$$0 \leqslant q \leqslant q' < \infty \quad \implies \quad N_{p,q,j}(\mathbf{x}) \leqslant N_{p,q',j}(\mathbf{x}) \leqslant \frac{1+q'}{1+q} N_{p,q,j}(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbb{F}^k,$$

with equality possible on $\mathbb{F}^k \setminus \{\mathbf{0}\}$ in either inequality (note that equality in the lower bound requires $k > 1$). Therefore $d_{\mathcal{S}_k(\mathbb{F})}(N_{p,q,j}, N_{p,q',j}) = \log(1+q') - \log(1+q)$. This concludes the proof. □

3.2. **Proof of the main result.** With Proposition 3.8 and the functorial analysis in the previous section in hand, we can show our main result.

Proof of Theorem 1.8. Begin by fixing any compact subset

$$X \subset \mathbb{F}^k \setminus \{\mathbf{0}\} \cong \mathbb{R}^{dk} \setminus \{\mathbf{0}\} \qquad \text{satisfying:} \qquad \forall \mathbf{x} \in \mathbb{F}^k \setminus \{\mathbf{0}\}, \; \exists \alpha_{\mathbf{x}} \in \mathbb{F}^\times \text{ such that } \alpha_{\mathbf{x}}\mathbf{x} \in X. \tag{3.9}$$

(For instance, X could be the unit sphere S^{dk-1}.) The bulk of the proof involves showing the claim that the space $\mathcal{S}_k(\mathbb{F})$ is a closed metric subspace of the Banach space $C(X,\mathbb{R})/\sim \; = C_b(X,\mathbb{R})/\sim$ (see Corollary 2.12, noting that X is compact). In particular, $\mathcal{S}_k(\mathbb{F})$ is complete.

To show the claim, we construct the embedding $\Psi : \mathcal{N}(\mathbb{F}^k) \to C(X,\mathbb{R})$ as follows: given a norm $N \in \mathcal{N}(\mathbb{F}^k)$, define $\Psi(N) := \log N|_X \in C(X,\mathbb{R})$. Since N is a norm, it is uniquely determined by its restriction to X, whence Ψ is injective. Moreover, the respective notions of $\sim$ are compatible via taking the logarithm, whence Ψ induces an injection $[\Psi] : \mathcal{S}_k(\mathbb{F}) \hookrightarrow C(X, \mathbb{R})/\sim$ of metric spaces. It is easily verified that $[\Psi]$ is an isometry; recall here that the metric on $\mathcal{S}_k(\mathbb{F})$ is given by: $d([N], [N']) := \log(M_{N,N'}/m_{N,N'})$ (see Equations (1.5) and (1.3)).

It remains to show closedness. Suppose N_l are norms on $\mathbb{F}^k$ such that $[\log N_l|_X] \to [f]$ in $C(X, \mathbb{R})/\sim$ under the metric in (1.5) (recall Corollary 2.12 here). As above, one can choose representative norms N_l on $\mathbb{F}^k$ and a function $f \in C(X, \mathbb{R})$ in their equivalence classes, such that for all $l > 0$,

$$(\log N_l) - f : X \to [0, \epsilon_l], \quad \epsilon_l \geqslant 0, \tag{3.10}$$

with $\epsilon_l \to 0^+$ as $l \to \infty$. In particular, $N_l \to \exp(f)$ pointwise on X. Define

$$N(\mathbf{0}) := 0, \qquad N(\mathbf{x}) := |\alpha_{\mathbf{x}}|^{-1} \exp(f(\alpha_{\mathbf{x}}\mathbf{x})), \ \forall \mathbf{x} \in \mathbb{F}^k \setminus \{\mathbf{0}\},$$

where $\alpha_{\mathbf{x}}$ comes from the defining property of X. The above claim is proved if one shows that N is a norm on $\mathbb{F}^k$. First note that N is indeed well-defined: if $\alpha_{\mathbf{x}}$ and $\beta_{\mathbf{x}}$ are such that $\alpha_{\mathbf{x}}\mathbf{x}, \beta_{\mathbf{x}}\mathbf{x} \in X$ for some $\mathbf{x}$, then

$$|\alpha_{\mathbf{x}}|^{-1} \exp(f(\alpha_{\mathbf{x}}\mathbf{x})) = \lim_{l\to\infty} |\alpha_{\mathbf{x}}|^{-1} N_l(\alpha_{\mathbf{x}}\mathbf{x}) = \lim_{l\to\infty} |\beta_{\mathbf{x}}|^{-1} N_l(\beta_{\mathbf{x}}\mathbf{x}) = |\beta_{\mathbf{x}}|^{-1} \exp(f(\beta_{\mathbf{x}}\mathbf{x})).$$

Next, N is homogeneous: given any scalar $\beta \in \mathbb{F}$ and vector $\mathbf{x} \in \mathbb{F}^k \setminus \{\mathbf{0}\}$,

$$\frac{N(\beta\mathbf{x})}{N(\mathbf{x})} = \frac{|\alpha_{\beta\mathbf{x}}|^{-1} \exp f(\alpha_{\beta\mathbf{x}}\beta\mathbf{x})}{|\alpha_{\mathbf{x}}|^{-1} \exp f(\alpha_{\mathbf{x}}\mathbf{x})} = \lim_{l\to\infty} \frac{|\alpha_{\beta\mathbf{x}}|^{-1} N_l(\alpha_{\beta\mathbf{x}}\beta\mathbf{x})}{|\alpha_{\mathbf{x}}|^{-1} N_l(\alpha_{\mathbf{x}}\mathbf{x})} = \lim_{l\to\infty} |\beta| = |\beta|.$$

Finally, observe that N is sub-additive, i.e., $N(\mathbf{x}+\mathbf{y}) \leqslant N(\mathbf{x}) + N(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{F}^k$. Indeed, this is immediate if any of $\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}$ is zero, so assume this does not happen, and compute:

$$\begin{aligned}
N(\mathbf{x}+\mathbf{y}) &= |\alpha_{\mathbf{x}+\mathbf{y}}|^{-1} \lim_{l\to\infty} N_l(\alpha_{\mathbf{x}+\mathbf{y}}(\mathbf{x}+\mathbf{y}))\\
&\leqslant |\alpha_{\mathbf{x}+\mathbf{y}}|^{-1} \lim_{l\to\infty} N_l(\alpha_{\mathbf{x}+\mathbf{y}}\mathbf{x}) + N_l(\alpha_{\mathbf{x}+\mathbf{y}}\mathbf{y})\\
&= |\alpha_{\mathbf{x}+\mathbf{y}}|^{-1} \lim_{l\to\infty} \left(\frac{|\alpha_{\mathbf{x}+\mathbf{y}}|}{|\alpha_{\mathbf{x}}|} N_l(\alpha_{\mathbf{x}}\mathbf{x}) + \frac{|\alpha_{\mathbf{x}+\mathbf{y}}|}{|\alpha_{\mathbf{y}}|} N_l(\alpha_{\mathbf{y}}\mathbf{y}) \right)\\
&= N(\mathbf{x}) + N(\mathbf{y}).
\end{aligned}$$

The closedness of $\mathcal{S}_k(\mathbb{F})$ now follows, whence by Theorem 2.3 and Corollaries 1.7 and 2.12, we have a chain of inclusions with closed images

$$\mathcal{S}_k(\mathbb{F}) \hookrightarrow \mathcal{S}_{dk}(\mathbb{R}) \hookrightarrow C_b(X, \mathbb{R})/\sim \hookrightarrow F_b(X, \mathbb{R})/\sim;$$

The above claim now follows; hence $\mathcal{S}_k(\mathbb{F})$ is complete. Next, Lemma 1.1 implies that $\mathcal{N}(\mathbb{F}^k)$ is convex, hence path-connected, whence so is $\mathcal{S}_k(\mathbb{F})$. Moreover, clearly $\mathcal{S}_1(\mathbb{F})$ is a point. Finally, assuming $k > 1$, Proposition 3.8 shows that $\mathcal{S}_k(\mathbb{F})$ is unbounded. □

The above proof shows that the metric space of norms embeds into $C(X, \mathbb{R})/\sim$ for many different compact topological subspaces $X \subset \mathbb{F}^k$ (see (3.9)). We conclude by exploring these embeddings in greater detail, fixing $\mathbb{F} = \mathbb{R}$ for convenience. Specifically, if $X = S^{k-1}$ denotes the unit sphere, then under the embedding $\mathcal{S}_k(\mathbb{R}) \hookrightarrow \mathbb{B} := C(S^{k-1}, \mathbb{R})/\sim$, the origin in the Banach space $\mathbb{B}$ is precisely the image of the 2-norm $\|\cdot\|_2$. More generally, we have such an identification of the origin for *every* norm – but not for every space X.

Proposition 3.11. *Given a norm $N : \mathbb{R}^k \to \mathbb{R}$ and any radius $r > 0$, let $X_{N,r}$ denote the sphere*

$$X_{N,r} := \{\mathbf{x} \in \mathbb{R}^k : N(\mathbf{x}) = r\}.$$

Now fix a norm N on $\mathbb{R}^k$, as well as a subset $X \subset \mathbb{R}^k$ such that every nonzero vector is a positive real multiple of a point in X. Then the following are equivalent:

(1) *$\mathcal{S}_k(\mathbb{R})$ embeds as a closed subset in the Banach space $C(X,\mathbb{R})/\sim$ for some topological space X via $[N'] \mapsto [\log N'|_X]$; and this embedding maps the equivalence class $[N]$ to the origin.*
(2) *$X = X_{N,r}$ for some $r > 0$.*

However, there exist compact sets $X \subset \mathbb{R}^k$ such that the image of the embedding $\mathcal{S}_k(\mathbb{R}) \hookrightarrow C(X,\mathbb{R})/\sim$ avoids the origin.

Proof. If (1) holds, then $N|_X$ must be constant, whence $X \subset X_{N,r}$ for some $r > 0$. Moreover, if $\mathbf{x} \in X_{N,r}$, then $\alpha\mathbf{x} \in X$ for some $\alpha > 0$. Since $N(\alpha\mathbf{x}) = \alpha r$, it follows that $\alpha = 1$ and hence $X = X_{N,r}$.

Conversely, suppose (2) holds. Since all norms on $\mathbb{R}^k$ are equivalent, the space $X_{N,r}$ is compact and hence satisfies (3.9), whence the proof of Theorem 1.8 applies to it. In particular, the image of N under the embedding : $\mathcal{S}_k(\mathbb{R}) \hookrightarrow C(X_{N,r},\mathbb{R})/\sim$ is a constant function, whose image under $\sim$ is the trivial class.

Finally, given any point $\mathbf{y} \neq 0$, define the sphere

$$X_{\mathbf{y}} := \{\mathbf{x} \in \mathbb{R}^k : \|\mathbf{x} - \mathbf{y}\|_2^2 = 1 + \|\mathbf{y}\|_2^2\}.$$

It is clear that $\mathbf{0}$ is in the 'interior' of the sphere. Also notice that for every unit direction $\mathbf{v} \in S^{k-1}$, there exists a unique $\alpha > 0$ such that $\alpha\mathbf{v} \in X_{\mathbf{y}}$. Indeed, from the conditions

$$\|\alpha\mathbf{v} - \mathbf{y}\|_2^2 = 1 + \|\mathbf{y}\|_2^2, \qquad \alpha > 0,$$

one derives: $\alpha = \sqrt{1 + \langle \mathbf{v}, \mathbf{y}\rangle^2} + \langle \mathbf{v}, \mathbf{y}\rangle$. In particular, $X_{\mathbf{y}}$ satisfies (3.9) and hence the proof of Theorem 1.8 applies to it. However, no norm maps via the embedding $\mathcal{S}_k(\mathbb{R}) \hookrightarrow \mathbb{B} := C(X_{\mathbf{y}},\mathbb{R})/\sim$ to the origin in the Banach space $\mathbb{B}$. Indeed, if $[N] \mapsto 0_{\mathbb{B}}$, then the norm N would restrict to a constant on $X_{\mathbf{y}}$. But this is false: $X_{\mathbf{y}}$ intersects the line $\mathbb{R}\mathbf{y}$ at the two points $(\|\mathbf{y}\|_2 \pm \sqrt{1 + \|\mathbf{y}\|_2})\mathbf{y}$, and as these are not negatives of one another, evaluating N yields unequal values. □

4. The normed space of metrics on a finite set

We now study a parallel setting to the metric space of norms on $\mathbb{F}^k$, in which the above functorial approach is also applicable. Given a finite set $[n] := \{1,\dots,n\}$ with $n \geqslant 2$, it is possible to impose a pseudometric on the space of metrics on $[n]$ in the same way as above: given metrics $\rho, \rho' : X \times X \to [0,\infty)$, define

$$d_{[n]}(\rho,\rho') := \log\ \max_{j \neq k} \frac{\rho'(x_j,x_k)}{\rho(x_j,x_k)}\ -\ \log\ \min_{j \neq k} \frac{\rho'(x_j,x_k)}{\rho(x_j,x_k)};$$

note that $\exp \circ d_{[n]}$ is termed the *distortion* in metric geometry and computer science.

We cite the well-known surveys [13, 15] for further details and reading on the numerous applications of distortion and metric geometry to computer science, combinatorics, and other fields. Also note that there is a different, well-studied pseudometric on the space of metrics on $[n]$, or more generally on all compact metric spaces. This is the Gromov–Hausdorff metric (which is not comparable to $d_{[n]}$, as we see below). Nevertheless, the metric $d_{[n]}$, as well as the connection between diameter norms and (log-)distortion, do not seem to be studied or recorded in the literature. This motivates the present section.

We begin with a result that is parallel to Theorem 1.8 for $\mathcal{S}_k(\mathbb{F})$, and again follows from the above functorial analysis. In particular, it shows that the metric $d_{[n]}$ is also a diameter norm:

Theorem 4.1. *Fix an integer $n \geqslant 2$.*

(1) *The map $d_{[n]}$ is a pseudometric on the space of metrics on $[n]$, with equivalence classes precisely consisting of proportional metrics.*
(2) *The quotient metric space $\mathcal{S}([n])$ is a complete, path-connected, metric subspace of the Banach space $\mathbb{R}^{\binom{[n]}{2}}/\sim\ = C(\binom{[n]}{2},\mathbb{R})/\sim$ with the diameter norm, where $\binom{[n]}{2}$ denotes the discrete set of two-element subsets of $[n]$.*

(3) *The space $\mathcal{S}([n])$ is a singleton if $n = 2$, and unbounded otherwise.*

Proof. We only point out why for $n > 2$ the space $\mathcal{S}([n])$ is unbounded. Indeed, for each $m \geqslant 1$ let $X_m := \{1, 2, \dots, n-1; n+m+1\}$ be the (induced) metric subspace of $(\mathbb{R}, |\cdot|)$, and let $\rho_{n,m} : [n] \times [n] \to \mathbb{R}$ be the metric induced by the unique rank/order-preserving map : $X_m \to [n] \subset \mathbb{R}$. Compare $\rho_{n,m}$ to the discrete metric $\rho(x, y) = 1 - \delta_{x,y}$: the log-distortion between them is $\log(n+m)$, and m can grow without bound. $\square$

Remark 4.2. The metric on $\mathcal{S}([n])$ – henceforth denoted by $d_{\mathcal{S}([n])}$ – is not comparable to the well-studied Gromov–Hausdorff metric on compact metric spaces:

$$d_{GH}(X_1, X_2) := \inf_{Z, \iota_1, \iota_2} d_H(\iota_1(X_1), \iota_2(X_2)),$$

where one runs over all metric spaces Z and isometric embeddings : $\iota_j : X_j \hookrightarrow Z$; and where d_H denotes the Hausdorff distance in Z. To see why d_{GH} is not comparable to the above metric $d_{\mathcal{S}([n])}$, for any $n > 2$ choose two different pairs of points from $[n]$, say $\{a, b\} \neq \{c, d\} \subset [n]$. Now define metrics ρ, ρ' on $[n]$ via: $\rho(a, b) = \rho'(c, d) = 1/2$ and all other nonzero values of ρ, ρ' are 1. These two metric spaces are clearly isometric under $a \leftrightarrow c$, $b \leftrightarrow d$, and all other points left unchanged. However, the metrics are not proportional. Going the other way, proportional but unequal metrics on $[n]$ do not admit an isometry between them.

Our next few results are meant to help better understand the metric space $\mathcal{S}([n])$. The following result parallels Proposition 3.8, and illustrates how a certain one-parameter family of metrics on $[n]$ can be understood through a more standard model:

Proposition 4.3. *Fix integers $n > 0$ and $1 \leqslant j \leqslant \binom{n}{2}$, as well as any bijection to identify the set of pairs $\binom{[n]}{2}$ (i.e. edges between points in $[n]$) with the set $\left[\binom{n}{2}\right] = \{1, \dots, \binom{n}{2}\}$. Given $a \in (0, 1]$, let $\rho_{j,a} : [n] \times [n] \to \mathbb{R}$ denote the metric in which all nonzero distances in $[n]$ are 1, except for the distance corresponding to the edge j, which is a. Define*

$$\mathcal{S}'([n]) := \{\rho_{j,a} : j \in \left[\tbinom{n}{2}\right],\ a \in (0, 1]\}.$$

Then $\mathcal{S}'([n])$ (or its set of equivalence classes) embeds isometrically into $\mathbb{R}^{\binom{n}{2}}$ with the ℓ^1-norm, via: $\rho_{j,a} \mapsto (\log a)\mathbf{e}_j$. The image is the union of the non-positive coordinate semi-axes.

The set $\mathcal{S}'([n])$ comprises the metrics in which $[n]$ may be viewed as a weighted graph with all edge weights but one equal and at least as large as the remaining edge weight.

Proof. Write $N := \binom{n}{2}$ for convenience. Viewing each metric ρ on $[n]$ as a function : $[N] \to (0, \infty)$, say $(\rho^{(1)}, \dots, \rho^{(N)})^T$, it follows that

$$d_{\mathcal{S}([n])}(\rho_{j,a}, \rho_{j',a'}) = \log \max_{1 \leqslant k \leqslant N} \frac{\rho_{j,a}^{(k)}}{\rho_{j',a'}^{(k)}} - \log \min_{1 \leqslant k \leqslant N} \frac{\rho_{j,a}^{(k)}}{\rho_{j',a'}^{(k)}}.$$

Now if $j = j'$ then the distance is $|\log a - \log a'|$, else the distance is $-\log a - \log a'$. $\square$

Remark 4.4. Notice that the function $\rho_{j,a}$ is a metric on $[n]$ if and only if $a \in (0, 2]$. Thus, one can compute the distance between $\rho_{j,a}$ and $\rho_{j',a'}$ for $a, a' \in (0, 2]$. If $j = j'$ then we once again obtain $|\log a - \log a'|$, but if $j \neq j'$ then we have:

$$d_{\mathcal{S}([n])}(\rho_{j,a}, \rho_{j',a'}) = \begin{cases} |\log a| + |\log a'|, & \text{if either } a, a' \geqslant 1 \text{ or } a, a' \leqslant 1; \\ \max\{|\log a|, |\log a'|\}, & \text{otherwise.} \end{cases}$$

We next study *embeddings* in $\mathcal{S}([n])$ of metric spaces. In doing so, we are motivated by recent work [7], where it was shown that every finite metric space of size at most $\binom{n}{2}$ embed isometrically into the Gromov–Hausdorff space of isometry classes of n-element metric spaces. In other words, a representative from the Gromov–Hausdorff equivalence class of every metric space of size at most $\binom{n}{2}$ embeds isometrically into Gromov–Hausdorff space. The following result is parallel in spirit, for the space $\mathcal{S}([n])$:

Theorem 4.5. *Let (X, d) be a finite metric space, and $n \geqslant 3$ an integer such that $X \leqslant \binom{n}{2}$. Then there exists an equivalent metric space to (X, d) – i.e., a rescaling of d – that admits an isometric embedding into $\mathcal{S}([n])$.*

Note that the result fails to hold for $n = 2$, since $\mathcal{S}([2])$ is a singleton.

Before proving Theorem 4.5, we recall that its Gromov–Hausdorff analogue in [7] was stated using the *smallest* n such that $|X| \leqslant \binom{n}{2}$. While our variant does not *a priori* have this extra restriction, we point out that the two versions are equivalent for $\mathcal{S}([n])$, because of the following result:

Proposition 4.6. *For all $n \geqslant 2$, the metric space $\mathcal{S}([n])$ isometrically embeds into $\mathcal{S}([n+1])$.*

Proof. Given a finite metric space (X, d) with $|X| = n$, embed it into a metric space $X \sqcup \{n+1\}$, where $n+1$ is an additional point with distance $\operatorname{diam}(X)$ from every $x \in X$. A straightforward computation (perhaps rescaling both diameters to 1 for convenience) now shows that this defines an isometry from $\mathcal{S}([n])$ into $\mathcal{S}([n+1])$. ☐

We now prove the above theorem.

Proof of Theorem 4.5. In fact we will construct the embedding : $(X, \alpha \cdot d) \hookrightarrow \mathcal{S}([n])$ for a specific $\alpha > 0$, using several 'natural' tools. We begin by describing these tools.

Observe that every metric on $[n]$ can be viewed as an element of $(0, \infty)^N$ where $N = \binom{n}{2}$. Via taking logarithms, the space $\mathcal{S}([n])$ is in bijection with the set $\Psi([n])$ of all tuples $(\psi_{ij})^T \in \mathbb{R}^{\binom{[n]}{2}} \cong \mathbb{R}^N$ (here $i < j$) such that

$$\exp(\psi_{ij}) + \exp(\psi_{jk}) \geqslant \exp(\psi_{ik}), \qquad \forall 1 \leqslant i < j < k \leqslant n. \tag{4.7}$$

Note that rescaling the metric by $\alpha > 0$ is equivalent to translating all ψ_{ij} by $\log \alpha$. In other words, $\Psi([n])$ sits inside $\mathbb{R}^N / \sim$, where $\sim$ denotes additive translations by scalar multiples of $(1, \dots, 1)^T$.

The next observation is that for an integer $p > 0$, the space $(\mathbb{R}^p, \|\cdot\|_\infty)$ is isometrically isomorphic as a Banach space to $\mathbb{R}^{p+1} / \sim$ with the diameter norm diam, where $\sim$ denotes quotienting by additive translation by multiples of $(1, \dots, 1)^T$. More generally, given integers $0 < p < q$, the map

$$\Psi_{p,q} : (x_1, \dots, x_p)^T \mapsto (x_1, \dots, x_p, 0_{1 \times (q-p)})^T + \mathbb{R}(1, \dots, 1)_q^T$$

is an isometric linear embedding of $(\mathbb{R}^p, \|\cdot\|_\infty)$ into $(\mathbb{R}^q / \sim, \operatorname{diam})$. For this reason, note in the previous paragraph that the bijection of sets

$$\log[-] : (\mathcal{S}([n]), d_{\mathcal{S}([n])}) \to (\Psi([n]), \operatorname{diam})$$

is in fact an isometry of metric spaces.

Finally, we recall the *Fréchet embedding* [5], which maps an N-element metric space $(X = \{x_0, \dots, x_{N-1}\}, d)$ isometrically into $\mathbb{R}^{N-1}$ with the sup-norm, via: $x_j \mapsto (d(x_1, x_j), \dots, d(x_{N-1}, x_j))^T$ for $0 \leqslant j \leqslant N-1$. Let us denote this embedding by $Fr : X \to \mathbb{R}^{|X|-1}$.

With these ingredients in hand, we claim:

Proposition 4.8. *Fix an integer $n \geqslant 3$ and a metric space (X, d) such that $3 \leqslant |X| \leqslant N = \binom{n}{2}$. If X has diameter at most $\log 2$, then the composite map*

$$\Psi_{|X|-1,N} \circ Fr : (X, d) \hookrightarrow (\mathbb{R}^{|X|-1}, \|\cdot\|_\infty) \hookrightarrow (\mathbb{R}^N / \sim, \operatorname{diam})$$

has image inside the metric space $\Psi([n]) \simeq \mathcal{S}([n])$. The converse holds if X is a three-element set.

Notice that Proposition 4.8 implies Theorem 4.5, by rescaling the metric on X by $(\log 2)/\operatorname{diam}(X)$. (The case of $|X| = 2$ is straightforward.) ☐

To complete the proof, it remains to show the preceding proposition.

Proof of Proposition 4.8. If $\operatorname{diam} X \leqslant \log 2$, then we claim for each $x \in X$ that any three coordinates of the Fréchet tuples $(d(x', x))_{x' \in X}$ satisfy (4.7). Indeed, if $x_1, x_2, x_3 \in X$ then

$$\exp d(x_3, x) \leqslant 2 \leqslant \exp d(x_1, x) + \exp d(x_2, x).$$

This shows $\Psi_{|X|,N} \circ Fr(X) \subset \Psi([n])$. Conversely, let $X = \{x, y, z\}$; then we are assuming that

$$\Psi_{|X|,N} \circ Fr(X) = \{(d(x,y), d(x,z), 0, \dots, 0)^T,\ (0, d(y,z), 0, \dots, 0)^T,\ (d(z,y), 0, 0, \dots, 0)^T\}$$

is contained in $\Psi([n])$. Hence the last of the three points in $\mathbb{R}^N$ satisfies (4.7), which in turn implies:

$$\exp d(z, y) \leqslant 1 + 1 = 2.$$

The same argument using the other two Fréchet embeddings of X shows that $\operatorname{diam} X \leqslant \log 2$. $\square$

5. Norms on arbitrary Banach spaces; concluding remarks and questions

5.1. **Parallel settings.** As shown in Section 2, diameter norms offer a unified and functorial framework, which subsumes and explains both Theorem 1.8 about norms on $\mathbb{F}^k$, as well as Theorem 4.1 about metrics on $[n]$. This treatment also applies more generally, and we begin this final section by stating (without proofs, and for completeness) two parallel results: in an arbitrary Banach space and in a class of discrete metrics on an arbitrary set.

Proposition 5.1. *Let $\mathbb{B}$ be an arbitrary Banach space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $X \subset \mathbb{B} \setminus \{\mathbf{0}\}$ be a subset such that for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{B}$, there exists $\alpha_{\mathbf{x}} \in \mathbb{F}^\times$ such that $\alpha_{\mathbf{x}}\mathbf{x} \in X$. Then the set of equivalence classes of norms on $\mathbb{B}$ (i.e., up to scaling by $(0,\infty)$) whose restriction to X is bounded away from $0, \infty$ can be isometrically realized as a complete, path-connected metric subspace of the Banach space $C_b(X, \mathbb{R})/\sim$ with the diameter norm.*

Notice that if moreover $\mathbb{B}$ is finite-dimensional and X is compact then this reduces to Theorem 1.8. Similarly, for any set X we have the following extension of Theorem 4.1:

Proposition 5.2. *For any nonempty set X of size at least 3, the equivalence classes (again by scaling) of metrics d on X bounded away from $0, \infty$ outside the diagonal – i.e., such that*

$$0 < \inf_{x \neq x' \in X} d(x, x') \leqslant \sup_{x \neq x' \in X} d(x, x') < \infty$$

form a complete, path-connected unbounded metric subspace of the Banach space $C_b(\binom{X}{2}, \mathbb{R})/\sim$. Here $\binom{X}{2}$ denotes the discrete set of pairs of elements in X.

5.2. **Further questions.** Following Theorems 1.8 and 4.1 studying the norms on $\mathbb{F}^k$ and the metrics on $[n]$ respectively, it may be interesting to further explore the spaces $\mathcal{S}_k(\mathbb{F})$ and $\mathcal{S}([n])$; exploring the former may provide additional insights into the Banach–Mazur compactum quotient space. Thus, we conclude with some observations and questions in both of the above settings.

(1) Are there more standard mathematical (geometric) models with which one can identify the metric spaces $\mathcal{S}_k(\mathbb{F})$ and $\mathcal{S}([n])$? What can one say about their geometric properties?

(2) What are the automorphism groups of these spaces? (Depending on the category under consideration, one may wish to study homeomorphisms, isometries, ...) For instance, by the final assertion in Lemma 1.1, $\mathcal{S}_k(\mathbb{F})$ is equipped with the group $PGL_{dk}(\mathbb{R})$ of isometries, under a real-linear identification of $\mathbb{F}^k$ with $\mathbb{R}^{dk}$.[1] An additional observation (by Terence Tao in recent discussions) is that $\mathcal{S}_k(\mathbb{F})$ also carries an isometric involution, which arises from considering dual norms. A parallel observation is that the space $\mathcal{S}([n])$ is equipped with an obvious symmetry group S_n of automorphisms. (In contrast, the Gromov–Hausdorff

[1] It is easy to verify here that for $A \in GL_{dk}(\mathbb{R})$, $d_{\mathcal{S}_k(\mathbb{F})}(N(A\cdot -), N'(A \cdot -)) = d_{\mathcal{S}_k(\mathbb{F})}(N(\cdot), N'(\cdot))$.

space has no isometries [9].) One may also consider local isometries of $\mathcal{S}([n])$ and $\mathcal{S}_k(\mathbb{F})$, as previously done for the Gromov–Hausdorff space in [8].

Remark 5.3. For completeness we point out that individual norms can indeed be unchanged under precomposing by elements of $PGL_{dk}(\mathbb{R})$. For instance, for the $\|\cdot\|_2$-norm one has the image of $O_{dk}(\mathbb{R})$, while for $p \in [1,\infty] \setminus \{2\}$, results of Banach [2] and Lamperti [12] show that 'generalized permutation matrices' are isometries of $\|\cdot\|_p$. These consist of the products of permutation matrices with diagonal orthogonal or unitary matrices for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ respectively. (When $\mathbb{F} = \mathbb{R}$, this is precisely the Weyl group of type B or C, i.e. the hyperoctahedral group $S_2 \wr S_n$ of signed permutations.)

At the same time, say for $\mathbb{F} = \mathbb{R}$ there is no nontrivial matrix $A \in GL_k(\mathbb{R}),\ A \notin \mathbb{R}^\times \cdot \mathrm{Id}$, whose precomposition fixes all of $\mathcal{S}_k(\mathbb{F})$. Indeed, using a $\|\cdot\|_p$-norm for $p \neq 2$, by the previous paragraph A must be a nonzero scalar multiple of some signed permutation matrix $A' \in S_2 \wr S_n$, say $A = cA'$. Suppose the nonzero entries of A' correspond to the (signed) permutation $\sigma \in S_n$. Now let $N(\mathbf{x}) := \sum_{j=1}^k j|x_j|$ for $\mathbf{x} \in \mathbb{R}^k$. If $\mathbf{e}_1, \dots, \mathbf{e}_k$ comprise the standard basis elements of $\mathbb{R}^k$, and $N(A\mathbf{x}) \equiv c'N(\mathbf{x})$ on $\mathbb{R}^k$ for some $c' > 0$, then

$$N(A\mathbf{e}_j) = c'N(\mathbf{e}_j)\ \forall j \quad \implies \quad c'j = |c|\sigma^{-1}(j),\ \forall j.$$

Multiplying these inequalities yields: $|c| = c'$. Now evaluating at $\mathbf{e}_j + \mathbf{e}_{\sigma^{-1}(j)}$ yields:

$$j + \sigma^{-1}(j) = \sigma^{-1}(j) + \sigma^{-2}(j),\ \forall j \in [k].$$

Hence σ has order at most 2. Using this and evaluating at $\mathbf{e}_j + 2\mathbf{e}_{\sigma^{-1}(j)}$ yields:

$$j + 2\sigma^{-1}(j) = \sigma^{-1}(j) + 2j,\ \forall j$$

and we conclude that $\sigma = \mathrm{Id}$. Finally, suppose two diagonal entries of A are unequal, say $a_{11} = c', a_{22} = -c'$. Define the norm $N(\mathbf{x}) := \|\mathbf{x}\|_1 + |x_1 + x_2|$, and evaluate it at $\mathbf{x} = (1,1,0,\dots,0)^T$:

$$0 = c'N(\mathbf{x}) - N(A\mathbf{x}) = 4c' - 2c'.$$

Since $c' > 0$, our supposition must therefore be false, concluding the proof. ☐

(3) How does the space $\mathcal{S}_k(\mathbb{F})$ relate to $\mathcal{S}_{k+1}(\mathbb{F})$? Observe by Proposition 3.2 that the p-norms isometrically map to the p-norms, provided one rescales the metric/norm on each $\mathcal{S}_k(\mathbb{F})$ by $\log(k)$. Alternately, without rescaling any of the norms on $\mathcal{S}_k(\mathbb{F})$, is it possible to compute the fibers of 'the' restriction map : $\mathcal{S}_{k+1}(\mathbb{F}) \to \mathcal{S}_k(\mathbb{F})$?

On a related note (say with $\mathbb{F} = \mathbb{R}$ for convenience), is this restriction map a surjection? I.e., is there a "Hahn–Banach" extension of every norm on $\mathbb{R}^k$ to one on $\mathbb{R}^{k+1}$, say minimally increasing/without increasing the (log-)distortion relative to some reference norm?

(4) Notice that the previous question has a variant for $\mathcal{S}([n])$ with a positive answer, by Proposition 4.6. Moreover, the fibers of the restriction of norms from $[n+1]$ to $[n]$ are solution sets to finite systems of inequalities. It may be interesting to study the structures of these solution spaces.

(5) To understand the 'sizes' and growth of balls in these spaces, one can also explore their metric entropy. Recall for a metric space X and a radius $r > 0$, the metric entropy of $E \subset X$ is the largest number of points in E that are r-separated. This is related to the internal and external covering numbers and the packing number of E; we refer the reader to [18] for a detailed introduction to these ideas.

(6) What is the smallest Banach space inside which these spaces (or distinguished subsets therein) can be isometrically embedded? Of course if we restrict to finite subsets X then the classic observation of Fréchet [5] shows that (X,d) isometrically embeds into $\mathbb{R}^{|X|-1}$ with the supnorm (and into $\mathbb{R}^{|X|-2}$ if $|X| \geqslant 4$) – see the discussion prior to Proposition 4.8.

If instead of the supnorm one is interested in Euclidean space embeddings – for subsets of $\mathcal{S}_k(\mathbb{F})$ or for $\mathcal{S}([n])$ – the classic paper of Schoenberg [17] (following related works in metric geometry by Menger, Fréchet, von Neumann, and others) provides the following result for finite metric spaces X:

Theorem 5.4 (Schoenberg [17], 1935)**.** *Fix integers $n, r \geqslant 1$, and a finite set $X = \{x_0, \dots, x_n\}$ together with a metric d on X. Then (X, d) isometrically embeds into $\mathbb{R}^r$ (with the Euclidean distance/norm) but not into $\mathbb{R}^{r-1}$ if and only if the $n \times n$ matrix*

$$A := (d(x_0, x_j)^2 + d(x_0, x_k)^2 - d(x_j, x_k)^2)_{j,k=1}^n \tag{5.5}$$

is positive semidefinite of rank r.

We also refer the reader to [17] for more general results for separable X, and [3, 14] for more recent, well-known variants with constraints on the 'embedding dimension' r.

(a) We end with some examples and comments in each of the two settings, starting with $\mathcal{S}([n])$. Note that Proposition 4.3 shows an isometric embedding into $\mathbb{R}^{\binom{n}{2}}$ with the 1-norm for a subset of $\mathcal{S}([n])$. It would be interesting to explore into what Banach space can the larger subset of norms explored in Remark 4.4 be isometrically embedded. Note that this is the restriction of the following metric on the union of the X, Y-axes:

$$d((x,0),(0,y)) := \begin{cases} \|(x,y)\|_1, & \text{if } xy \geqslant 0; \\ \|(x,y)\|_\infty, & \text{otherwise,} \end{cases}$$

and d restricted to the X or Y axis is the usual Euclidean distance. Can this metric space be (better) understood in terms of an isometrically embedding into a Banach space?
Another question is if Theorem 4.5 can be strengthened, to characterize the finite metric spaces on at most $\binom{n}{2}$ elements, which can be embedded isometrically – i.e., without scaling the metric – into $\mathcal{S}([n])$.

(b) Here are some examples of 'finite-dimensional embeddings' for infinite subsets of $\mathcal{S}_k(\mathbb{F})$. Recall from Proposition 3.8 that for each $p \in [1, \infty]$, the family of norms $\{N_{p,q,j} : q \in [0, \infty),\ j \in [k]\}$ isometrically embeds into $\mathbb{R}^k$ with the 1-norm. Next, by Proposition 3.2 the p-norms isometrically embed inside a one-dimensional real normed space (in fact, inside $[0, \log k]$). On the other hand for the p-norms, one can show that the image $\mathcal{S}'_k(\mathbb{R})$ of the p-norms in $C(\mathbb{R}^k \setminus \{\mathbf{0}\}, \mathbb{R})/\sim$ (akin to (3.9)) has affine hull of infinite – in fact uncountable – dimension. However, this is a consequence of the specific embedding and not an intrinsic property of $\mathcal{S}'_k(\mathbb{R})$. Thus, it is not clear what is the smallest (dimensional) Banach space containing an isometric copy of $\mathcal{S}_k(\mathbb{F})$.

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