THE LATTICE OF NIL-HECKE ALGEBRAS OVER REAL AND COMPLEX REFLECTION GROUPS

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Abstract. Associated to every complex reflection group, we construct a lattice of quotients of its braid monoid-algebra, which we term nil-Hecke algebras, and which are obtained by killing all braid words that are "sufficiently long", as well as some integer power of each generator. These include usual nil-Coxeter algebras, nil-Temperley–Lieb algebras, and their variants, and lead to symmetric semigroup module categories which necessarily cannot be monoidal.

Motivated by classical work of Coxeter (1957) and the Broué–Malle–Rouquier freeness conjecture [*J. reine Angew. Math.* 1998], and continuing beyond work of the second author [*Trans. Amer. Math. Soc.* 2018], we obtain a complete classification of the finite-dimensional nil-Hecke algebras for all complex reflection groups W. These comprise the usual nil-Coxeter algebras for W of finite type, their "fully commutative" analogues for W of FC-finite type, three exceptional algebras (of types F_4, H_3, H_4), and three exceptional series (of types B_n and A_n, two of them novel). In particular, we find the first – and only two – finite-dimensional nil-Hecke algebras over discrete complex reflection groups; this breaks from the nil-Coxeter case (where no braid words are further killed, and) where Marin [*J. Pure Appl. Alg.* 2014] and Khare [*Trans. Amer. Math. Soc.* 2018] showed that such algebras do not exist.

In addition to these algebras, and also algebraic connections (to PBW deformations and non-monoidal tensor categories), we further uncover combinatorial bases of algebras, both known (fully commutative elements) and novel ($\overline{1}2$-avoiding signed permutations). Our classification draws from and brings together results of Popov [*Comm. Math. Inst. Utrecht* 1982], Stembridge [*J. Alg. Combin.* 1996, 1998], Malle [*Transform. Groups* 1996], Postnikov via Gowravaram–Khovanova (2015), Hart [*J. Group Th.* 2017], and Khare [*Trans. Amer. Math. Soc.* 2018].

Contents

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1. Introduction and main results

Throughout this paper, $\mathbb{k}$ *will denote a fixed unital commutative ground ring, and by* $\dim M$ *we mean the* $\mathbb{k}$*-rank of a free* $\mathbb{k}$*-module* M*.*

In this paper, we study a class of graded rings over a given Coxeter group W (finite or infinite). We term such rings *nil-Hecke algebras*. Most of the rings that we study turn out to be the associated graded rings for certain quotients of the corresponding (generic) Iwahori–Hecke algebras – which include, for instance, Temperley–Lieb algebras. Thus, the quotients we study include the corresponding nil-Coxeter as well as nil-Temperley–Lieb algebras.

Our focus in the present work is on exploring the finite-dimensionality of these graded rings. In this, we are motivated by connections to Coxeter group combinatorics, semigroup tensor categories, and flatness of deformations (which in the complex case is related to the Broué–Malle–Rouquier (BMR) freeness conjecture). We elaborate on these connections below.

To introduce nil-Hecke algebras, recall that any Coxeter group W is accompanied by:

- an associated braid group $\mathcal{B}_W \twoheadrightarrow W$;
- a finite index set I as well as a set $\mathbf{S} := \{\mathbf{s}_i : i \in I\}$ that generates $\mathcal{B}_W$ and yields the simple reflections under the map $\mathcal{B}_W \twoheadrightarrow W$; and
- a finite index set J (in fact, $J \subseteq \binom{I}{2}$) as well as relations $\mathcal{R} := \{\mathbf{v}_j - \mathbf{w}_j : j \in J\}$, where for each j, both sides denote words in $\mathbf{S}$ of equal (and finite) length,

such that $\langle \mathbf{S} | \mathcal{R} \rangle$ is a presentation of $\mathcal{B}_W$. (This presentation was extended by Brieskorn [10].)

We now define our main object of interest: a lattice of algebras corresponding to W.

Definition 1.1. Fix a Coxeter group W, with associated data $I, J, \mathbf{S}, \mathcal{R}$ (so $J \subseteq \binom{I}{2}$).

(1) Given $J_0 \subseteq J$ and an integer tuple $\mathbf{d} = (d_i)_{i \in I}$ with $1 \leqslant d_i \leqslant \infty \ \forall i$, define the corresponding *nil-Hecke algebra* $n\mathcal{H}(W, \mathbf{d}, J_0)$ to be the $\mathbb{k}$-algebra generated by $\mathbf{S}$ with relations:
$$\mathbf{s}_i^{d_i} = 0 \ \forall i, \qquad \mathbf{v}_j = \mathbf{w}_j \ \forall j \in J_0, \qquad \mathbf{v}_j = 0 = \mathbf{w}_j \ \forall j \in J \setminus J_0.$$

(2) For $k \geqslant 1$, define $J_{<k} := \{j \in J : \ell(\mathbf{v}_j) = \ell(\mathbf{w}_j) < k\}$. Notice that $J_{<1} = J_{<2} = \emptyset$, so that $n\mathcal{H}(W, \mathbf{d}, J_{<1}) = n\mathcal{H}(W, \mathbf{d}, J_{<2})$. Also define $J_{<\infty} := J$.

Remark 1.2. Here we omit the relation $\mathbf{s}_i^{d_i} = 0$ if $d_i = \infty$. Also note that all the relations in $\mathcal{R}$ still hold in $n\mathcal{H}(W, \mathbf{d}, J_0)$, but the emphasis is on the additional relations $\mathbf{v}_j = 0 = \mathbf{w}_j$, $j \in J \setminus J_0$.

In other words, $n\mathcal{H}(W, \mathbf{d}, J_0)$ is a quotient of the Artin monoid-algebra of W by the order relations $\mathbf{s}_i^{d_i} = 0$ (see [45]). Two special cases of this construction are:

- ($k = \infty$.) The *generalized nil-Coxeter algebra* associated to the above data is $NC_W(\mathbf{d}) := n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$. These algebras were defined and studied in previous work [33, 34] (see also the extended abstract [35]). In the special case where $d_i = 2 \ \forall i$, $NC_W((2, \dots, 2))$ is the "usual" nil-Coxeter algebra associated to W, with $J = \binom{I}{2}$. These algebras are well-studied in connection to flag varieties [8, 40], categorification [36, 37], symmetric function theory [7], and Schubert polynomials [24, 41].
- ($k = 3$.) We define the *generalized nil-Temperley–Lieb algebra* associated to the above data to be $NTL_W(\mathbf{d}) := n\mathcal{H}(W, \mathbf{d}, J_{<3})$. In the special case where $d_i = 2 \ \forall i$, $NTL_W((2, \dots, 2))$ is the "usual" nil-Temperley–Lieb algebra associated to W. These latter are quotients of nil-Coxeter algebras as well as associated graded algebras of Temperley–Lieb algebras (see e.g. [6, 23]), and have connections to fermionic particle configurations and to quantum Schubert calculus [39, 48]. They were defined and

studied in [21, 22, 26]. In the special case of the Coxeter (multi)graph of W being simply laced, the algebras $NTL_W((2,\dots,2))$ were rediscovered by Postnikov as *XYX algebras* [25].

For a given group W with presentation as above, the set of nil-Hecke $\Bbbk$-algebras forms a lattice isomorphic to the product of the power set 2^J with I copies of $\mathbb{Z}^{\geqslant 1} \sqcup \{\infty\}$. Two algebras in this lattice are comparable if and only if one surjects onto the other; the extremal points in the lattice are the source $\Bbbk\mathcal{B}_W$ (for $J_0 = J$ and $d_i = \infty\ \forall i$) and the rank-1 algebra $\Bbbk$ (whenever $d_i = 1\ \forall i$). For each $\mathbf{d} \in (\mathbb{Z}^{\geqslant 1})^I$, the corresponding algebras form a sub-lattice with unique source $n\mathcal{H}(W, \mathbf{d}, \emptyset)$ and source the generalized nil-Coxeter algebra $NC_W(\mathbf{d})$.

We now come to the results in this paper, which classify all nil-Hecke $\Bbbk$-algebras $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ for $1 \leqslant k < \infty$ that are finite-dimensional (for $\Bbbk$ a field, or more precisely, those of finite $\Bbbk$-rank) of the form $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ with $1 \leqslant k \leqslant \infty$. In this, we are motivated by classical work of Coxeter [15], in which he considered quotients of the type A braid group by the relations $s_i^p = 1\ \forall i \geqslant 1$, and classified the pairs (n, p) for which the resulting "generalized Coxeter group" is finite (see also [2]). Our work studies the "Iwahori–Hecke" analogue of Coxeter's problem – and in the complex groups case, it can be thought of as related to the "Temperley–Lieb" analogue of the Broué–Malle–Rouquier freeness conjecture [11, 12].

At the same time, our work also complements (by proving results for all finite k) the work of the second author [34], who studied the $k = \infty$ case. Furthermore, we unify under one umbrella several classical and recent works, including ([34], as well as) results by Stembridge [55, 56, 57], Gowravaram–Khovanova (who attribute their construction to Postnikov) [25], and Hart [28], and show how they fit into this picture.

1.1. **Classification of finite-dimensional cases.** We begin with the recent work [34], which classified all finite-dimensional objects among the generalized nil-Coxeter algebras (i.e., corresponding to $k = \infty$). Interestingly, this yields a "non-usual" type A family of such algebras – and it turns out to be the only one:

Theorem 1.3 ([34])**.** *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, and integers $d_i \geqslant 2\ \forall i$. The corresponding nil-Hecke (or generalized nil-Coxeter) algebra $n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$ is a finitely generated $\Bbbk$-module if and only if exactly one of the following occurs:*

(1) W is a finite Coxeter group and $\mathbf{d} = (2, \dots, 2)$ (the "usual" nil-Coxeter algebras); or
(2) W is of type A, and $\mathbf{d} = (d, 2, \dots, 2)$ or $(2, \dots, 2, d)$ for some $d > 2$. We denote this nil-Hecke algebra by $NC_A(n, d) := n\mathcal{H}(S_{n+1}, (d, 2, \dots, 2), J_{<\infty})$.

Remark 1.4. Note that setting any $d_i = 1$ yields degeneracies in the original Definition 1.1, since the generator $\mathbf{s}_i$ is rendered superfluous. Moreover, if $d_i = \infty$ then $n\mathcal{H}(W, \mathbf{d}, J_0)$ contains the $\Bbbk$-span of $\mathbf{s_i}^n$ for $n \geqslant 0$. As our goal in this paper is to classify the finite-rank nil-Hecke algebras, throughout this paper we assume $2 \leqslant d_i < \infty\ \forall i \in I$.

Following Theorem 1.3 from [34], the present work obtains the corresponding classification results for $J_{<k}$ with $k < \infty$. Our first result is for $k = 1, 2$:

Theorem A ($k = 1, 2$)**.** *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, and integers $d_i \geqslant 2\ \forall i$. The corresponding nil-Hecke $\Bbbk$-algebra $n\mathcal{H}(W, \mathbf{d}, J_{<1}) = n\mathcal{H}(W, \mathbf{d}, J_{<2})$ is a finitely generated $\Bbbk$-module if and only if exactly one of the following occurs:*

(1) The Coxeter (multi)graph of W is a tree with exactly one multiple edge $4 \leqslant m_{i_0 j_0} < \infty$, and $d_i = 2\ \forall i$. In this case, if disconnecting i_0, j_0 yields trees T, T' with a, b nodes

respectively, then

$$\dim n\mathcal{H}(W,\mathbf{d},J_{<2}) = \begin{cases} \frac{1}{2}m_{i_0j_0}|I|^2+1-2ab, & \text{if } m_{i_0j_0} \text{ is even,}\\ \frac{1}{2}(m_{i_0j_0}-1)|I|^2+1, & \text{if } m_{i_0j_0} \text{ is odd.}\end{cases}$$

(2) *The Coxeter (multi)graph of W is a simply laced tree, and at most one node $i_0 \in I$ satisfies: $d_{i_0} \geqslant 3$. In this case,*

$$\dim n\mathcal{H}(W,\mathbf{d},J_{<2}) = 1+|I|^2(d_{i_0}-1).$$

Remark 1.5. Here and below, by the statement "$\dim n\mathcal{H}(W,\mathbf{d},J_0)=m$" for an integer $m>0$, we mean that $n\mathcal{H}(W,\mathbf{d},J_0)$ is a free $\Bbbk$-module of rank m.

Thus for $k=1,2$, there are infinitely many finite-dimensional nil-Hecke algebras corresponding to "non-Coxeter" matrices (i.e., where $d_{i_0}>2$). In contrast, our next result shows that for $k\geqslant 3$, this already gets reduced to precisely one non-Coxeter family, which for $k=\infty$ corresponds to $NC_A(n,d)$ in Theorem 1.3(2).

Theorem B ($k=3$)**.** *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, and integers $d_i \geqslant 2\ \forall i$. The corresponding nil-Hecke (or generalized nil-Temperley–Lieb) $\Bbbk$-algebra $n\mathcal{H}(W,\mathbf{d},J_{<3}) = NTL_W(\mathbf{d})$ is a finitely generated $\Bbbk$-module if and only if exactly one of the following occurs:*

(1) *W is a finite Coxeter group and $\mathbf{d}=(2,\dots,2)$ (these are "usual" nil-Temperley–Lieb algebras);*
(2) *W is a Coxeter group of type $E_n (n\geqslant 9)$, $F_n (n\geqslant 5)$, or $H_n (n\geqslant 5)$ (see Figure 1.1), and $\mathbf{d}=(2,\dots,2)$ (these are also "usual" nil-Temperley–Lieb algebras); or*
(3) *W is of type A, and $\mathbf{d}=(d,2,\dots,2)$ or $(2,\dots,2,d)$ for some $d>2$.*

In the first two cases, $\dim NTL_W(\mathbf{d})$ is precisely the number of fully commutative elements in the corresponding Coxeter group W, i.e., the number of words $w\in W$ for which any reduced word can be obtained from any other without using the "non-commutative" braid relations ($m_{ij}\geqslant 3$). In the last case of $W=W(A_n)$ and $d_1>2=d_2=\dots=d_n$, we have:

$$\dim NTL_W(\mathbf{d}) = (d-1)C_{n+1}-(d-2)C_n+(d-2)\sum_{j=1}^{n-1} jC_{n-j} \tag{1.6}$$

where C_n is the n^{th} Catalan number.

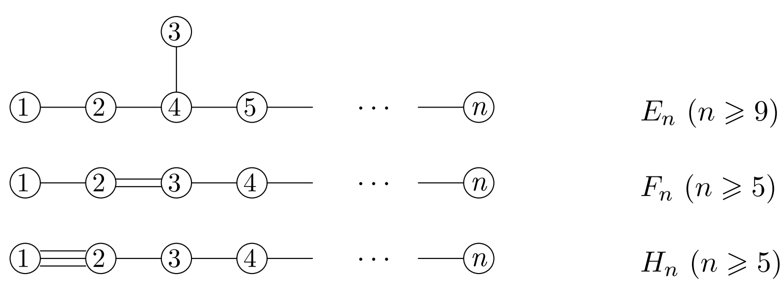

FIGURE 1.1. Dynkin graphs with $|W|$ infinite but $\dim(NTL_W)$ finite

Remark 1.7. Following Postnikov's calling $NTL_{W(A_n)}((2,\dots,2))$ an *XYX-algebra* (see [25]), we term $NTL_{W(A_n)}((d,2,\dots,2)) = n\mathcal{H}(W(A_n),(d,2,\dots,2),J_{<3})$ a *generalized XYX-algebra.*

Its $\Bbbk$-rank (1.6) involving Catalan numbers, generalizes the $d = 2$ fact that the type A XYX-algebra has rank C_{n+1} – a fact first observed in [4]. We also remark that the final sum in (1.6) is the partial sum of the partial sum of the Catalan numbers, and its first few terms can be found in [54].

In particular, Theorem B together with [34, Theorem B] implies Theorem 1.3 for all generalized Coxeter matrices. This because all Coxeter group algebras $\Bbbk W$ are flat deformations of the corresponding nil-Coxeter algebras $NC_W((2,\dots,2))$.

As the algebras $NTL_W(\mathbf{d})$ are not our primary focus in the present paper, we will not go beyond the proof of Theorem B; however, we remark that the proof of Theorem B is constructive, and reveals the analogue of fully commutative elements in the "non-usual" case of the generalized XYX-algebras $NTL_{A_n}((2,\dots,2,d))$, $d > 2$. Recall that such elements were introduced and studied by Stembridge in [55, 56], and enumerated in his sequel [57]. In [55], Stembridge also computed the Coxeter groups which contain only finitely many fully commutative elements (see also [20, 26]): these are the finite Coxeter groups as well as the families E_n, F_n, H_n as above. Theorem B now constructs and enumerates such elements in the "non-usual" generalized nil-Coxeter algebras $NC_A(n,d)$ (see Theorem 1.3(2)) introduced and studied in [34], adding to our knowledge of them.

Theorem B classified the finite $\Bbbk$-rank nil-Hecke algebras for $k = 3$. Next, we look at the case $k = 4$, followed by an immediate corollary for $k = 5$.

Theorem C ($k = 4$)**.** *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, and integers $d_i \geqslant 2$ $\forall i$. The corresponding nil-Hecke $\Bbbk$-algebra $n\mathcal{H}(W, \mathbf{d}, J_{<4})$ is a finitely generated $\Bbbk$-module if and only if exactly one of the following occurs:*

(1) W is a finite Coxeter group with $\mathbf{d} = (2,\dots,2)$; or
(2) W is of type A, and $\mathbf{d} = (d, 2,\dots,2)$ or $(2,\dots,2,d)$ for some $d > 2$.

In the first case, $\dim n\mathcal{H}(W, \mathbf{d}, J_{<4})$ *is given by the following table:*

W	$\dim n\mathcal{H}(W, \mathbf{d}, J_{<4})$
A_n, D_n, E_6, E_7, E_8	$\lvert W\rvert$
B_n	$\sum_{k=0}^{n} \binom{n}{k}^2 k!$
F_4	304
H_3	76
H_4	1460
$I_2(m)$	$\begin{cases} 2m & \text{if } m < 4 \\ 2m-1 & \text{if } m \geqslant 4 \end{cases}$

In the second case, we have

$$\dim n\mathcal{H}(W, \mathbf{d}, J_{<4}) = n!(1 + n(d-1)).$$

Corollary 1.8 ($k = 5$)**.** *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, and integers $d_i \geqslant 2$ $\forall i$. The corresponding nil-Hecke $\Bbbk$-algebra $n\mathcal{H}(W, \mathbf{d}, J_{<5})$ is a finitely generated $\Bbbk$-module if and only if exactly one of the following occurs:*

(1) W is a finite Coxeter group with $\mathbf{d} = (2,\dots,2)$; or
(2) W is of type A, and $\mathbf{d} = (d, 2,\dots,2)$ or $(2,\dots,2,d)$ for some $d > 2$.

In the first case, $\dim n\mathcal{H}(W, \mathbf{d}, J_{<5})$ *is given by the following table:*

W	$\dim n\mathcal{H}(W, \mathbf{d}, J_{<5})$
$A_n, B_n, D_n, E_6, E_7, E_8, F_4$	$\|W\|$
H_3	76
H_4	1460
$I_2(m)$	$\begin{cases} 2m & \textit{if } m < 5 \\ 2m-1 & \textit{if } m \geqslant 5 \end{cases}$

In the second case, we have

$$\dim n\mathcal{H}(W, \mathbf{d}, J_{<5}) = n!(1 + n(d-1)).$$

Remark 1.9. In particular, this work provides some refined combinatorial information that (to the best of our knowledge) was not written down for "usual" nil-Coxeter algebras – nor for their "non-usual" (aka generalized, i.e. with $\mathbf{d} \neq (2, \dots, 2)$) counterparts in type A.

- For "usual" nil-Coxeter algebras, we prove the above results by computing not just the dimension (or $\Bbbk$-rank), but by isolating – via Theorem 2.3 – a distinguished subset of words S'_k inside the "usual" monomial basis, which provides a $\Bbbk$-basis. For $k = 1, 2$, the set S'_k is simply the full basis (of size $|W|$), and for $k = 3$ it corresponds to the subset $W_{\rm fc}$ of fully commutative elements. However, for $k = 4, 5$, S'_k was not isolated earlier – even for types B_n, F_4, H_3, H_4 if we restrict to S'_k finite.

  There is also combinatorial information, which to the best of our knowledge was not previously connected to algebra. Namely, in type B_n in Theorem C, the monomials in S'_4 that are a basis of the algebra $n\mathcal{H}(W, \mathbf{d}, J_{<4})$ – i.e., which avoid braid relations that are "long enough" – turn out to be precisely the "$\overline{1}2$-avoiding signed permutations".
- Moving from combinatorics to algebra: in addition to our identifying a subset of words with some combinatorial properties – our work moreover isolates a multiplication structure on their span. This is akin to the set of fully commutative words inside W – while their closure under multiplication in W is unclear, this indeed happens when looking at their images inside the corresponding nil-Temperley–Lieb algebras. Similarly, the construction of "usual" nil-Hecke algebras in this work reveals a multiplication structure on the $\Bbbk$-span of the set S'_k for $k = 4, 5$, which is not revealed when considering S'_k as a subset of the Coxeter group W.
- In parallel, in the "non-usual" case, the work [34] that introduced the algebras $NC_A(n, d)$ (see Theorem 1.3(2)) could not detect the analogue of the fully commutative elements, which we now uncover (via Theorem 2.4) in the proof of Theorem B.

Returning to the classification of finite-rank nil-Hecke algebras, our final result is for $k \geqslant 6$.

Corollary 1.10 ($6 \leqslant k \leqslant \infty$)**.** *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, integers $d_i \geqslant 2$ $\forall i$ and $6 \leqslant k \leqslant \infty$. The corresponding nil-Hecke $\Bbbk$-algebra $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is a finitely generated $\Bbbk$-module if and only if W is either a finite Coxeter group, with $\mathbf{d} = (2, \dots, 2)$ or W is of type A with $\mathbf{d} = (d, 2, \dots, 2)$ or $(2, \dots, 2, d)$ for some $d > 2$. In the first case, the corresponding nil-Hecke algebra has $\Bbbk$-rank equal to $|W|$ unless it is of the type $I_2(m)$, in which case it is given by*

$$\dim n\mathcal{H}(W, \mathbf{d}, J_{<k}) = \begin{cases} 2m & \textit{if } m < k, \\ 2m-1 & \textit{if } m \geqslant k. \end{cases}$$

In the second case, the $\Bbbk$-rank is $n!(1 + n(d-1))$.

For ease of reference, we tabulate the above results for $3 \leqslant k \leqslant \infty$ in Table 1, and follow it up with a few comments.

W	$J_{<3}$	$J_{<4}$	$J_{<5}$	$J_{<k}$ $(6 \leqslant k \leqslant \infty)$
A_n, D_n, E_6, E_7, E_8	$\|W_{\rm fc}\|$	$\|W\|$	$\|W\|$	$\|W\|$
B_n	$\|W_{\rm fc}\|$	$\sum_{k=0}^{n} \binom{n}{k}^2 k!$	$\|W\|$	$\|W\|$
F_4	$\|W_{\rm fc}\|$	304	$\|W\|$	$\|W\|$
H_3	$\|W_{\rm fc}\|$	76	76	$\|W\|$
H_4	$\|W_{\rm fc}\|$	1460	1460	$\|W\|$
$I_2(m)$	$\|W\|$ if $m < 3$, else $\|W_{\rm fc}\| = \|W\| - 1$	$\|W\|$ if $m < 4$, else $\|W\| - 1$	$\|W\|$ if $m < 5$, else $\|W\| - 1$	$\|W\|$ if $m < k$, else $\|W\| - 1$
E_n $(n \geqslant 9)$, F_n $(n \geqslant 5)$, H_n $(n \geqslant 5)$	$\|W_{\rm fc}\|$			
A_n, $\mathbf{d} = (d, 2, \dots, 2)$, $d > 2$	(see (1.6))	$n!(1 + n(d-1))$	$n!(1 + n(d-1))$	$n!(1 + n(d-1))$

TABLE 1. Classification of Coxeter groups with finite-dimensional nil-Hecke algebras $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ for $3 \leqslant k \leqslant \infty$. Except the last row, $\mathbf{d} = (2, \dots, 2)$. $W_{\rm fc}$ denotes the fully commutative elements in W. Each cell entry is the dimension/rank over $\Bbbk$.

Remark 1.11. The fully commutative words in all Coxeter groups were studied and enumerated by Stembridge in [57]. In particular, for W dihedral (i.e., of type $I_2(m)$), $W_{\rm fc} = W \setminus \{w_\circ\}$ comprises all non-longest elements. Thus in Table 1, the $\Bbbk$-rank of $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ over $W = W(I_2(m))$ with $m \geqslant 3$ is $|W|$ if $m < k$, and $|W| - 1 = |W_{\rm fc}|$ otherwise.

This shows that the finite rank nil-Hecke algebras have ranks either (a) equal to $|W|$ or $|W_{\rm fc}|$, or (b) belonging to the three exceptional cases F_4, H_3, H_4 with $k = 4$, or (c) belonging to the three exceptional families: B_n with $k = 4$, or A_n with $k = 3, 4$ and $\mathbf{d} = (2, \dots, 2, d)$ for some $d \geqslant 2$.

Finally, note that as $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is a quotient of $n\mathcal{H}(W, \mathbf{d}, J_{<l})$ for $1 \leqslant k \leqslant l \leqslant \infty$, the list of finite rank examples for any value of k – i.e., in any non-initial column in Table 1 – is contained in the rows occupied in the previous column. (This is useful in the proofs below.) In particular, the result in [34] that there is only one family of finite rank generalized nil-Coxeter algebras with $\mathbf{d} \neq (2, \dots, 2)$, already follows from the $J_{<4}$ column in Table 1.

Our next result provides an equivalent condition to the finite-dimensionality of nil-Hecke algebras, motivated by considerations of deformation theory (see the end of Section 2):

Theorem D. *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, and integers $d_i \geqslant 2$ $\forall i$, and let k be an integer such that $1 \leqslant k \leqslant \infty$. Then the following are equivalent:*

1. *The corresponding nil-Hecke $\Bbbk$-algebra $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is a finitely generated $\Bbbk$-module.*
2. *The nil-Hecke algebra is found in the lists in Theorems A, B, C and Corollaries 1.8, 1.10 – or equivalently, in Theorem A and Table 1.*

(3) The two-sided augmentation ideal $\mathfrak{m} \subseteq n\mathcal{H}(W, \mathbf{d}, J_{<k})$*, which is generated by* $\{\mathbf{s}_i : i \in I\}$*, is nilpotent.*

If $\Bbbk$ *is moreover a field, then the finite-dimensional* $\Bbbk$*-algebra* $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ *is local, with unique maximal ideal* $\mathfrak{m}$*.*

Note that Theorem D generalizes the assertion [34, Theorem D(3)], which is simply the $k = \infty$ special case.

1.2. **Complex reflection groups; Frobenius algebras.** We next explore the analogous picture when working with nil-Hecke $\Bbbk$-algebras over complex reflection groups. Recall that such groups also have Coxeter-type presentations, listed e.g. in [12, 43, 46]. In fact this holds not only for the finite complex reflection groups (e.g. classified by Sheppard–Todd [51]), but also for the infinite discrete groups generated by affine unitary reflections – these were classified by Popov [46]. In the sequel, we term these (finite or infinite) groups as *discrete complex reflection groups.*

Given such a complex reflection group – which we continue to denote by W – one can define a lattice of nil-Hecke type algebras $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ over $\Bbbk$, and we define these following [9, 12] (and as in [34]):

Definition 1.12. Suppose W is a discrete (finite or infinite) complex reflection group, together with a finite generating set of complex reflections $\{s_i : i \in I\}$, the order relations $s_i^{m_{ii}} = 1 \ \forall i$, a set of braid relations $\{R_j : j \in J\}$ – each involving words with at least two distinct reflections s_i – and for the infinite non-Coxeter complex reflection groups W listed in [43, Tables I, II], one more order relation $R_0^{m_0} = 1$. Now define $I_0 := I \sqcup \{0\}$ for these infinite non-Coxeter complex reflection groups W, and $I_0 := I$ otherwise. Given an integer vector $\mathbf{d} \in \mathbb{N}^{I_0}$ with $d_i \geqslant 2 \ \forall i$, define the corresponding *nil-Hecke algebra* to be

$$n\mathcal{H}(W, \mathbf{d}, J_{<k}) := \frac{\Bbbk\langle \mathbf{s}_i, i \in I\rangle}{(\{R'_j = 0, j \in J_{<k}\}, \mathbf{s}_i^{d_i} = 0 \ \forall i \in I, \ (R'_0)^{d_0} = 0)}, \tag{1.13}$$

where the braid relations R_j are replaced by the corresponding relations R'_j in the alphabet $\{\mathbf{s}_i : i \in I\}$, and similarly for R'_0 if $R_0^{m_0} = 1$ in W; and where the two braid words on either side of the relations R'_j that are of equal length $\geq k$ (i.e., $j \notin J_{<k}$) are both set to equal zero. There is also the notion of the corresponding *braid diagram* as in [12, Tables 1–4] and [43, Tables I, II]; this is no longer always a Coxeter graph.

Remark 1.14. As Popov explains in [46, Section 1.6], one needs to work in the preceding definition with a specific presentation for W, since there is no canonical (minimal) set of generating reflections. Some related results are found in [3].

We now present our next main theorem. Parallel to the above results, it is natural to ask which of these algebras have finite $\Bbbk$-rank, even if (parallel to the BMR freeness conjecture) this never happens for $k = \infty$. For $k = \infty$, this was explored in previous work [34], where it was shown that no such group yields finite-rank Hecke algebras. This result subsumes an interesting observation of Marin [44], where he says that a key difference between real and complex reflection groups W is that there are no nil-Coxeter $|W|$- (or finite-)dimensional algebras over complex reflection groups.

Our next theorem explores the (larger) lattice of nil-Hecke algebras – and shows that Marin's observation, as well as the result in [34], hold more generally for all $k > 2$ – for *finite* complex reflection groups. However, in a break with these results, we provide the first – and only! – two examples of finite-dimensional nil-Hecke algebras, and they arise over infinite discrete complex reflection groups:

Theorem E. *Suppose* $3 \leq k \leq \infty$*, and* W *is any irreducible discrete complex reflection group, i.e.,* W *is a complex reflection group with connected braid diagram and presentation given as in* [12, Tables 1–4]*,* [43, Tables I, II]*, or* [46, Table 2]*. Also fix an integer vector* $\mathbf{d}$ *with* $d_i \geqslant 2\ \forall i$ *(including possibly for the additional order relation as in* [43]*).*

(1) If W *is finite or genuine crystallographic (this notation is explained in the proof), then* $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ *has infinite* $\Bbbk$*-rank.*
(2) If W *is noncrystallographic then either* $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ *has infinite* $\Bbbk$*-rank, or* $k = 3$ *and* W *is the complexification of an affine Coxeter group of type* E_9 *or* F_5*.*

In particular, the $k = 4$ case of this result implies the $5 \leq k < \infty$ cases – as well as the $k = \infty$ case, which was one of the main results in previous work [34].

Remark 1.15. We also discuss the remaining case in [46] of irreducible discrete complex reflection groups – wherein W is "non-genuine" crystallographic. As we explain using recent results in [32], the Coxeter-type presentation that we use (from [32]) involves relations whose left and right sides have unequal lengths, and so we do not consider these algebras any further.

Our final main result discusses which of the finite-dimensional $\Bbbk$-algebras are Frobenius. (For this result we assume that $\Bbbk$ is a field.) This line of investigation was first explored by Khovanov, who pointed out [36] that the "usual" nil-Coxeter algebra over any finite Coxeter group W is always Frobenius. To proceed further, we need recall additional notation.

Definition 1.16. Given a nil-Hecke algebra $n\mathcal{H}(W, \mathbf{d}, J_{<k})$, we say that an element $x \in n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is *left-primitive* if $\mathfrak{m}x = 0$, and *right-primitive* if $x\mathfrak{m} = 0$, where $\mathfrak{m}$ is the ideal defined in Theorem D. We call x *primitive* if it is both right-primitive and left-primitive.

Now our final main result characterizes the Frobenius algebras among all finite-dimensional nil-Hecke algebras: (i) in terms of their primitive elements; and (ii) for all $k \geq 1$. The latter means that our result again extends the classification in the $k = \infty$ case in [34]. In light of Theorem E, we only consider Coxeter groups.

Theorem F. *Fix a field* $\Bbbk$*, a Coxeter group* W *with related data* $I, J, \mathbf{S}, \mathcal{R}$*, integers* $d_i \geqslant 2\ \forall i$*, and* $1 \leqslant k \leqslant \infty$*, such that the corresponding nil-Hecke algebra* $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ *is finite-dimensional. Then the following are equivalent:*

(1) The algebra $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ *is Frobenius.*
(2) The space of right-primitive (equivalently, left-primitive) elements is one-dimensional.
(3) The spaces of right-primitive, left-primitive, and primitive elements are all the same and have dimension one.
(4) Exactly one of the following holds: (a) W *is a Coxeter group with no braid relation of length bigger than* k*, and* $d_i = 2$ *for all* i*; or (b)* W *is of type* A_1*, with* $\mathbf{d} = (d)$ *for some* $d \geqslant 3$*.*

Once again, the $k = 3$ case subsumes "one implication" of the $k = \infty$ case shown originally in [34]. Moreover, Theorem F shows that the nil-Coxeter algebras occupy a distinguished position among the nil-Temperley–Lieb and other nil-Hecke algebras (even when all $d_i = 2$): the others are not Frobenius.

Organization. This paper is organized as follows. We begin by providing several motivations and connections to prior works in the following section, as well as by producing a monomial word $\Bbbk$-basis for $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ for general W, $\mathbf{d} \in (\mathbb{Z}^{\geqslant 2})^I$, and $1 \leqslant k \leqslant \infty$. The remaining sections are devoted to proving the main theorems above. In an Appendix, we provide SAGE code that helped verify some of our results computationally.

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2. Flat deformations, other connections, and a word basis

In this section we explain how the problem under consideration, and more broadly, nil-Hecke algebras, are connected to several other areas. We begin with a categorical connection: even before considering finite-dimensionality, the entire lattice of nil-Hecke algebras $\{n\mathcal{H}(W, \mathbf{d}, J_0) : \mathbf{d} \in (\mathbb{Z}^{\geqslant 2})^I,\ J_0 \subseteq J\}$ yields examples of co-commutative algebras with coproduct, which are not bialgebras.

Proposition 2.1. *Let $A := n\mathcal{H}(W, \mathbf{d}, J_0)$ for some $\mathbf{d} \in (\mathbb{Z}^{\geqslant 2})^I$ and $J_0 \subseteq J$. If $\Bbbk$ is a field of characteristic zero, then A is not a bialgebra.*

Proof. We provide a sketch, as the proof is similar to that of [34, Proposition 3.1]. Since each $d_i < \infty$, the only possible counit $\varepsilon : A \to \Bbbk$ sends each $\mathbf{s}_i$ to 0. Let $\mathfrak{m}$ denote the two-sided augmentation ideal in A, generated by $\{\mathbf{s}_i : i \in I\}$; then one shows that any coproduct $\Delta : A \to A \otimes A$ must satisfy:

$$\Delta(\mathbf{s}_i) \in 1 \otimes \mathbf{s}_i + \mathbf{s}_i \otimes 1 + \mathfrak{m} \otimes \mathfrak{m}.$$

But now,

$$0 = \Delta(\mathbf{s}_i)^{d_i} = \sum_{k=1}^{d_i - 1} \binom{d_i}{k} \mathbf{s}_i^k \otimes \mathbf{s}_i^{d_i - k} + \text{ higher degree terms},$$

and this is impossible because $n\mathcal{H}(W, \mathbf{d}, J_0)$ surjects onto the quotient of the usual nil-Coxeter algebra $NC_W((2, \dots, 2))$ by some braid words (each of length at least 2), so the $\mathbf{s}_i$ have nonzero images in this quotient. □

A consequence of this result is that the module category $\operatorname{Rep} A$ (with $A = n\mathcal{H}(W, \mathbf{d}, J_0)$) cannot be a tensor category. But since $\Delta : A \to A \otimes A$ is co-commutative, $\operatorname{Rep} A$ is a symmetric semigroup category. We refer the reader to [34, Theorem A] and [19, Proposition 14.2 and Theorem 18.3] for more on the Tannakian formalism behind such categories. Additionally, we mention for completeness how to pass to an "honest" tensor category from A. Define a central extension of the graded algebra A by an element $\mathbf{s}_\infty$:

$$0 \to \Bbbk \mathbf{s}_\infty \to \widetilde{A} \to A \to 0,$$

with the non-graded $\Bbbk$-algebra $\widetilde{A}$ generated by $\mathbf{s}'_i, i \in I$ and $\mathbf{s}_\infty$ via

$$(\mathbf{s}'_i)^{d_i} = \mathbf{s}'_i \mathbf{s}_\infty = \mathbf{s}_\infty \mathbf{s}'_i = \mathbf{s}_\infty^2 = \mathbf{s}_\infty,\ \forall i \in I,$$

and the analogues for the $\mathbf{s}'_i$ of the braid word relations:

$$\mathbf{v}'_j - \mathbf{w}'_j = 0,\ \forall j \in J_0, \qquad \mathbf{v}'_l = \mathbf{w}'_l = \mathbf{s}_\infty, \forall l \in J \setminus J_0.$$

Then $\widetilde{A}$ is indeed a bialgebra, under:

$$\Delta(\mathbf{s}'_i) := \mathbf{s}'_i \otimes \mathbf{s}'_i, \quad \Delta(\mathbf{s}_\infty) := \mathbf{s}_\infty \otimes \mathbf{s}_\infty, \quad \varepsilon(\mathbf{s}_i) = \varepsilon(\mathbf{s}'_\infty) = 1, \qquad \forall i \in I,$$

and hence $\operatorname{Rep} \widetilde{A}$ is a monoidal category. We refer the reader to [34, Section 3] for more details in the special case $J_0 = J$. Notice also that the algebras $\widetilde{A} = \widetilde{n\mathcal{H}}(W, \mathbf{d}, J_0)$ form a lattice of bialgebras that is isomorphic to the product of 2^J with I copies of $\mathbb{Z}^{\geqslant 2}$.

A second theme, extensively explored for decades in the Coxeter/Hecke/Lie theory literature, involves classifying finite-dimensional objects in various settings. Such classifications have been of enormous interest in recent and earlier times – including complex simple Lie algebras; real and complex reflection groups [14, 51] and their nil-Coxeter and more general Hecke algebras; finite type quivers, the McKay–Slodowy correspondence, and Kleinian singularities. This evergreen theme has seen recent additions, including for finite-dimensional Nichols algebras [27, 30, 29]; and finite-dimensional pointed Hopf algebras [1], which are intimately connected to small quantum groups. There are also other combinatorial-type results, including by Stembridge and Hart, that are discussed presently; and the second author's recent work classifying the finite-dimensional generalized nil-Coxeter algebras [34]. The present work is a sequel to this last reference [34], connecting it with the works by Stembridge and Hart, and also going beyond [34] via refined combinatorial phenomena for $k = 4, 5$ even in the "usual" (i.e., $\mathbf{d} = (2, \dots, 2)$) Coxeter types B_n, F_4, H_3, H_4.

We next mention a connection with the question of *flatness of deformations*. In this paper, our goal is to explore the flatness question for e.g. the Temperley–Lieb algebras and their nil-Hecke versions. For instance, in their theses Fan and Graham studied Temperley–Lieb algebras $TL_W((2, \dots, 2))$ for Coxeter groups W (see [21, 22, 26]), say with $\mathbb{k} = \mathbb{Z}[u, u^{-1}]$. These are quotients of the Iwahori–Hecke algebras $\mathcal{H}_u(W)$; the authors showed in *loc. cit.* that $TL_W((2, \dots, 2))$ has a $\mathbb{k}$-basis in bijection with the fully commutative words in W. Now Theorem B shows that $TL_W((2, \dots, 2))$ is indeed a flat deformation of $NTL_W((2, \dots, 2))$, meaning that the following diagram is (i) valid, and (ii) a commuting square:

$$\begin{array}{ccc} \mathcal{H}_u(W) & \xrightarrow{\text{gr}} & NC_W((2,\dots,2)) \\ \big\downarrow & & \big\downarrow \\ TL_W((2,\dots,2)) & \xrightarrow{\text{gr}} & NTL_W((2,\dots,2)) \end{array}$$

More strongly, we will show in Theorem 2.3 that the above flatness phenomenon holds for nil-Hecke algebras $n\mathcal{H}(W, (2, \dots, 2), J_{<k})$ for *all* k. We then extend this to arbitrary $\mathbf{d}$ in Theorem 2.4.

Our next connection is to the combinatorics of Coxeter groups W. Stembridge [55] studied the fully commutative elements W_{fc}(see Theorem B for the definition) in connection with the Bruhat and weak orderings. In [38] the authors study the 2-sided Kazhdan–Lusztig cell formed by the words $W_1 \subseteq W$ that have a unique reduced expression. These words have been counted recently in [28]. In fact all words in the group W have been studied for the above reasons.

We now introduce a more general notion of word-sets $W(J_0) \subseteq W$ for any Coxeter group, of which the above sets W_{fc}, W_1, W are special cases for $J_0 = J_{<3}, J_{<2}, J_{<\infty}$ respectively.

Definition 2.2. Given a Coxeter matrix M with corresponding group W, define for $J_0 \subseteq \binom{I}{2}$ the subset of elements $W(J_0) \subseteq W$, to consist of all $w \in W$ such that no reduced expression for w has a substring of positive length that occurs in the set $\{\mathbf{v}_j, \mathbf{w}_j : j \in J \setminus J_0\}$.

With this notation in hand, we can state the promised result on flatness of deformations of Temperley–Lieb algebras – and more generally, relate every subset $W(J_0)$ to the corresponding nil-Hecke algebra $n\mathcal{H}(W,(2,\dots,2),J_0)$.

Theorem 2.3. *Suppose M is a Coxeter matrix, and $J_0 \subseteq \binom{I}{2}$ as above. Then the nil-Hecke algebra $n\mathcal{H}(W,(2,\dots,2),J_0)$ is a free $\Bbbk$-module with basis $\{\overline{\mathbf{s}_w} : w \in W(J_0)\}$, where $\{\mathbf{s}_w : w \in W\}$ is the "canonical" basis of the nil-Coxeter algebra $NC_W((2,\dots,2))$.*

In particular, by [21, 22, 26], Temperley–Lieb algebras are flat deformations of their nil-Temperley–Lieb analogues.

Theorem 2.3 thus provides a broader setting into which the families of words $W(J_0)$ fit: bases of nil-Hecke algebras corresponding to Coxeter matrices. Even more generally, we will consider (bases of) nil-Hecke algebras for generalized Coxeter matrices, i.e. where $\mathbf{s}_i^{d_i} = 0$ for $d_i \geqslant 2$. Beyond proving Theorem 2.3, in the remaining sections we will characterize when such bases are finite in size.

Proof of Theorem 2.3. Begin by observing that the subset M in $NC_W((2,\dots,2))$ consisting of the words

$$M := \{\mathbf{s}_w : w \in W\} \sqcup \{0\}$$

forms a monoid under multiplication; and moreover, $M \setminus \{0\}$ forms a $\Bbbk$-basis of $NC_W((2,\dots,2))$ (see e.g. [31]). Now let $I(J_0)$ be the $\Bbbk$-span in $NC_W((2,\dots,2))$ of the words

$$\{\alpha \mathbf{v}_j \beta,\ \alpha \mathbf{w}_j \beta \ :\ j \in J \setminus J_0,\ \alpha, \beta \in M\}.$$

Then $I(J_0)$ clearly contains the elements $\mathbf{s}_w,\ w \notin W(J_0)$; moreover, it is a two-sided ideal of $NC_W((2,\dots,2))$. Now suppose a linear combination $\sum_{w \in W} c_w \mathbf{s}_w \in I(J_0)$. By definition/$\Bbbk$-freeness, if $c_w \neq 0$ then $w \notin W(J_0)$, and $I(J_0)$ intersects the $\Bbbk$-span of $\{\mathbf{s}_w : w \in W(J_0)\}$ trivially. This concludes the proof. ☐

The above theorem provides a natural basis for nil-Hecke algebras with $\mathbf{d} = (2,2,\dots,2)$. We now strengthen this result to one which applies to all of our nil-Hecke algebras, and which is key to proving the finite-dimensionality in all of our main results.

Theorem 2.4. *Let W be a Coxeter group generated by $\{s_i : i \in I\}$, with corresponding nil-Hecke algebra $n\mathcal{H}(W,\mathbf{d},J_{<k})$, where $\mathbf{d} \in (\mathbb{Z}^{\geqslant 2})^I$ and $1 \leqslant k \leqslant \infty$. Consider equivalence classes of strings in the s_i, where two strings are equivalent if one can be reached from another by applying the braid relations finitely many times. Let S denote the set of classes such that no string in the class contains any of the substrings $s_i^{d_i}$ or the braid words of length k or more. Then the monomials corresponding to some choice of class representatives from S form a free $\Bbbk$-module basis of $n\mathcal{H}(W,\mathbf{d},J_{<k})$.*

Proof. Since any monomial not coming from S can be reduced to zero, these monomials clearly span $n\mathcal{H}(W,\mathbf{d},J_{<k})$. It remains to prove that $\dim n\mathcal{H}(W,\mathbf{d},J_{<k}) \geqslant |S|$, and we would be done.

Consider a free $\Bbbk$-module $\mathcal{M}$ with basis elements indexed by S. We now define a natural $n\mathcal{H}(W,\mathbf{d},J_{<k})$-module structure on $\mathcal{M}$ as follows: the element $\mathbf{s}_j$ acts on the class $[s_{i_1}\cdots s_{i_n}]$ by sending it to the class $[s_j s_{i_1}\cdots s_{i_n}]$ if it is in S, and to zero otherwise. This extends to an action of the free associative algebra $\Bbbk\langle \mathbf{s}_i \mid i \in I\rangle$.

We now verify this action indeed satisfies the defining relations in $n\mathcal{H}(W,\mathbf{d},J_{<k})$. Indeed, if applying the $\mathbf{s}_i$-action d_i times on a basis element yields a non-zero basis element, then this class contains a string with d_i consecutive leading s_i's and thus cannot be in S, a contradiction. Similarly, multiplying by braid words of length k or more gives the zero vector. Finally, the

braid relations of lengths less than k hold: indeed, multiplying a basis element by either side of the braid equations either yields elements in the same equivalence class by definition, or yields elements not in S, giving the zero vector on either side.

Since $\mathcal{M}$ is generated as a $n\mathcal{H}(W, \mathbf{d}, J_{<k})$-module by the basis vector corresponding to the empty string, this gives a surjection $n\mathcal{H}(W, \mathbf{d}, J_{<k}) \twoheadrightarrow \mathcal{M}$. This, combined with the first paragraph, completes our proof. □

Following Theorem 2.4 on a word basis of $n\mathcal{H}(W, \mathbf{d}, J_{<k})$, we describe here a final connection of this work, to the study of PBW bases and deformations – more broadly than the commuting square in this section. The study of PBW (Poincaré–Birkhoff–Witt) type deformations has seen intense activity for several decades now, including foundational works on Drinfeld Hecke algebras [16], graded affine Hecke algebras [42], symplectic reflection algebras [18] (including rational Cherednik algebras), as well as on infinitesimal Hecke algebras, quantum analogues, and their generalizations. The recent program of Shepler and Witherspoon (and their coauthors) has led to a profusion of activity; we mention e.g. [52, 53, 58] here. In all of these works, a bialgebra A (which is most often a Hopf algebra) acts on a symmetric algebra S_V of some vector space V, and one is interested in understanding which deformations of the smash-product algebra $A \ltimes S_V$ are "PBW", or flat.

When A is a nil-Hecke algebra, there are several significant features to note. First, these are not bialgebras by Proposition 2.1 (but possess a coproduct Δ); and yet, a variant of the "PBW theorem" nevertheless holds for the algebras $A \ltimes S_V$, which subsumes the PBW theorems in various previous works. (See the previous paper [33] by the second author for the details.) Thus the nil-Hecke algebras $n\mathcal{H}(W, \mathbf{d}, J_0)$ widen beyond [34] the class of examples of such "algebras with coproduct" (A, Δ), which do not possess an antipode or even a counit, yet fit into the framework of the aforementioned works.

Second, there also are technical consequences of finite-dimensionality. We mention two of these; in both, we will assume $\Bbbk$ to be a field. It was shown in [53], [33] that if (A, Δ) is finite-dimensional, one can characterize those graded $\Bbbk[t]$-deformations of $A \ltimes S_V$, for which the fiber at $t = 1$ satisfies the PBW property. This provides a PBW-theoretic motivation to classify the finite-dimensional nil-Hecke algebras.

Even more is true. It was shown in [33] that if (A, Δ) is any algebra with coproduct (e.g. $A = n\mathcal{H}(W, \mathbf{d}, J_{<k})$), and if it is local with *nilpotent* augmentation ideal $\mathfrak{m}$ such that $\Delta(\mathfrak{m}) \subseteq \mathfrak{m} \otimes \mathfrak{m}$, then one obtains much additional information regarding the deformations of $A \ltimes S_V$ – their abelianization, center, and (simple) modules. See e.g. [33, Section 6.1]. Thus, we have a second motivation from the perspective of PBW deformations – to understand which nil-Hecke algebras $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ have nilpotent augmentation ideal $\mathfrak{m}$. This is precisely the point of Theorem D; and it shows that this question turns out to be *equivalent* to the main results of the paper: $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ must be finite-dimensional.

3. Proof of Theorem A: finite rank nil-Hecke algebras with $k = 1, 2$

The remainder of the paper is devoted to proving the above classification theorems on finite-dimensional nil-Hecke algebras. To do so, we will employ the diagrammatic calculus utilized in [34]. We begin by showing Theorem A in this section; the proof is in steps.

Step 1: By Theorem 2.3, $n\mathcal{H}(W, \mathbf{d}, J_{<2})$ has finite $\Bbbk$-rank if and only if $W(J_{<2})$ is also finite. By using the results of [28], this proves the result if $\mathbf{d} = (2, \dots, 2)$, i.e., when dealing with quotients of usual nil-Coxeter algebras.

Step 2: In the remainder of the proof, at least one d_i is 3 or more. Since $n\mathcal{H}(W, \mathbf{d}, J_0) \twoheadrightarrow n\mathcal{H}(W, (2, \dots, 2), J_0)$, by the previous step the Coxeter (multi)graph of W must be a tree with no $m_{ij} = \infty$ and at most one $m_{ij} \geqslant 4$.

Suppose first that we have a simply laced tree, and $d_\alpha, d_\gamma \geqslant 3$ for $\alpha \neq \gamma \in I$. As the graph of W is connected, suppose

$$\alpha \quad \longleftrightarrow \quad \beta_1 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \beta_{m-1} \quad \longleftrightarrow \quad \gamma$$

is a path in I (in the figures below, we write $m' = m - 1$). Now define a free $\Bbbk$-module $\mathscr{M}$ with basis vectors

$$\{A_r, B_{1r}, \dots, B_{mr}, C_r, B'_{1r}, \dots, B'_{mr} : r \geqslant 1\}.$$

Give $\mathscr{M}$ an $n\mathcal{H}(W, \mathbf{d}, J_{<2})$-module structure by letting every $\mathbf{s}_i$ kill all basis vectors, except for the actions described by the first diagram in Figure 3.1. Explicitly, the basis vectors on which the action is nonzero are:

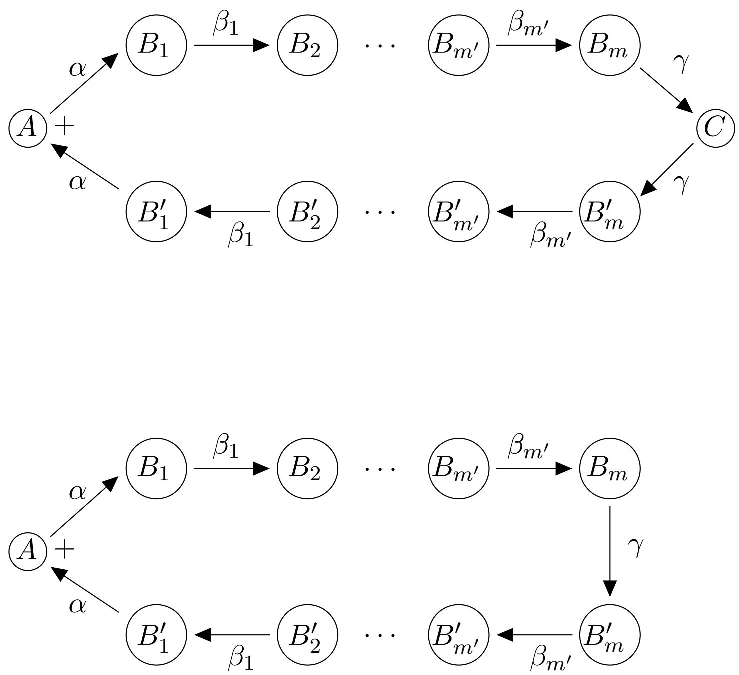


FIGURE 3.1. The modules $\mathscr{M}$ for the nil-Hecke algebras in Steps 2 and 3

$$\mathbf{s}_\alpha(A_r) := B_{1r}, \quad \mathbf{s}_{\beta_j}(B_{jr}) := B_{j+1,r} \ (1 \leqslant j \leqslant m-1), \quad \mathbf{s}_\gamma(B_{mr}) := C_r,$$
$$\mathbf{s}_\gamma(C_r) := B'_{mr}, \quad \mathbf{s}_{\beta_j}(B_{jr}) := B_{j-1,r} \ (2 \leqslant j \leqslant m), \quad \mathbf{s}_\alpha(B'_{1r}) := A_{r+1}.$$

Thus the "+" at the head of an arrow refers to the index r increasing by 1. One verifies that the defining relations for $n\mathcal{H}(W, \mathbf{d}, J_{<2})$ hold on each basis vector, hence in $\operatorname{End}_\Bbbk(\mathscr{M})$. Thus $\mathscr{M}$ is a cyclic $n\mathcal{H}(W, \mathbf{d}, J_{<2})$-module generated by A_{11}. Now $\mathscr{M}$, hence $n\mathcal{H}(W, \mathbf{d}, J_{<2})$, is not a finitely generated $\Bbbk$-module.

Step 3: Suppose next that some $m_{ij} > 3$ and some $d_{i_0} > 2$. Without loss of generality there exist nodes nodes $\beta_1, \dots, \beta_{m-1}$ for some $m \geqslant 1$ (in the figure we write $m' := m - 1$), such that

$$\alpha = i_0 \quad\longleftrightarrow\quad \beta_1 \quad\longleftrightarrow\quad \cdots \quad\longleftrightarrow\quad \beta_{m-1} = i \quad\longleftrightarrow\quad \gamma = j$$

Now construct a free $\Bbbk$-module $\mathscr{M}$ as in Step 2, this time giving it an $n\mathcal{H}(W, \mathbf{d}, J_{<2})$-module structure using the second diagram in Figure 3.1. One verifies that the action on $\mathscr{M}$ satisfies the defining relations. Hence the nil-Hecke algebra is once again of infinite $\Bbbk$-rank.

Step 4: It remains to show the assertion when the Coxeter (multi)graph of W is a simply laced tree with node set I, and $d_{i_0} \geqslant 3$ for a unique $i_0 \in I$. Denote the corresponding nil-Hecke algebra by $n\mathcal{H}(W, i_0, d_{i_0}, J_{<2})$. For nodes $s, t \in I$, let $[s,t]$ denote the path from source $s \in I$ to target $t \in I$. If $s = i_1, \dots, i_n = t$ enumerates sequentially the nodes in $[s,t]$ for some $n \geqslant 1$, define:

$$\mathbf{s}_{[s,t]} := \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_n}. \tag{3.1}$$

We begin by enumerating a spanning set for $n\mathcal{H}(W, i_0, d_{i_0}, J_{<2})$ of the correct size. Notice that the product $\mathbf{s}_i \mathbf{s}_j$ (with $i, j \in I$) vanishes unless $i = j = i_0$ or $i \neq j$ are adjacent in I. It follows that the set

$$\{1\} \sqcup \{\mathbf{s}_{[s,t]} : s,t \in I\} \sqcup \{\mathbf{s}_{[s,i_0]} \mathbf{s}_{i_0}^{k-2} \mathbf{s}_{[i_0,t]} : s,t \in I,\ 2 \leqslant k \leqslant d_{i_0} - 1\}$$

spans $n\mathcal{H}(W, i_0, d_{i_0}, J_{<2})$. Now by the proof of Theorem 2.4, this is indeed a basis, and we are done. □

4. Proof of Theorem B: finite rank nil-Hecke algebras with $k = 3$

Proof of Theorem B. We begin by showing that the only "adjacency graphs", corresponding to Coxeter groups W that admit finite $\Bbbk$-rank nil-Hecke algebras for $NTL_W(\mathbf{d}) = n\mathcal{H}(W, \mathbf{d}, J_{<3})$ (these algebras are defined following Definition 1.1), are Dynkin diagrams of finite type or type $E_n, n \geqslant 9$. Here, first, by the *adjacency multigraph* of a Coxeter group associated to a Coxeter matrix $(m_{ij})_{i,j=1}^n$ with $n = |I|$, we mean the multigraph on n nodes with $m_{ij} - 2$ edges between nodes $i \neq j \in \{1, \dots, n\}$. Next, the corresponding *adjacency graph* is the simple graph on n nodes with $\mathbf{1}(m_{ij} > 2)$ edges between nodes $i \neq j$.

If $\mathbf{d} = (2, \dots, 2)$, then W is a Coxeter group, and by Theorem 2.3, the problem reduces to finding all W with finitely many fully commutative elements. This was worked out in [20, 26, 55]. In what follows, we therefore assume there exists at least one i such that $d_i \geqslant 3$. Let this i correspond to vertex α.

Step 1. Suppose the adjacency multigraph of W contains a cycle on nodes $\beta_1, \dots, \beta_m$ with $m \geqslant 3$, i.e., $m_{\beta_i, \beta_{i+1}} > 2$ for $0 < i < m$ and $m_{\beta_m, \beta_1} > 2$. Now construct a free $\Bbbk$-module $\mathscr{M}$ with basis given by the countable set $\{A_{1r}, \dots, A_{mr} : r \geqslant 1\}$, and define the following $NTL_W(\mathbf{d})$-action on it: $\mathbf{s}_{\beta_i}$ kills A_{jr} except that $\mathbf{s}_{\beta_i}(A_{ir}) := A_{i+1,r}$ for $0 < i < r$ and $\mathbf{s}_{\beta_m}(A_{mr}) := A_{1,r+1}$. It is easy to verify that the defining relations of $NTL_W(\mathbf{d})$ hold in $\operatorname{End}_{\Bbbk}(\mathscr{M})$, as they hold on each A_{ir}. Therefore $\mathscr{M}$ is a cyclic $NTL_W(\mathbf{d})$-module generated by A_{11}, which is not finitely generated as a $\Bbbk$-module. Hence neither is $NTL_W(\mathbf{d})$, as claimed.

The analysis in this step can be conveniently expressed by a diagram of a cycle. We will do so for other cases in the remainder of the proof.

Step 2. The remaining adjacency multigraphs are those whose underlying simple graphs are trees. We next claim there is no vertex adjacent to 4 nodes, and at most one vertex adjacent to 3 nodes. Indeed, if β_1 is adjacent to nodes $\alpha_1, \alpha_2, \gamma_1, \gamma_2$, then we appeal to Figure 4.1 with $m = 2$, to generate a $NTL_W(\mathbf{d})$-module $\mathscr{M}$ with basis $\{A_r, B_r, B_{1r}, B_{2r}, B'_{1r}, B'_{2r}, C_r, D_r :$

$r \geqslant 1\}$. The module relations are read off of the diagram. Namely, $\mathbf{s}_i$ kills all basis vectors for all $i \in I$, with the following exceptions:

$$\mathbf{s}_{\beta_1}(B_{1r}) = B_{2r}, \quad \mathbf{s}_{\gamma_1}(B_{2r}) = D_r, \quad \mathbf{s}_{\gamma_2}(B_{2r}) = C_r, \quad \mathbf{s}_{\gamma_1}(C_r) = B'_{2r}, \quad \mathbf{s}_{\gamma_2}(D_r) = B'_{2r},$$
$$\mathbf{s}_{\beta_1}(B'_{2r}) = B'_{1r}, \quad \mathbf{s}_{\alpha_2}(B'_{1r}) = B_r, \quad \mathbf{s}_{\alpha_1}(B_r) = B_{1r}, \quad \mathbf{s}_{\alpha_2}(A_r) = B_{1r}, \quad \mathbf{s}_{\alpha_1}(B'_{1r}) = A_{r+1}.$$

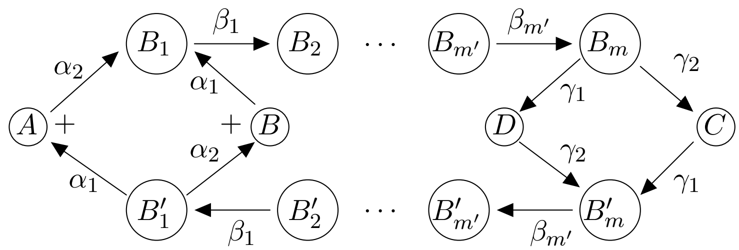


FIGURE 4.1. Diagrammatic calculus for the algebras $NTL(M)$; here $m' = m - 1$.

(The "+" at the head of an arrow again refers to the index increasing by 1.) It is easy to verify that the defining relations of $NTL_W(\mathbf{d})$ hold at each node in the diagram (i.e., at the corresponding basis vector) and hence in $\mathrm{End}_{\Bbbk}(\mathscr{M})$. This yields a cyclic $NTL_W(\mathbf{d})$-module $\mathscr{M}$ of infinite $\Bbbk$-rank, as desired.

Next if $\deg \alpha = \deg \gamma = 3$, let

$$\alpha = \beta_1 \quad \longleftrightarrow \quad \beta_2 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \beta_{m-2} \quad \longleftrightarrow \quad \gamma = \beta_{m-1}$$

be the path between them, for suitable m. Also suppose α is adjacent to $\beta_2, \alpha_1, \alpha_2$, and γ is adjacent to $\beta_{m-2}, \gamma_1, \gamma_2$. We again appeal to Figure 4.1 to generate a representation $\mathscr{M}$ which has infinite $\Bbbk$-rank.

Remark 4.1. A brief remark for the reader, about checking that the relations hold in Figure 4.1 and the other figures in this paper: (a) On each commuting sub-diagram, one only needs to check the braid relations in $J_{<k}$, and account for the '+' signs. (b) On each directed cycle in the diagrams, one needs to check that no braid word in $J \setminus J_{<k}$ occurs, nor any word $\mathbf{s}_i^{d_i}$.

Step 3. Thus the Dynkin multigraph is in fact acyclic, with at most one vertex of degree 3 and no vertex of degree $\geqslant 4$. First, we use the first diagram in Figure 3.1 to show there exists a unique node i such that $d_i \geqslant 3$. Second, use Figure 4.2 with $m = 1$ to show this node has degree 1 (i.e., is pendant).

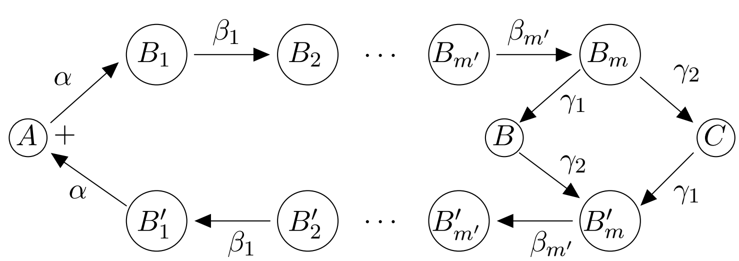


FIGURE 4.2. Diagrammatic calculus for Step 3; here $m' = m - 1$.

Third, use the second diagram in Figure 3.1 to show that the Dynkin multigraph of W is in fact simply laced. Hence from above, it is a tree graph.

Next, suppose there is a vertex β of degree 3. Let

$$\alpha = \beta_1 \quad \longleftrightarrow \quad \beta_2 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \beta_{m-2} \quad \longleftrightarrow \quad \beta = \beta_{m-1}$$

be a path from α to β; further, let γ_1 and γ_2 be two more vertices adjacent to β. Then the module pictured in Figure 4.2 again has infinite $\Bbbk$-rank, a contradiction. This shows that the adjacency multigraph is necessarily a path graph. As we have seen, $\mathbf{d} = (d, 2, \dots, 2)$ for some $d \geqslant 3$, and so we obtain a quotient of the algebra $NC_A(n, d)$ as desired.

Step 4. To finish the proof, we need to determine the $\Bbbk$-rank of $NTL_W(\mathbf{d})$, where W is of type A_n and $\mathbf{d} = (d, 2, \dots, 2)$. By Theorem 2.4, it suffices to enumerate all the monomials in $x_1, \dots, x_n$ that are not equivalent (via commutation relations $x_i x_j = x_j x_i$ for $|i - j| > 1$) to any monomial containing the strings x_1^d, x_i^2 for $i > 1$, or containing any of the length 3 braid words. We extend the argument in [25] to accomplish this.

Consider a nonzero monomial in $NTL_W(\mathbf{d})$, and in its equivalence class under the commutation relations ($x_i x_j = x_j x_i$ for $|i - j| > 1$), look at the lexicographically smallest monomial $\mathbf{w}$ as in [25]. We claim that all the x_1's occur in a contiguous string in $\mathbf{w}$. Indeed, suppose not. Then by the arguments in [25], the monomial $\mathbf{w}$ can be assumed to be composed of decreasing runs, where in each run the indices go down by one, except possibly where x_1 is followed by x_1. Since x_1's can only occur at the ends of such runs, if there are two x_1's that are not consecutive, a closest such pair must be part of a substring that looks like $x_1 x_j x_{j-1} \cdots x_2 x_1$. But this can be reduced to $x_j x_{j-1} \cdots x_1 x_2 x_1$ and thus zero, proving our claim.

One can repeat the arguments in [25, Section 3] and verify that Lemmas 1, 3, and 4 there still hold, with the following exceptions:

1. There might be x_1 followed by x_1 in the monomial $\mathbf{w}$, contrary to Lemma 1; or
2. The indices of the peaks might not be strictly increasing, precisely when the first run starts with x_i for $i \geq 2$, ends with x_1^j for $j \geq 2$, and is followed by x_2, contrary to Lemma 2.

Let us set aside the monomials that come under exception (2) for the moment. Then the monomials we need to count are precisely those counted by [25], with the added caveat that there might be a contiguous block of up to $d - 1$ x_1's. Now if the monomial contains no x_1, then this is simply a monomial in $NTL_W((2, \dots, 2))$ on the generators $x_2, \dots, x_n$; by the results of [25], there are C_n such monomials. If the monomial does contain a nonempty block of x_1's, then deleting all but one x_1 gives a monomial containing x_1 in $NTL_W((2, \dots, 2))$; conversely, given a monomial in $NTL_W((2, \dots, 2))$ containing x_1, one can form $d-1$ different valid monomials from there by adding x_1's. The number of $NTL_W((2, \dots, 2))$-monomials containing an x_1 is simply $C_{n+1} - C_n$ (since there are C_n monomials not having x_1), so the total number is

$$C_n + (d - 1)(C_{n+1} - C_n) = (d - 1)C_{n+1} - (d - 2)C_n.$$

Finally, we count the monomials that exhibit exception (2). Such a monomial must start with $x_i x_{i-1} \cdots x_2 x_1^j x_2$, and then either the next term is x_3, or the remaining string is a nonzero monomial on the generators $x_{i+1}, \dots, x_n$. Indeed, the peak of the next run cannot be x_k for $4 \leq k \leq i$, else we can move it to the left by the commutation relations and the entire monomial reduces to zero. The next run cannot even contain such a x_k, else it would have to contain x_i and thus the initial segment of this run till x_i can be commuted leftward to reduce the monomial to zero, proving our claim. Continuing this argument, we see that the x_1^j in our monomial is followed by a string of the form $x_2 x_3 \cdots x_\ell$, with $2 \leq \ell \leq i$, and a nonzero (possibly empty) monomial in the generators $x_{i+1}, \dots, x_n$. Thus the number of such monomials is $(i-1)C_{n-i+1}$. Summing over all possible i and j, the total number becomes

$$(d-2)\sum_{i=2}^{n}(i-1)C_{n-i+1} = (d-2)\sum_{j=1}^{n-1} jC_{n-j}.$$

As in [25], we can verify these indeed correspond to non-zero monomials, so adding this to the previous count completes the proof. □

5. Proof of Theorem C and its corollaries: $k \geq 4$

We now consider the remaining cases $4 \leqslant k \leqslant \infty$, starting with $k = 4$.

Proof of Theorem C. Since $n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$ surjects onto $n\mathcal{H}(W, \mathbf{d}, J_{<4})$, it follows that $n\mathcal{H}(W, \mathbf{d}, J_{<4})$ has finite $\Bbbk$-rank if $n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$ does; these cases are listed in Theorem 1.3. On the other hand, $n\mathcal{H}(W, \mathbf{d}, J_{<4}) \twoheadrightarrow n\mathcal{H}(W, \mathbf{d}, J_{<3})$, so $n\mathcal{H}(W, \mathbf{d}, J_{<4})$ has infinite $\Bbbk$-rank whenever $n\mathcal{H}(W, \mathbf{d}, J_{<3})$ does. The cases where this is not so are listed in Theorem B. Comparing these two lists, we see that only the cases where W is of type $E_n (n \geqslant 9)$, $F_n (n \geqslant 5)$ or $H_n (n \geqslant 5)$ with $\mathbf{d} = (2, \dots, 2)$ remain to be checked for finiteness.

If W has type E_n, there are no braid relations of length 4 or more, and so $n\mathcal{H}(W, \mathbf{d}, J_{<4}) \cong n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$, which is known by [34] to have infinite $\Bbbk$-rank for $n \geqslant 9$. Next, assume W is of type F_n with $n \geqslant 5$. We will prove that $n\mathcal{H}(W, \mathbf{d}, J_{<4})$ has infinite $\Bbbk$-rank for $n = 5$, from which the $n > 5$ cases would follow trivially.

Let W be of type F_5, and label the generators as in Figure 5.1.

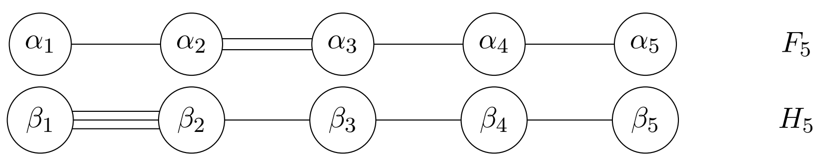


Figure 5.1. Dynkin graphs for F_5 and H_5

Then the infinite rank module defined by the diagram in Figure 5.2 proves our claim.

For the remaining case, it suffices to show a similar module for H_5. This is demonstrated in Figure 5.3.

Next, we compute the ranks of the finite rank nil-Hecke algebras. First suppose W is of type A and $\mathbf{d} = (d, 2, \dots, 2)$ or $(2, \dots, 2, d)$ for some $d > 2$. As all braid relations have length 3, $n\mathcal{H}(W, \mathbf{d}, J_{<4}) = n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$, and so we are done by Theorem 1.3.

Otherwise, we fix $\mathbf{d} = (2, \dots, 2)$ henceforth. Now if W is simply laced (of type ADE) or of type $I_2(m)$ with $m < 4$, then there are no braid relations of length 4 or more, so $n\mathcal{H}(W, \mathbf{d}, J_{<k}) \cong n\mathcal{H}(W, \mathbf{d}, J_{<\infty}) = NC_W((2, \dots, 2))$. This has $\Bbbk$-rank precisely $|W|$. If instead W is of type $I_2(m)$, $m \geqslant k = 4$, then both sides of the braid relation (i.e., the longest word in $NC_W((2, 2))$) get killed in $n\mathcal{H}(W, \mathbf{d}, J_{<4})$, while the remaining $2m - 1$ words are not killed in the quotient. The result now follows by Theorem 2.3.

We tackle the case of B_n next. Let the generators of the corresponding Weyl group $W(B_n)$ be $s_0, s_1, \dots, s_{n-1}$, where the labeling is chosen such that $m_{01} = 4$, and $m_{i,i+1} = 3$ for all $i \in \{1, \dots, n-2\}$. By Theorem 2.3, we need to count the set $B_{n,4}$ of elements $w \in W(B_n)$, for which no reduced expression contains the substring $s_0 s_1 s_0 s_1$ or $s_1 s_0 s_1 s_0$. For this, we use the following well-known combinatorial description of $W(B_n)$ (see, for example, [5]):

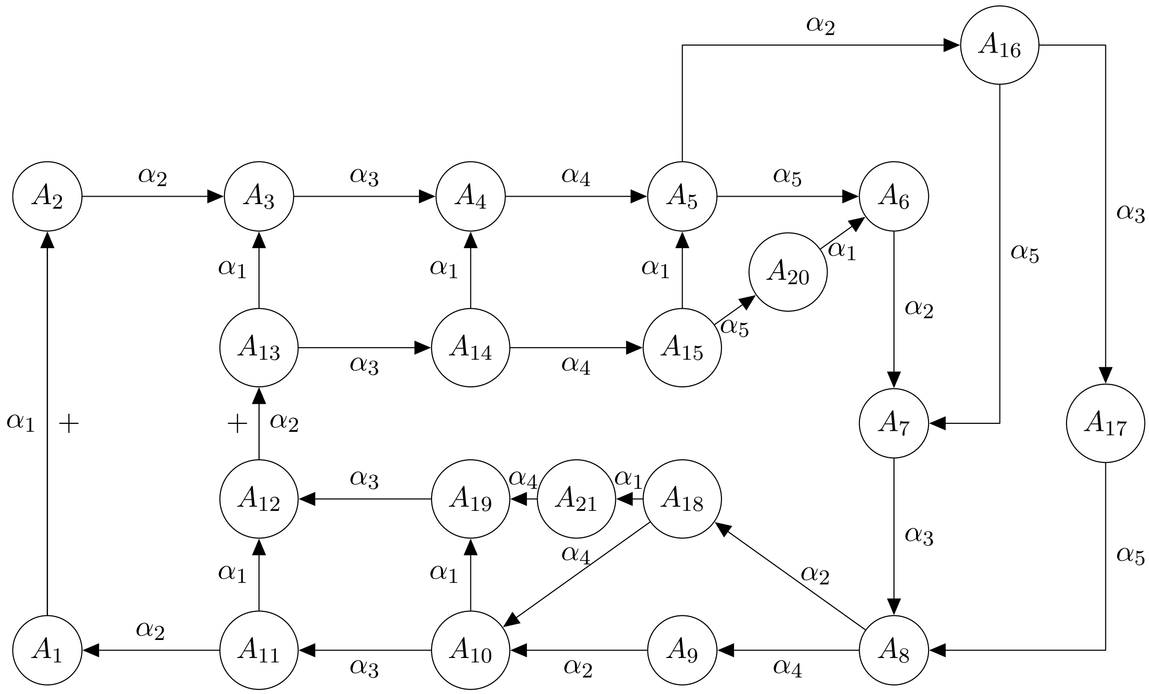


FIGURE 5.2. An infinite rank module for F_5

Theorem 5.1. *Let W be the group of all permutations w on $S = \{\pm 1, \dots, \pm n\}$ such that $w(-a) = -w(a)$ $\forall a \in S$. Then $W = W(B_n)$ is the Coxeter group of type B_n, with generators*

$$s_i = [1, \dots, i-1, i+1, i, i+2, \dots, n],\ 1 \leqslant i \leqslant n-1; \qquad s_0 = [-1, 2, \dots, n],$$

where we write $w = [a_1, a_2, \dots, a_n] \in W(B_n)$ if $w(i) = a_i$ $\forall 1 \leqslant i \leqslant n$.

Further, given any $w \in W$, the length function $\ell(\cdot)$ satisfies the following properties:

(1) $\ell(ws_0) = \ell(w) + 1$ if and only if $w(1) < 0$; and
(2) $\ell(ws_i) = \ell(w) + 1$ if and only if $w(i) < w(i+1)$ for $i \in \{1, \dots, n-1\}$.

Using this, we now prove the following lemma:

Lemma 5.2. *Let $w = [a_1, \dots, a_n] \in W(B_n)$ be as above. Then $w \in B_{n,4}$ if and only the following holds: no pair of indices $i < j$ satisfies $a_i > a_j$ and $a_i, a_j < 0$.*

Proof. Suppose $w \in W(B_n)$ does not satisfy the condition: thus, there exist $i < j$ such that $a_j < a_i < 0$. Call such a pair of indices a *bad* pair. One can pick i, j with minimal $|i - j|$. Perform the following operations on w: if the leftmost negative number in w occurs in position $i > 1$, multiply by the transposition s_{i-1}. If it occurs in position 1, multiply by s_0. Each move decreases the length by 1, and after sufficiently many applications, this brings a_i to position 1. By another sequence of length-decreasing transpositions, we can bring a_j to 2. Now one verifies that multiplying by $s_0 s_1 s_0 s_1$ further reduces the length by 4. Thus $ww' = w''$, where w' contains the string $s_0 s_1 s_0 s_1$, and $\ell(w) - \ell(w') = \ell(w'')$. This implies $w'^{-1} w''$ is a reduced expression for w with the substring $s_1 s_0 s_1 s_0$, and thus $w \notin B_{n,4}$.

For the other direction, suppose w satisfies the given condition. We will prove $w \in B_{n,4}$ by induction on $\ell(w)$. The base cases of $\ell(w) \leqslant 1$ are clear. For the induction step, suppose to the contrary that w has a reduced expression with $s_0 s_1 s_0 s_1$ in it, and let s_i be the rightmost generator in that expression. Then $w = w' s_i$, with $\ell(w') = \ell(w) - 1$. There are two cases:

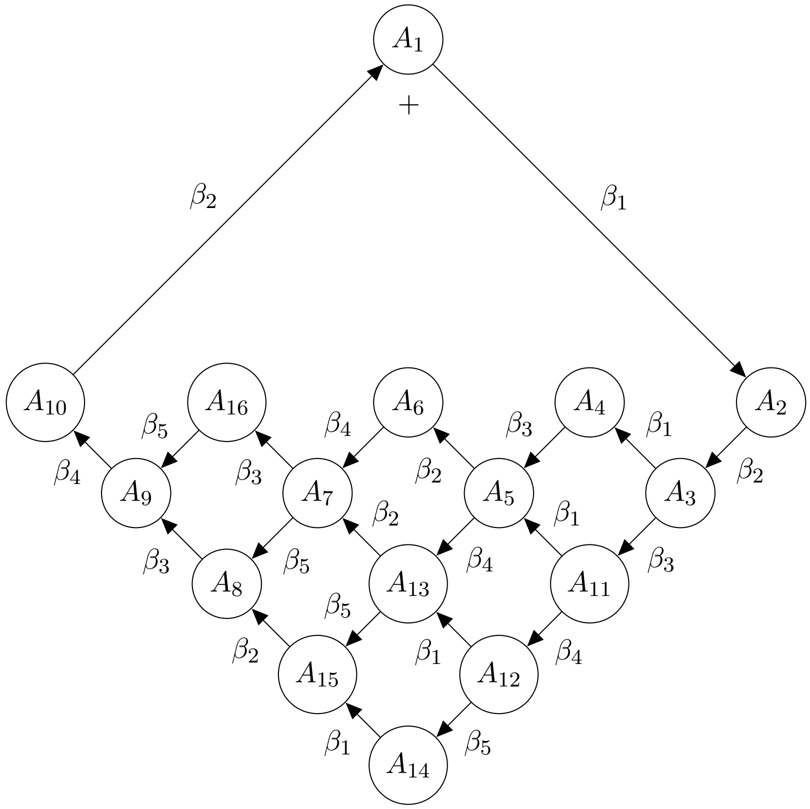


FIGURE 5.3. An infinite rank module for H_5

Case 1: First suppose $s_0 s_1 s_0 s_1$ does not occur at the rightmost position in this expression for w, so it has to occur in the expression for w'. Now ws_i has smaller length than s, so if $i \geqslant 1$, $w(i) > w(i{+}1)$. Multiplying by s_i swaps these, so if w does not have a bad pair, swapping $w(i)$ with $w(i+1)$ cannot create further bad pairs. Thus by the induction hypothesis, $ws_i = w'$ does not have a reduced expression containing $s_0 s_1 s_0 s_1$, a contradiction. Similarly, if $i = 0$, $w(1)$ must be negative, and swapping its sign cannot create more bad pairs. By a similar reasoning as above, we are done in this case.
Case 2: Otherwise, $w = w'' s_0 s_1 s_0 s_1$, with $\ell(w'') = \ell(w) - 4$. As this is a reduced expression, $\ell(ws_1) < \ell(w)$, implying $w(1) > w(2)$. Now $w_1 = ws_1 = [w(2), w(1), \ldots]$, so $\ell(w_1 s_0) < \ell(w_1)$ implies $w_1(1) = w(2) < 0$. Next, $w_2 = w_1 s_0 = [-w(2), w(1), \ldots]$, so $w_3 = w_2 s_1 = [w(1), -w(2), \ldots]$, and $w_4 = w_3 s_0 = [-w(1), -w(2), \ldots]$. Since $\ell(w_3 s_0) < \ell(s_3)$, we have $w_3(1) = w(1) < 0$. Thus $(1, 2)$ is a bad pair in w, a contradiction. This finishes the proof. □

Returning to the main proof, we want to enumerate the elements in $W(B_n)$ without a bad pair. This is counted in [50] as the $\overline{1}\overline{2}$-avoiding signed permutations, and the answer is precisely $\sum_{k=0}^{n} \binom{n}{k}^2 k!$. This shows the result for type B_n.

The only remaining cases are F_4, H_3 and H_4: all of finite $\Bbbk$-rank. By computer checking and use of Theorem 2.3, one verifies the numbers stated in Theorem C. The proof is now complete. □

We conclude by proving the two corollaries to Theorem C.

Proof of Corollary 1.8. Since $n\mathcal{H}(W, \mathbf{d}, J_{<5}) \twoheadrightarrow n\mathcal{H}(W, \mathbf{d}, J_{<4})$, it suffices to check the cases in Theorem B. The dihedral case is easy to verify, and the case of type A with $\mathbf{d} = (d, 2, \dots, 2)$ or $(2, \dots, 2, d)$ is immediate since there are no braid relations of length 4 or more in this case. As the same fact holds for types A, B, D, E, F, hence for these types (with $\mathbf{d} = (2, \dots, 2)$) it follows that $n\mathcal{H}(W, \mathbf{d}, J_{<5}) = n\mathcal{H}(W, \mathbf{d}, J_{<\infty}) = NC_W((2, \dots, 2))$ is the usual nil-Coxeter algebra, and hence has $\Bbbk$-rank $|W|$.

Finally, suppose W is of type H_3 or H_4, and $\mathbf{d} = (2, \dots, 2)$. As all braid words have length either 3 or 5, it follows that $J_{<3} = J_{<4}$ in both cases, and so the algebras and hence their $\Bbbk$-ranks remain unchanged when passing from $k = 4$ to $k = 5$. ☐

Proof of Corollary 1.10. Since $n\mathcal{H}(W, \mathbf{d}, J_{<k}) \twoheadrightarrow n\mathcal{H}(W, \mathbf{d}, J_{<5})$, to identify the finite $\Bbbk$-rank nil-Hecke algebras, we need only consider the cases in Corollary 1.8, i.e., for $k = 5$. Once again the case of type A with $\mathbf{d} = (d, 2, \dots, 2)$ or $(2, \dots, 2, d)$ is immediate, and the dihedral case of $I_2(m)$ is again easily checked. Since none of the remaining cases have braid relations of length 6 or more, in all non-dihedral cases with $\mathbf{d} = (2, \dots, 2)$, we have $n\mathcal{H}(W, \mathbf{d}, J_{<k}) = n\mathcal{H}(W, \mathbf{d}, J_{<\infty}) = NC_W((2, \dots, 2))$ as above. This completes the proof. ☐

6. Proof of Theorem D: nilpotence of the augmentation ideal

We next show that the finite-dimensionality of a nil-Hecke algebra is equivalent to the nilpotence of the augmentation ideal $\mathfrak{m}$.

Proof of Theorem D. The key equivalence to be shown here is (1) $\iff$ (3); that (1) $\iff$ (2) was shown in earlier sections. First assume $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ has infinite $\Bbbk$-rank. Recall from above that every such algebra was shown to have infinite rank by demonstrating an explicit cyclic $n\mathcal{H}(W, \mathbf{d}, J_{<k})$-module of infinite $\Bbbk$-rank; moreover, every diagram above that described such a module has at least one directed cycle that is "accompanied" by a "+" symbol. Let the generators along the edges of any such fixed cycle be $\alpha_1, \alpha_2, \dots, \alpha_m$, and suppose a basis element v in that module is associated to the initial node of the edge corresponding to α_1. Take an arbitrary positive integer N. Then the element $\mathbf{s} = (\mathbf{s}_{\alpha_m} \cdots \mathbf{s}_{\alpha_2}\mathbf{s}_{\alpha_1})^j$ belongs to $\mathfrak{m}^N$ for large enough $j \gg 0$, and $\mathbf{s}v$ is non-zero by how the module action is defined. This implies $\mathbf{s} \in \mathfrak{m}^N$ is non-zero; and since N was arbitrary, $\mathfrak{m}$ is not nilpotent.

For the other direction, assume $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ has finite $\Bbbk$-rank. Applying Theorem 2.4, one obtains a finite set S_{fin} of monomials corresponding to equivalence classes in S (as defined in Theorem 2.4) that gives a $\Bbbk$-basis of the algebra in question. Thus, S_{fin} has an element of maximal length, say ℓ. We claim that $\mathfrak{m}^{\ell+1} = 0$, which would prove the desired nilpotency.

Indeed, consider a monomial $\mathbf{s}'$ in $\mathfrak{m}^{\ell+1}$: this must be a product of strictly more than ℓ generators. If $\mathbf{s}'$ equals some monomial composed of ℓ generators or less, one can reduce this via using the defining relations of $n\mathcal{H}(W, \mathbf{d}, J_{<k})$. Since applying the braid relations does not change the number of monomials, this can be reduced to some monomial containing the string $\mathbf{s}_i^{d_i}$ or a braid word of length $\geqslant k$ at some point, and hence it is zero in $n\mathcal{H}(W, \mathbf{d}, J_{<k})$.

On the other hand, if $\mathbf{s}'$ is not equal in $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ to any monomial composed of ℓ generators or less, the corresponding string of $\mathbf{s}_i$'s cannot belong to any equivalence class in S, hence reduces to 0 (as in the proof of Theorem 2.4).

This completes the proof of the equivalence. Finally, suppose these conditions hold and $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is finite-dimensional, where $\Bbbk$ is now a field. It is clear that if $x \in \mathfrak{m}$ and $\mathfrak{m}^{\ell+1} = 0$, then $(1 - x)^{-1} = 1 + x + \cdots + x^\ell$, and so $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is local. ☐

Remark 6.1. In light of Theorem E (proved in the next section), one can ask if Theorem D holds for the finite-rank nil-Hecke algebras over complex reflection groups. This is indeed the case, because – as we explain in the next section – there are precisely two such algebras, and they essentially arise from real affine reflection groups, so Theorem D applies to them too.

7. Proof of Theorem E: nil-Hecke algebras over complex reflection groups

We now turn to the assertion – extending Marin's assertion [44] for nil-Coxeter algebras and its extension in [34] to generalized nil-Coxeter algebras – that finite complex reflection groups admit no finite-dimensional nil-Hecke algebras. This is "half" of Theorem E, and we then produce (exactly) two finite-dimensional examples over infinite complex groups. After the proof, we turn to the remaining case of "non-genuine" crystallographic groups W.

Proof of Theorem E. We begin by discussing the finite (irreducible) complex reflection groups, for which a presentation can be found in [12]; note that $k \geq 3$ here. Now a straightforward verification using this presentation reveals that all corresponding nil-Hecke algebras of the form $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ have infinite $\Bbbk$-rank. Indeed, the arguments in [34, Section 6, Cases 10–12] (which prove the same statement in the special case $k = \infty$) work for $3 \leq k < \infty$ as well. The only point where our proof diverges from [34] is when $W = G_{29}$.

In this exceptional case, we note that the corresponding ("usual") nil-Coxeter algebra is generated by the generators $\mathbf{s}_s, \mathbf{s}_t, \mathbf{s}_u, \mathbf{s}_v$ subject to the following relations:

$$\mathbf{s}_s^2 = \mathbf{s}_t^2 = \mathbf{s}_u^2 = \mathbf{s}_v^2 = 0, \quad \mathbf{s}_s\mathbf{s}_v = \mathbf{s}_v\mathbf{s}_s, \quad \mathbf{s}_s\mathbf{s}_u = \mathbf{s}_u\mathbf{s}_s,$$
$$\mathbf{s}_s\mathbf{s}_t\mathbf{s}_s = \mathbf{s}_t\mathbf{s}_s\mathbf{s}_t, \quad \mathbf{s}_v\mathbf{s}_t\mathbf{s}_v = \mathbf{s}_t\mathbf{s}_v\mathbf{s}_t, \quad \mathbf{s}_u\mathbf{s}_v\mathbf{s}_u = \mathbf{s}_v\mathbf{s}_u\mathbf{s}_v,$$
$$\mathbf{s}_t\mathbf{s}_u\mathbf{s}_t\mathbf{s}_u = \mathbf{s}_u\mathbf{s}_t\mathbf{s}_u\mathbf{s}_t, \quad \mathbf{s}_v\mathbf{s}_t\mathbf{s}_u\mathbf{s}_v\mathbf{s}_t\mathbf{s}_u = \mathbf{s}_t\mathbf{s}_u\mathbf{s}_v\mathbf{s}_t\mathbf{s}_u\mathbf{s}_v.$$

The corresponding nil-Hecke algebras $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ are obtaining as a quotient of this by (first replacing the Coxeter relations/exponents 2 by d_i, and then) killing suitable braid words depending on k. One notes that for all of these, the diagram in Figure 7.1 defines a suitable infinite-rank $\Bbbk$-module. This completes the proof for finite complex reflection groups.

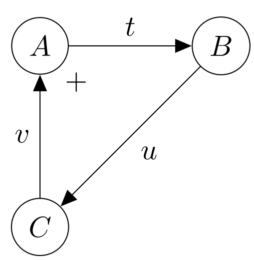


Figure 7.1. An infinite rank module for G_{29}

We next turn to the infinite irreducible groups (i.e., associated to connected braid diagrams). These were described in [43, 46], and are each associated to a complex affine space E with group of translations V. We fix a basepoint $v_0 \in E$ and identify $GL(V) \ltimes V \cong \mathrm{Aff}(E)$, the affine transformation group of E. As is explained in *loc. cit.*, $W \leq \mathrm{Aff}(E)$. Now invoke the results on [46, pp. 30], to note that if $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ has finite rank, then the same holds with $\mathbf{d}$ replaced by $(2, 2, \dots, 2)$. But then $W = W_{\mathbb{R}}$ is an irreducible affine Coxeter group that acts on a real form $E_{\mathbb{R}}$ of E, and the action of W on $E = \mathbb{C} \otimes_{\mathbb{R}} E_{\mathbb{R}}$ is simply its complexification. Now $n\mathcal{H}(W, \mathbf{d}, J_{<k}) = n\mathcal{H}(W_{\mathbb{R}}, \mathbf{d}, J_{<k})$ has finite $\Bbbk$-rank, hence so does $n\mathcal{H}(W_{\mathbb{R}}, (2, 2, \dots, 2), J_{<k})$. As $W_{\mathbb{R}}$ is affine and $k \geq 3$, our results in previous sections (see the penultimate row of Table 1) show that $k = 3$ and $W = W_{\mathbb{R}}$ is affine of type E_9 or F_5.

The remaining case is when W is a genuine crystallographic group, i.e., E/W is compact (with $W \leq \mathrm{Aff}(E)$) and $W/(W \cap V)$ is not the complexification of a real reflection group (where $V \leq \mathrm{Aff}(E)$ consists of the translations). For such groups, Coxeter-type presentations were provided by Malle in [43, Tables I, II]; and in Case 14 in the proof of [34, Theorem D], it was shown that there were no generalized nil-Coxeter algebras (i.e., with arbitrary $\mathbf{d}$ but with $k = \infty$). One can now verify that the proofs in [34] in *loc. cit.* for the $k = \infty$ case, also go through verbatim if $3 \leq k < \infty$. □

7.1. **Non-genuine, crystallographic complex reflection groups.** As explained in [46], there is a third class of (irreducible) discrete infinite complex reflection groups, which were termed "non-genuine crystallographic" in [34, Section 6] – and so it is natural to ask if the nil-Hecke algebras over these groups also have infinite $\Bbbk$-rank. As we now explain, a "specific" Coxeter-type presentation for these groups – arising out of work of Popov [46] and Ion–Sahi [32] – involves a braid-type relation equating two braid words of unequal lengths, and so the nil-Hecke algebras here are not defined.

We now elaborate on this. First, like the other complex groups, the non-genuine crystallographic complex reflection groups also admit a Coxeter-type presentation, as was explained to the second author by Popov [47]. For the details, we refer the reader to Case 15 in the proof of [34, Theorem D]. In brief: retaining the notation in the preceding proof, $W \leq \mathrm{Aff}(E) = GL(V) \ltimes V$ is now given by $W = \mathrm{Lin}(W) \ltimes \mathrm{Tran}(W)$, where $\mathrm{Lin}(W) := W \cap V$ and $\mathrm{Tran}(W) := W/(W \cap V)$. Moreover, $W' := \mathrm{Lin}(W)$ is a finite irreducible real reflection group (in fact a Weyl group), and $\mathrm{Tran}(W) \cong \Lambda_1 \oplus \tau\Lambda_1$ for some $\mathbb{Z}$-lattice Λ_1 of full rank, and a scalar $\tau \in \mathbb{C} \setminus \mathbb{R}$ (which we may thus take to lie in the upper half-plane). Moreover, the affine Weyl group $\widetilde{W}$ over W satisfies:

$$W' \ltimes \Lambda_1 \cong \widetilde{W} \cong W' \ltimes \tau\Lambda_1,$$

so that W is, in a sense, a "double affine Weyl group".

Now fix a simple base Π for the root system Φ of W', and let θ denote the unique highest root for Φ. Define $a_{01} := t(\theta)s_\theta$ to be the extra affine reflection in $W' \ltimes \Lambda_1$, where $t(\theta)$ is the translation by θ in $E_\mathbb{R}$ and $s_\theta \in O(E_\mathbb{R})$ is the reflection sending θ to $-\theta$. Similarly, let $a_{02} := t(\tau\theta)s_\theta$ denote the extra affine reflection in $W' \ltimes \tau\Lambda_1$. Then the non-genuine crystallographic infinite irreducible discrete complex reflection group W (which is a double affine Weyl group) is generated by $\{s_\alpha : \alpha \in \Pi\}$ and a_{01}, a_{02}. This does have a Coxeter-type presentation, as was explained to the second author by Sahi [49]. To obtain this, first note that the map

$$a_{03} := a_{01}s_\theta a_{02} = t((1+\tau)\theta)s_\theta$$

also squares to the identity map on $E_\mathbb{R}$. In particular, we obtain the further relation

$$a_{01}a_{03}a_{02} = s_\theta; \tag{7.1}$$

moreover, W' and a_{03} also generate an affine Weyl group, again isomorphic to $\widetilde{W}$.

Now Ion and Sahi have shown – see the end of [32, Section 5] – that W has a Coxeter-type presentation, with generators given by $\{s_\alpha : \alpha \in \Pi\} \sqcup \{a_{01}, a_{02}, a_{03}\}$; the relations are that the subset $\{s_\alpha : \alpha \in \Pi\} \sqcup \{a_{0j}\}$ satisfies the Coxeter presentation for $\widetilde{W}$ for $j = 1, 2, 3$, and the additional relation (7.1) holds. (Note that $a_{0j}a_{0j'}$ has infinite order in W for $j \neq j'$.)

With this presentation at hand, we return to our original question of interest: to examine the finite dimensionality (or $\Bbbk$-rank) of the associated nil-Hecke algebra $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ for $k \geq 3$. Note that if this happens then the same holds with $\mathbf{d}$ replaced by $(2, \dots, 2)$, and moreover with W replaced by $\widetilde{W}$ (by killing a_{02}, a_{03}). As in the proof of Theorem E, this

implies that $k = 3$ and W' is finite of type E_8 or F_4. (In particular, there is a unique node $\alpha \in \Pi$ to which the three affine nodes a_{0j} are attached, each by a single edge.) But now the braid words on both sides of the extra relation (7.1) have unequal lengths, and so the algebra $n\mathcal{H}(W, \mathbf{d}, J_{<3})$ is not defined.

8. Proof of Theorem F: classification of Frobenius nil-Hecke algebras

The goal of this final section is to show Theorem F. We **assume** throughout this section that $\Bbbk$ is a field. Below, we will use without further mention the fact that the spaces of left-primitive and right-primitive elements of $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ are linearly isomorphic – working over any Coxeter group W – via the anti-involution on $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ that sends each generator $\mathbf{s}_i$ to itself. As a first step, we classify the cases where the set of primitive elements have dimension 1.

Theorem 8.1. *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, integers $d_i \geqslant 2$ $\forall i$ and $1 \leqslant k \leqslant \infty$. Suppose the corresponding nil-Hecke $\Bbbk$-algebra $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is finite-dimensional. Then the space of primitive elements in $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is one-dimensional if and only if one of the following holds:*

(1) $d_i = 2$ for all i, and W contains no braid relations of length k or more;
(2) $k \in \{1, 2\}$, W is of type B or A_1, and $d_i = 2$ for all i;
(3) $k \in \{1, 2\}$, W is of type A, $d_i = 3$ for for exactly one of the pendant vertices of the Coxeter graph of W and $d_i = 2$ for all other vertices;
(4) $k = 3$, W is of the type A_1 or H_3 and $d_i = 2$ for all i;
(5) $k = 3$, W is of type A_2 with $d_i = 3$ for exactly one vertex and $d_i = 2$ for the other;
(6) $k \geqslant 3$ and W is of type A_1 with $d_i > 2$ for the only vertex present.

Proof. We first show that the cases other than those mentioned above yield at least two linearly independent primitive elements – in fact, two distinct primitive monomials.

Part 1: Suppose $k \in \{1, 2\}$. Suppose first the associated Coxeter graph is a simply laced tree with at least two vertices, and no vertex i has $d_i > 2$. This tree necessarily has a maximal path of length at least two, say

$$v_1 \quad \longleftrightarrow \quad v_2 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad v_n.$$

Letting $\mathbf{s}_i$ be the generator associated to vertex v_i, one can check that $\mathbf{s}_1\mathbf{s}_2 \cdots \mathbf{s}_n$ and $\mathbf{s}_n\mathbf{s}_{n-1} \cdots \mathbf{s}_1$ are two distinct primitive monomials. In case the graph contains a multiple edge (respectively, a vertex with $d_i > 2$) that is not at a pendant vertex, one can again form a maximal path containing this edge (respectively, this vertex) at some point other than the two ends. Then the same argument as above gives again two distinct primitive monomials.

Now suppose this graph has a multiple edge at its end, with all $d_i = 2$. If there is some vertex v with degree at least 3, then deleting this gives rise to three or more disconnected components, only one of which contains the multiple edge. Suppose v is connected to v_1 and v_2, each of which belong to a component not containing the multiple edge. Extending the path $v_1 v v_2$ gives us a maximal path with no multiple edge, and as before, we can conclude this does not correspond to a Frobenius algebra.

Thus we can only have a path graph with a multiple edge as one end. Label the vertices $v_1, v_2, \ldots, v_n$ where $m = m_{v_{n-1}, v_n} > 3$. Since this is not of type B, $m \geq 5$. If $m = 5$, again calling the generator associated to vertex v_i as $\mathbf{s}_i$, we see that $\mathbf{s}_1\mathbf{s}_2 \cdots \mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_{n-1} \ldots \mathbf{s}_2\mathbf{s}_1$ and $\mathbf{s}_1\mathbf{s}_2 \cdots \mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_{n-1}\mathbf{s}_n$ are distinct and primitive. If $m \geq 6$, then $\mathbf{s}_1\mathbf{s}_2 \cdots \mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_{n-1} \ldots \mathbf{s}_2\mathbf{s}_1 \neq \mathbf{s}_1\mathbf{s}_2 \cdots \mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_{n-1}\mathbf{s}_{n-2} \ldots \mathbf{s}_2\mathbf{s}_1$ are both primitive.

According to Theorem A and the previous analysis, the only remaining case is when the Coxeter graph is a simply laced tree, with $d_v > 2$ for a pendant vertex v. Using the above arguments, one can show that this graph is necessarily a path graph. Serially label its vertices $v_1, v_2, \ldots, v_{n-1}, v_n = v$. If $d_v > 3$, then again $\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_{n-1}\mathbf{s}_n^2\mathbf{s}_{n-1}\ldots\mathbf{s}_2\mathbf{s}_1 \neq \mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_{n-1}\mathbf{s}_n^3\mathbf{s}_{n-1}\ldots\mathbf{s}_2\mathbf{s}_1$ are two distinct primitive monomials. Thus the only remaining possibility is $d_v = 3$, which is accounted for in (3).

Part 2: Next, assume $k = 3$. For each case where $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is finite-dimensional as described in Theorem B, we exhibit two distinct primitive monomials, except for the cases mentioned in the statement of Theorem 8.1. These monomials are tabulated in Table 2.

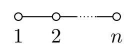
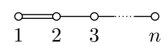
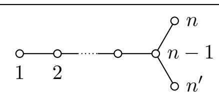
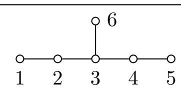
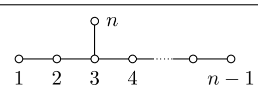
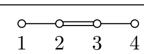
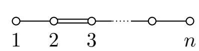
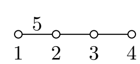
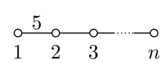
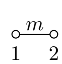
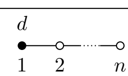
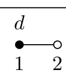
Type	Dynkin diagram	Primitive monomials
$A_n (n \geqslant 2)$		$\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_n, \quad \mathbf{s}_n\cdots\mathbf{s}_2\mathbf{s}_1$
$B_n (n \geqslant 3)$		$\mathbf{s}_n\mathbf{s}_{n-1}\cdots\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_{n-1}\mathbf{s}_n,$ $\mathbf{s}_{n-1}\cdots\mathbf{s}_2\mathbf{s}_1\mathbf{s}_n\mathbf{s}_2\cdots\mathbf{s}_{n-1}$
$D_{n+1} (n \geqslant 3)$		$\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_{n'}\mathbf{s}_{n-1}\cdots\mathbf{s}_2\mathbf{s}_1,$ $\mathbf{s}_2\cdots\mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_1\mathbf{s}_{n'}\mathbf{s}_{n-1}\cdots\mathbf{s}_2$
E_6		$\mathbf{s}_5\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_6\mathbf{s}_3\mathbf{s}_1\mathbf{s}_6\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_5,$ $\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_6\mathbf{s}_3\mathbf{s}_5\mathbf{s}_6\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1$
$E_n (n \geqslant 7)$		$\mathbf{s}_{n-1}\cdots\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_7\mathbf{s}_1\mathbf{s}_3\mathbf{s}_7\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\cdots\mathbf{s}_{n-1},$ $\mathbf{s}_{n-2}\cdots\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_7\mathbf{s}_1\mathbf{s}_{n-1}\mathbf{s}_3\mathbf{s}_7\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\cdots\mathbf{s}_{n-2}$
F_4		$\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4, \quad \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_2\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1$
$F_n (n \geqslant 5)$		$\mathbf{s}_n\cdots\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\cdots\mathbf{s}_n,$ $\mathbf{s}_{n-1}\cdots\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_n\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\cdots\mathbf{s}_{n-1}$
H_4		$\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_4\mathbf{s}_3,$ $\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4$
$H_n (n \geqslant 5)$		$\mathbf{s}_n\cdots\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\cdots\mathbf{s}_n,$ $\mathbf{s}_{n-1}\cdots\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_n\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\cdots\mathbf{s}_{n-1}$
$I_2(m) (m \geqslant 4)$		$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\cdots$ ($m-1$ generators), $\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\cdots$ ($m-1$ generators)
$A_n (n \geqslant 3)$, $\mathbf{d} = (d, 2, \ldots, 2), d \geqslant 3$		$\mathbf{s}_n\cdots\mathbf{s}_2\mathbf{s}_1\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_n,$ $\mathbf{s}_{n-1}\cdots\mathbf{s}_2\mathbf{s}_1\mathbf{s}_n\mathbf{s}_1\mathbf{s}_2\mathbf{s}_{n-1}$
$A_2, \mathbf{d} = (d, 2)$, $d > 3$		$\mathbf{s}_2\mathbf{s}_1\mathbf{s}_1\mathbf{s}_2,$ $\mathbf{s}_2\mathbf{s}_1\mathbf{s}_1\mathbf{s}_1\mathbf{s}_2$

TABLE 2. Primitive monomials for $k = 3$

Part 3: Now let $k = 4$. For cases where the Coxeter graph has no edges of label 4 or more, the corresponding algebra $n\mathcal{H}(W, \mathbf{d}, J_{<4})$ is identical to $n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$, which has been analyzed in [34]. The only remaining cases are B_n, F_4, H_3 and H_4 and $I_2(m)$ for $m \geqslant 4$.

For B_n with $n \geqslant 2$, using the notation in Theorem 5.1, we see that the signed permutations

$$w' := [-n, -(n-1), \ldots, -1], \quad w := [n-1, n-2, \ldots, 2, 1, -n] \in W$$

both correspond to right-primitive monomials $\mathbf{s}(w'), \mathbf{s}(w) \in n\mathcal{H}(W, \mathbf{d}, J_{<4})$ (i.e., $x\mathfrak{m} = 0$ for $x = \mathbf{s}(w'), \mathbf{s}(w)$). Further, one can check that w', w are self-inverses, which easily implies that

$\mathbf{s}(w'), \mathbf{s}(w)$ are left-primitive as well (i.e., $\mathfrak{m}x = 0$), as desired. We tabulate the results for the remaining cases in Table 3.

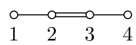
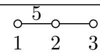
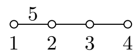
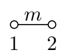
Type	Dynkin diagram	Primitive monomials
F_4		$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1, \quad \mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3$
H_3		$\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3, \quad \mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1$
H_4		$\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4,$ $\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1$
$I_2(m)$ $(m \geqslant 4)$		$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\cdots$ ($m-1$ generators), $\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\cdots$ ($m-1$ generators)

TABLE 3. Primitive monomials for $k = 4$

Part 4: First suppose $k = 5$. For $I_2(m)$, a similar argument as before applies. The cases of H_3 and H_4 are identical to those for $k = 4$, and the remaining cases are identical to those for $k = \infty$. Next, for $6 \leqslant k \leqslant \infty$, again the proof for $I_2(m)$ follows along similar lines as above, and the remaining cases are identical to their corresponding $k = \infty$ analogues.

Part 5: It remains to prove the cases listed in Theorem 8.1 indeed yield one-dimensional spaces of primitive elements. We consider each of the cases separately:

(1) In this case, $n\mathcal{H}(W, \mathbf{d}, J_{<k}) = n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$, and this is analyzed in [34].
(2) The case where W has type A_1 is easy to verify explicitly. Now suppose W has type B: say it contains the vertices $v_1, v_2, \ldots, v_n$ on a path in that order, and $m_{v_{n-1},v_n} = 4$. Then according to the results in [28] and Theorem 2.4, the non-zero monomials have one of the following forms:
   (i) $\mathbf{s}_i\mathbf{s}_{i+1}\cdots\mathbf{s}_j$ or $\mathbf{s}_j\mathbf{s}_{j-1}\cdots\mathbf{s}_i$ for $1 \leqslant i \leqslant j \leqslant n$. The first monomial can be right-multiplied by $\mathbf{s}_{j+1}$ if $j < n$ and by $\mathbf{s}_{n-1}$ if $j = n$ without yielding zero, and thus is not primitive. A similar logic holds for monomials of the second form.
   (ii) $\mathbf{s}_i\mathbf{s}_{i+1}\cdots\mathbf{s}_{n-1}\mathbf{s}_n\mathbf{s}_{n-1}\cdots\mathbf{s}_{j+1}\mathbf{s}_j$. This can be left-multiplied by s_{i-1} if $i > 1$ and can be right-multiplied by $\mathbf{s}_{j-1}$ if $j > 1$, so the only case where this is primitive is $i = j = 1$, and here it indeed yields a primitive monomial.

   Since we have considered all possible monomials, this completes the proof.
(3) Again, in this case, we can list out the possible monomials. Suppose the vertices are $v_1, v_2, \ldots, v_n$ on a path, with $d_{v_n} = 3$. The possible monomials are of the form:
   (i) $\mathbf{s}_i\mathbf{s}_{i+1}\cdots\mathbf{s}_j$ or $\mathbf{s}_j\mathbf{s}_{j-1}\cdots\mathbf{s}_i$ for $1 \leqslant i \leqslant j \leqslant n$. These can be eliminated as before.
   (ii) $\mathbf{s}_i\mathbf{s}_{i+1}\cdots\mathbf{s}_{n-1}\mathbf{s}_n^2\mathbf{s}_{n-1}\cdots\mathbf{s}_{j+1}\mathbf{s}_j$. As before, the only case where this is primitive is $i = j = 1$, and that proves the claim.
(4) and (5) are both small finite cases. The case of H_3 can be verified computationally, while the other two are simple enough for manual verification.
(6) Here, the algebra is simply $\Bbbk[\mathbf{s}]/\langle \mathbf{s}^{d_i} \rangle$, which has only one primitive monomial: $\mathbf{s}^{d_i - 1}$.

This concludes our proof. □

Next, we enumerate the cases where $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ has a one-dimensional space of right-primitive elements.

Theorem 8.2. *Fix a Coxeter group W with related data $I, J, \mathbf{S}, \mathcal{R}$, integers $d_i \geqslant 2$ $\forall i$ and $1 \leqslant k \leqslant \infty$. Suppose the corresponding nil-Hecke $\Bbbk$-algebra $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is finite-dimensional.*

Then the set of right-primitive elements in $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ has dimension one if and only if one of the following holds:

1. *$d_i = 2$ for all i, and W is a finite Coxeter group that contains no braid relations of length k or more;*
2. *W is of type A_1.*

Proof. Clearly if the space of right-primitive elements is one-dimensional, so is its subspace of primitive elements. Thus we only need to consider the cases listed in Theorem 8.1. We check each of them separately:

1. In this case the space of right-primitive elements is one-dimensional, as seen in [34].
2. If W is of type B_n with $n \geq 3$, label the vertices as in the proof of Theorem 8.1, part 5. Then $\mathbf{s}_2\mathbf{s}_1 \neq \mathbf{s}_3\mathbf{s}_2\mathbf{s}_1$ are two right-primitive monomials. On the other hand, if $n = 2$, then $\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1$ and $\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2$ are two distinct right-primitive monomials. This leaves us with the case of A_1, which can be checked to have a one-dimensional space of right-primitive elements as at the end of the proof of Theorem 8.1.
3. Again, if W has three or more vertices, then with the notation from the proof of Theorem 8.1, as before we see that $\mathbf{s}_2\mathbf{s}_1$ and $\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1$ are two distinct right-primitive monomials. If W has two nodes, then suppose v_1 has order $d \geqslant 3$. Then $\mathbf{s}_1\mathbf{s}_2$ and $\mathbf{s}_1^2\mathbf{s}_2$ are two right-primitive monomials.
4. If W is of type H_3, suppose the vertices are v_1, v_2, v_3 in that order with $m_{v_1,v_2} = 5$. Then $\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3$ and $\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3$ are both right-primitive.
5. Suppose the two vertices are v_1, v_2 with $d_1 = 3$ and $d_2 = 2$. Then $\mathbf{s}_1\mathbf{s}_2$ and $\mathbf{s}_1\mathbf{s}_1\mathbf{s}_2$ are right-primitive.
6. As this case has a commutative algebra, all right-primitive elements are primitive. □

With a bulk of the work done in the proofs of the above theorems, we conclude by showing the final outstanding main result.

Proof of Theorem F. The equivalence (2) $\iff$ (4) follows from Theorem 8.2. Let us now prove (1) $\implies$ (2) by extending the argument used to show [34, Theorem 5.2].

Suppose $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ is Frobenius; so there exists a nondegenerate invariant bilinear form σ on $n\mathcal{H}(W, \mathbf{d}, J_{<k})$. Now for each non-zero primitive p, there is $a_p \in n\mathcal{H}(W, \mathbf{d}, J_{<k})$ so that $0 \neq \sigma(p, a_p) = \sigma(pa_p, 1)$. Now if $a_p \in \mathfrak{m}$, $pa_p = 0$, so one may assume $a_p = 1$ for all p. Now the linear functional $\sigma(-, 1)$ gives an injective homomorphism from the set of right-primitive elements to $\Bbbk$: indeed, if we have $a \neq b$, both right-primitive, so that $\sigma(a, 1) = \sigma(b, 1) \iff \sigma(a - b, 1) = 0$, then we claim that $\sigma(a - b, c) = 0$ for all $c \in n\mathcal{H}(W, \mathbf{d}, J_{<k})$. By linearity, it suffices to prove this for $c = 1$ and $c \in \mathfrak{m}$. If $c = 1$, the conclusion is clear; if $c \in \mathfrak{m}$, $\sigma(a - b, c) = \sigma(ac - bc, 1) = \sigma(0, 1) = 0$ since a, b are right-primitive. This contradicts the non-degeneracy of σ, and thus injectivity must hold.

Now this clearly implies the space of right-primitive elements is at most one-dimensional. Since it has dimension at least one (for example, the longest non-zero monomial is necessarily right-primitive), the dimension is exactly one.

We now show (4) $\implies$ (1). Indeed, in the first case in (4), $n\mathcal{H}(W, \mathbf{d}, J_{<k}) = n\mathcal{H}(W, \mathbf{d}, J_{<\infty})$ is the usual nil-Coxeter algebra over W, and it is Frobenius by [36]. In the second case in (4), $n\mathcal{H}(W, \mathbf{d}, J_{<k}) = \Bbbk[\mathbf{s}]/\langle \mathbf{s}^d \rangle$ was shown to be Frobenius in [34], e.g. use the bilinear form σ obtained via $\sigma(\mathbf{s}^i, \mathbf{s}^j) = \mathbf{1}(i + j = d - 1)$.

It remains to show (2) $\iff$ (3). That (3) $\implies$ (2) is clear, and we showed (2) $\iff$ (4), so we now show (3) in the cases listed in (4). The first case is handled in [34], while the second case is trivial to check. The proof is now complete. □

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Appendix A. Sage codes

Here, we include the *Sage* programs for verifying some of the results that were obtained computationally.

The following calculates $\dim n\mathcal{H}(W, \mathbf{d}, J_{<4})$ for F_4, H_3 and H_4. To increase efficiency, this algorithm uses the fact that the set of all the group elements corresponding to the monomials to be enumerated form a weak order ideal.

```
F4,H3,H4=WeylGroup(['F',4]),CoxeterGroup(['H',3],implementation="coxeter3"),CoxeterGroup(['H',4],implementation="coxeter3")
checkF4=lambda w:all(['2323' not in "".join([str(i) for i in x]) for x in w.reduced_words()])
checkH3=lambda w:all(['23232' not in "".join([str(i) for i in x]) for x in w.reduced_words()])
checkH4=lambda w:all(['34343' not in "".join([str(i) for i in x]) for x in w.reduced_words()])
I1,I2,I3=F4.weak_order_ideal(predicate=checkF4),H3.weak_order_ideal(predicate=checkH3),H4.weak_order_ideal(predicate=checkH4)
print(I1.cardinality(),I2.cardinality(),I3.cardinality())
```

The following code returns a list of all primitive monomials in $n\mathcal{H}(W, \mathbf{d}, J_{<k})$ for $k = 3$, $W = H_3$, thereby proving the space of such elements is one-dimensional.

```
W=CoxeterGroup(['H',3],implementation="coxeter3")
s=W.simple_reflections()
checkH3_FC=lambda w:all(['23232' not in "".join([str(i) for i in x]) for x in w.reduced_words()]) and all(['121' not in "".join([str(i) for i in x]) for x in w.reduced_words()])
FClist=[w for w in W if checkH3_FC(w)]
def primitive(w):
    l=w.length()
    return all([not checkH3_FC(w*i) or (w*i).length()<l for i in s]) and all([not checkH3_FC(i*w) or (i*w).length()<l for i in s])
for w in FClist:
    if primitive(w):
        print(w)
```

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