MATRIX POSITIVITY PRESERVERS IN FIXED DIMENSION. II: POSITIVE DEFINITENESS AND STRICT MONOTONICITY OF SCHUR FUNCTION RATIOS

ALEXANDER BELTON, DOMINIQUE GUILLOT, APOORVA KHARE, AND MIHAI PUTINAR

Abstract. We continue the study of real polynomials acting entrywise on matrices of fixed dimension to preserve positive semidefiniteness, together with the related analysis of order properties of Schur polynomials.

Previous work has shown that, given a real polynomial with positive coefficients that is perturbed by adding a higher-degree monomial, there exists a negative lower bound for the coefficient of the perturbation which characterises when the perturbed polynomial remains positivity preserving.

We show here that, if the perturbation coefficient is strictly greater than this bound then the transformed matrix becomes positive definite given a simple genericity condition that can be readily verified. We identity a slightly stronger genericity condition that ensures positive definiteness occurs at the boundary.

The analysis is complemented by computing the rank of the transformed matrix in terms of the location of the original matrix in a Schubert cell-type stratification that we have introduced and explored previously. The proofs require enhancing to strictness a Schur monotonicity result of Khare and Tao, to show that the ratio of Schur polynomials is strictly increasing along each coordinate on the positive orthant and non-decreasing on its closure whenever the defining tuples satisfy a coordinate-wise domination condition.

1. Background and setup

The study of entrywise positivity preservers involves understanding the structure of functions of the form $f : I \to \mathbb{R}$, for some complex domain I, such that, if a complex Hermitian matrix $A = (a_{ij})$ with entries in I is positive semidefinite then so is the matrix $f[A] := \big(f(a_{ij})\big)$; when f is a power function, so that $f(x) \equiv x^\alpha$ for some α, we also use the Schur product notation $f[A] = A^{\circ\alpha}$.

This subject has a rich history, beginning with the Schur product theorem [21], which implies that all functions represented by power series with non-negative coefficients preserve positivity in this sense for square matrices of arbitrary size. The converse, that there are no other preservers in all dimensions, was first shown by Schoenberg [20] for continuous functions defined on $I = [-1, 1]$, and subsequently by several others. For domains of the form $I = (-\rho, \rho)$, with $0 < \rho \leqslant \infty$, we mention Rudin [19] and recent work [6] for variants with greatly reduced test sets in each dimension. The book [14] contains additional details and references.

The situation is more involved in a fixed dimension N, where the complete classification of the entrywise positivity preservers remains open to date even for 3×3 matrices, that is, when $N = 3$. For matrices with positive entries, the real powers which are entrywise

Date: October 27, 2023.

2010 *Mathematics Subject Classification.* 15B48 (primary); 05E05, 15A24, 15A45, 26C05 (secondary).

positivity preservers were classified by FitzGerald and Horn in [9]: these are the non-negative integers and all real powers beyond the threshold $N-2$, that is, elements of the set $\mathbb{Z}_+ \cup [N-2, \infty)$. If one considers polynomial preservers instead then no such preservers were known in fixed dimension beyond the case of non-negative coefficients, until the previous part of this work [3], which was subsequently extended by Khare and Tao [15].

1.1. **Polynomial preservers yield positive definite matrices.** We lay out here in a condensed form the key results from [3, 15] that are relevant to our work here, and provide from this context the first novel observation of this paper.

By a result of Loewner (see [12]), if $\rho > 0$ and the smooth function $f : (0, \rho) \to \mathbb{R}$ is such that $f[A]$ is positive semidefinite for any positive semidefinite $A \in (0, \rho)^{N \times N}$, then f, f', …, $f^{(N-1)}$ are non-negative on $(0, \rho)$, but this need not hold for any higher derivative of f. More generally, if f is a real polynomial preserver with exactly $N+1$ monomial terms, then the first N non-zero Maclaurin coefficients of f are positive. The question of whether the leading coefficient could be negative was eventually answered positively in [3, 15] with an explicit sharp negative lower bound in several slightly different settings. We begin here fixing some notation, introducing these settings and then providing the common bound which holds for all of them.

Definition 1.1. Given a domain $I \subseteq \mathbb{C}$ and positive integers N and k with $k \leqslant N$, denote by $\mathcal{P}_N^k(I)$ the set of positive semidefinite $N \times N$ matrices with entries in I and rank at most k; recall that any positive semidefinite complex matrix is automatically Hermitian. For convenience, we also set $\mathcal{P}_N(I) := \mathcal{P}_N^N(I)$. The Loewner partial order on $N \times N$ Hermitian matrices is defined by setting $A \geqslant B$ if and only if $A - B \in \mathcal{P}_N(\mathbb{C})$.

For any $\rho > 0$, let $\overline{D}(0, \rho)$ denote the closed disc in $\mathbb{C}$ with center 0 and radius ρ. We are interested in the entrywise action of the function

$$f(z) := \sum_{j=0}^{N-1} c_j z^{n_j} + c' z^M = h(z) + c' z^M \tag{1.1}$$

on some suitable set of test matrices $\mathcal{P}_0 \subseteq \mathcal{P}_N\big(\overline{D}\big((0, \rho)\big)$, where the number of terms N is a positive integer, the coefficients c_0, …, c_{N-1} and c' are real numbers and the powers are arranged in increasing order: $n_0 < n_1 < \cdots < n_{N-1} < M$. The test set $\mathcal{P}_0$ is may depend on the form of f, as follows.

(1) The minimal subset $\mathcal{P}_0 = \mathcal{P}_N^1\big((0, \rho)\big)$, for arbitrary real powers n_0, …, n_{N-1} and M.
(2) A subset $\mathcal{P}_0$ such that $\mathcal{P}_N^1\big((0, \rho)\big) \subseteq \mathcal{P}_0 \subseteq \mathcal{P}_N^1\big([0, \rho]\big)$, for non-negative powers n_0, …, n_{N-1} and M. Here and elsewhere we set $0^0 := 1$.
(3) A subset $\mathcal{P}_0$ such that $\mathcal{P}_N^1\big((0, \rho)\big) \subseteq \mathcal{P}_0 \subseteq \mathcal{P}_N([0, \rho]\big)$, where n_0, …, n_{N-1} and M are elements of the set $\mathbb{Z}_+ \cup [N-1, \infty)$. [1]
(4) A subset $\mathcal{P}_0$ such that $\mathcal{P}_N^1\big((0, \rho)\big) \subseteq \mathcal{P}_0 \subseteq \mathcal{P}_N\big(\overline{D}(0, \rho)\big)$, where n_0, …, n_{N-1} are successive non-negative integers (so that $n_j = n_0 + j$ for $j = 0$, …, $N-1$) and M is an integer.

[1] As mentioned previously, it is known [9] that all real powers $\alpha \geqslant N-2$ preserve positivity when acting entrywise on $\mathcal{P}_N\big([0, \rho]\big)$, but we need more for our purposes, namely, powers that preserve the Loewner order on $\mathcal{P}_0$: if A, $B \in \mathcal{P}_0$ with $A - B \in \mathcal{P}_N\big([0, \rho]\big)$ then $A^{\circ\alpha} - B^{\circ\alpha} \in \mathcal{P}_N(\mathbb{R})$. See [11, Theorem 5.1(ii)].

In the complex case (4) above, if the polynomial f has the form (1.1) with coefficients $c_0, \ldots, c_{N-1} > 0$ and $c_M < 0$, and the powers $n_0, \ldots, n_{N-1}$ are not successive non-negative integers then f does not preserve positive semidefiniteness entrywise on $\mathcal{P}_0$ for some $M > n_{N-1}$: see [15, Proposition 7.1].

Having described these possibilities, we recall the corresponding classification of entrywise polynomial preservers.

Theorem 1.2 ([3, Theorem 1.1] and [15, Section 1.3])**.** *Let f be as in (1.1), let $\rho > 0$ and set*

$$\mathcal{C} = \mathcal{C}(f, \rho) := \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_j}, \tag{1.2}$$

where the Vandermonde determinant

$$V(\mathbf{m}) := \prod_{1 \leqslant k < l \leqslant N} (m_l - m_k) \qquad \textit{for any } \mathbf{m} = (m_1, \ldots, m_N)$$

and the N-tuples

$$\mathbf{n}_j := (n_0, \ldots, \widehat{n_j}, \ldots, n_{N-1}, M) \qquad \textit{and} \qquad \mathbf{n} := (n_0, \ldots, n_{N-1}), \tag{1.3}$$

where $\widehat{n_j}$ indicates that n_j is omitted. Given a test set $\mathcal{P}_0$ according to Definition 1.1, the following are equivalent.

(1) The map $f[-]$ preserves positivity on $\mathcal{P}_0$.
(2) The coefficients of f satisfy either (a) $c_0, \ldots, c_{N-1}, c' \geqslant 0$, or (b) $c_0, \ldots, c_{N-1} > 0$ and $c' \geqslant -\mathcal{C}^{-1}$.
(3) The map $f[-]$ preserves positivity on the subset of Hankel matrices in $\mathcal{P}_N^1\big((0, \rho)\big)$.

Fundamentally, our work involves the constructive analysis of the largest eigenvalue for linear pencils of Hermitian matrices of the form

$$h[A] - \lambda A^{\circ M},$$

where h is the unperturbed polynomial adapted to the size of the positive matrix A and the power M exceeds the degree of h. One of the results we show in the present work is an enhancement of previous work to show the positive definiteness of $f[A]$ for generic A:

Theorem 1.3. *Let f, $\mathcal{C}$ and $\mathcal{P}_0$ be as in Theorem 1.2, with $c_0, \ldots, c_{N-1} > 0$ and $c' > \mathcal{C}^{-1}$. If all of the rows of $A \in \mathcal{P}_0$ are distinct and $n_0 = 0$ when A has a zero row then $f[A]$ is positive definite.*

This is stated and proved in Theorems 2.2 and 2.8 below.

To establish these two theorems, we rely on a lower-bound result, that if a positive semidefinite matrix A has distinct rows then it has a rank-one lower bound $\mathbf{u}$, such that $A \geqslant \mathbf{u}\mathbf{u}^T$, and $\mathbf{u}$ may be chosen to have distinct entries. In the complex setting this is elementary, but if A has non-negative entries and $\mathbf{u}$ is required to as well then we establish the existence of such a lower bound using Perron–Frobenius theory. This result, Theorem 2.12, may be of independent interest.

1.2. **Strict monotonicity of Schur polynomial ratios.** Next we switch tracks and focus on Schur polynomials from an order perspective. While this may seem a non sequitur, it is not: the proofs of Theorem 1.2 in [15] rely crucially on

(i) a combinatorial determinant formula involving Schur polynomials (Theorem 2.6) and
(ii) a Schur monotonicity lemma (Section 2.1).

We will now introduce some notation to facilitate the statement of the monotonicity lemma.

For any set of real numbers S, the collection of N-tuples of distinct elements of S is denoted by $S^N_{\neq}$ and its subset of N-tuples with entries in increasing order is denoted by $S^N_{<}$. Given vectors $\mathbf{u} = (u_i)_{i=1}^N \in (0,\infty)^N$ and $\mathbf{m} = (m_j)_{j=1}^N \in \mathbb{R}^N$, we let the matrix $\mathbf{u}^{\circ \mathbf{m}} := (u_i^{m_j})_{i,j=1}^N$.

Theorem 1.4 (Schur monotonicity lemma, [15, Corollary 8.7 and Proposition 8.1])**.** *Let* $\mathbf{m}$*,* $\mathbf{n} \in \mathbb{R}^N_{<}$ *be such that* $m_j \leqslant n_j$ *for all* j*, where* $N \geqslant 1$*. The symmetric function*

$$f : (0,\infty)^N_{\neq} \to \mathbb{R}; \ \mathbf{u} \mapsto \frac{\det \mathbf{u}^{\circ \mathbf{n}}}{\det \mathbf{u}^{\circ \mathbf{m}}}$$

is non-decreasing in each coordinate. If, moreover, the entries of the vectors $\mathbf{m}$ *and* $\mathbf{n}$ *are non-negative integers then* f *extends uniquely to the whole of* $(0,\infty)^N$ *and coordinate-wise monotonicity holds everywhere.*

To see the connection with Schur, we note that when $\mathbf{m}$ and $\mathbf{n}$ are composed of non-negative integers then $f(\mathbf{u}) \equiv s_{\mathbf{n}}(\mathbf{u})/s_{\mathbf{m}}(\mathbf{u})$, the ratio of Schur polynomials $s_{\mathbf{m}}$ and $s_{\mathbf{n}}$ as defined in (2.2) below.

Theorem 1.4 is interesting for multiple reasons. First, it provided the missing ingredient required to extend the positivity preserver results in [3] to general polynomials in [15]. Second, it led to novel characterizations in the theory of real inequalities [15]: of weak majorization, as well as of majorization for all real tuples, extending the integer-tuple case in [8, 22]. Third, this result admits several different proofs: via a log-supermodularity phenomenon and totally positive matrices [15], using a result of Lam, Postnikov and Pylyavskyy [16] from representation theory and the theory of symmetric functions [15], and relying on the theory of Chebyshev blossoming in Müntz spaces, as developed by Ait-Haddou and co-authors [1, 2].

In fact, the hypotheses of this theorem serve to deliver a stronger conclusion and this is our second main result:

Theorem 1.5. *With the hypotheses of Theorem 1.4, when* $\mathbf{m}$ *and* $\mathbf{n}$ *are distinct the function* f *is actually strictly increasing in each coordinate. Moreover, when* $\mathbf{m}$ *and* $\mathbf{n}$ *also have non-negative-integer entries, this coordinate-wise strict monotonicity holds for the extension of* f *to all of* $(0,\infty)^N$*.*

In fact, we show a stronger result than the final assertion here, by extending the function to parts of the boundary of the positive orthant. Moreover, it is not the generalized Vandermonde ratio with non-integer powers but the Schur polynomial ratio with integer exponents whose strict monotonicity has the more involved proof. See Theorems 2.9 and 2.11 for details.

Apart from its intrinsic interest, Theorem 1.5 is the key to proving Theorem 1.3 and its variations in Section 2. The proofs of both main results combine techniques from analysis with properties of Schur polynomials, which are inherently algebraic objects with a representation-theoretic flavour. Our exploration reinforces the need for further study of Schur functions from an analytical viewpoint. Prior work has already revealed the essential role of Schur functions in the investigation of positivity transforms (see [3, 15] and also [18]), and we can add two more contributions from recent work [13]. The first creates a bridge between analysis and algebra: the Schur polynomials lie within the Maclaurin expansion of $\det f[\mathbf{u}\mathbf{v}^T]$ for every smooth function f. The second walks across this bridge to contribute to algebra: the well-known determinant formula of Cauchy in symmetric function theory, its extension by Frobenius, and a determinant computation by Loewner [12] all admit a common extension, to power series over an arbitrary commutative ring.

While the main theme of our work is the classification of positivity transforms, at least two ingredients in the proofs below may be of independent interest: the strict monotonicity of certain ratios of Schur functions and the continuity of certain Rayleigh quotients on isogenic strata of positive matrices.

One conclusion that may be drawn from the present article is that applications of Schur functions to topics beyond algebra are far from being fully explored. Further discoveries and more surprises undoubtedly lie in wait.

Organisation of the remainder of this paper. Section 2 contains the statements and proofs of extended versions of the two new theorems stated above, Theorems 1.3 and 1.4. This section concludes by resolving the question of whether Loewner's necessary condition for smooth functions to preserve positive semidefiniteness in fixed dimension is also sufficient.

In Section 3, we recall the isogenic block stratification from [4, 5] and use this to find the rank of the matrix $f[A]$ for A in any given stratum and f as in Theorem 1.2(2)(b).

We conclude with Section 4, in which we recall the interpretation from [3] of the bound $\mathcal{C}$ in terms of a Rayleigh quotient. We prove that this Rayleigh quotient is continuous as a function of the underlying matrix when restricted to each isogenic stratum.

For the reader's convenience, we append before the bibliography a list of symbols used throughout this article.

2. Strictness of linear matrix inequalities for Hadamard powers, and the Schur strict monotonicity lemma

In this section, we obtain two variations on Theorem 1.2. We note first the following consequence of this theorem.

Corollary 2.1. *Let f, $\mathcal{C}$ and $\mathcal{P}_0$ be as in Theorem 1.2. If $c_0, \ldots, c_{N-1} > 0$ then*

$$A^{\circ M} \leqslant \mathcal{C} \sum_{j=0}^{N-1} c_j A^{\circ n_j} \qquad \text{for any } A \in \mathcal{P}_0, \tag{2.1}$$

where $\leqslant$ denotes the Loewner ordering, and the constant $\mathcal{C}$ is sharp.

It follows immediately from this Corollary that the matrix

$$f[A] = \sum_{j=0}^{N-1} c_j A^{\circ n_j} + c' A^{\circ M}$$

is positive semidefinite, whenever $c_0, \ldots, c_{N-1} > 0$ and $c' \geqslant -\mathcal{C}^{-1}$, for any $A \in \mathcal{P}_0$. We introduce and recall some notation for two important boundary cases:

$$g(z) = \sum_{j=0}^{N-1} c_j z^{n_j} - \mathcal{C}^{-1} z^M \qquad \text{and} \qquad h(z) = \sum_{j=0}^{N-1} c_j z^{n_j}.$$

It is natural to ask when the matrices $f[A]$, $g[A]$ and $h[A]$ are positive definite. The following strengthening of Theorem 1.2 shows that these matrices are generically positive definite in a strong sense, and zero only in the one-dimensional, degenerate case.

Theorem 2.2. *Let f and $\mathcal{P}_0$ be as in Definition 1.1(4), so that n_0 and M are non-negative integers and $n_j = n_0 + j$ for $j = 0, \ldots, N-1$. Suppose $c_0, \ldots, c_{N-1} > 0$ and $c' > -\mathcal{C}^{-1}$, where $\mathcal{C}$ is as in (1.2).*

(1) *Let $A \in \mathcal{P}_0$ and suppose $n_0 = 0$ if A has a zero row. The following are equivalent.*
   (a) *There exists a vector $\mathbf{u} \in \mathbb{C}^N$ with distinct entries such that $A \geqslant \mathbf{u}\mathbf{u}^*$ and $\mathbf{u}$ has a zero entry if and only if A has a zero row.*
   (b) *All of the rows of A are distinct.*
   (c) *The matrix $h[A]$ is positive definite.*
   (d) *The inequality (2.1) is strict, that is, $f[A]$ is positive definite.*

(2) *Suppose $A \in \mathcal{P}_0$ has a row with distinct entries and $n_0 = 0$ if any entry in this row is zero. Then $g[A]$ is positive definite.*

Furthermore, equality in (2.1) is attained on $\mathcal{P}_0$ if and only if either $N = 1$ and $A = \rho$, or $n_0 > 0$ and $A = \mathbf{0}_{N\times N}$.

Note that part (1)(a) of Theorem 2.2 does not depend on the coefficients $c_0, \ldots, c_{N-1}$ and c', and that the existence of $\mathbf{u}$ follows immediately from Proposition 2.4 if A is positive definite. Note also that "row" may be replaced with "column" throughout, as all the matrices are Hermitian.

Theorem 2.8 below provides a variation on Theorem 2.2 for the other three settings of Definition 1.1.

The proof of Theorem 2.2 relies on the following preliminary observations.

Lemma 2.3. *Suppose $N \geqslant 1$ and $C, D \in \mathcal{P}_N(\mathbb{C})$ with $C \geqslant D$. Then $C - tD$ has the same kernel and rank as C for all $t \in [0,1)$.*

Proof. Fix $t \in (0,1)$. If $C\mathbf{u} = 0$ for some $\mathbf{u} \in \mathbb{C}^N$, then, as $0 \leqslant C - tD \leqslant C$, it follows that

$$0 \leqslant \mathbf{u}^*(C - tD)\mathbf{u} \leqslant \mathbf{u}^* C \mathbf{u} = 0,$$

so $\ker C \subseteq \ker(C - tD)$. Conversely, if $(C - tD)\mathbf{u} = 0$ for some $\mathbf{u} \in \mathbb{C}^N$, then

$$0 = \mathbf{u}^*(C - tD)\mathbf{u} = \mathbf{u}^* C\mathbf{u} - t(\mathbf{u}^* D\mathbf{u}) \qquad \Longrightarrow \qquad \mathbf{u}^* C\mathbf{u} = t(\mathbf{u}^* D\mathbf{u}).$$

Now, if $\mathbf{u}^* C\mathbf{u} > 0$ then $\mathbf{u}^* D\mathbf{u} > 0$, so

$$\mathbf{u}^*\Big(C - \frac{1+t}{2} D\Big)\mathbf{u} = \frac{t-1}{2}\mathbf{u}^* D\mathbf{u} < 0,$$

which is impossible as $0 \leqslant C - D \leqslant C - \frac{1+t}{2} D$. Thus $C\mathbf{u} = 0$, proving the reverse inclusion. We are now done, by the rank-nullity theorem. $\square$

Proposition 2.4 ([3, Proposition 4.2])**.** *Suppose $N \geqslant 1$ and C, $D \in \mathcal{P}_N(\mathbb{C})$. The following are equivalent.*

1. *If $\mathbf{v}^* C \mathbf{v} = 0$ for some $\mathbf{v} \in \mathbb{C}^N$, then $\mathbf{v}^* D \mathbf{v} = 0$.*
2. *The inclusion $\ker C \subseteq \ker D$ holds.*
3. *There exists a constant $t > 0$ such that $C \geqslant tD$.*

It follows immediately from the previous result that if C, $D \in \mathcal{P}_N$ with $C \geqslant D$ and D is positive definite, so invertible, then C is also invertible, so positive definite.

While Theorem 2.2(2) is a result on positive definiteness, its proof uses connections to Schur polynomials and Young tableaux. The key step in this respect is Theorem 2.6, which requires the following definition (which adopts a different convention to that often found in the literature [17]).

Definition 2.5. As above, if S is any subset of real numbers, we let $S^N_<$ denote the set of all increasing N-tuples of the form $\mathbf{n} = (n_0 < \ldots < n_{N-1})$ with entries in S. For such an N-tuple $\mathbf{n}$, we let $|\mathbf{n}| := n_0 + \cdots + n_{N-1}$.

Given any $\mathbf{n} \in (\mathbb{Z}_+)^N_<$, the corresponding *Schur polynomial* $s_{\mathbf{n}}(u_1, \ldots, u_N)$ is the unique polynomial extension of the rational expression

$$s_{\mathbf{n}}(u_1, \ldots, u_N) := \frac{\det(u_i^{n_{j-1}})_{i,j=1}^N}{\det(u_i^{j-1})_{i,j=1}^N}. \tag{2.2}$$

Note that the denominator is precisely the Vandermonde determinant

$$V(\mathbf{u}) = V(u_1, \ldots, u_N) := \det(u_i^{j-1})_{i,j=1}^N = \prod_{1 \leqslant k < l \leqslant N} (u_l - u_k)$$

and we can write $s_{\mathbf{n}}(\mathbf{u}) V(\mathbf{u}) = \det \mathbf{u}^{\circ \mathbf{n}}$, where the matrix $\mathbf{u}^{\circ \mathbf{n}} := (u_i^{n_{j-1}})_{i,j=1}^N$. Since the right-hand side of (2.2) is unchanged after swapping any two elements of $\mathbf{u}$, each Schur polynomial is a symmetric function.

For any $q \neq 0$ we have the product identity [23, ((7.105)]

$$s_{\mathbf{n}}(1, q, \ldots, q^{N-1}) = \frac{\det\big((q^{n_{j-1}})^{i-1}\big)_{i,j=1}^N}{\det(q^{j-1})^{i-1})_{i,j=1}^N} = \prod_{1 \leqslant k < l \leqslant N} \frac{q^{n_{l-1}} - q^{n_{k-1}}}{q^{l-1} - q^{k-1}} \tag{2.3}$$

as the numerator and denominator are both Vandermonde determinants. Taking $q \to 0$ leads to the specialisation

$$s_{\mathbf{n}}(u \mathbf{1}_T^N) = u^{|\mathbf{n} - \boldsymbol{\delta}|} \prod_{1 \leqslant k < l \leqslant N} \frac{n_{l-1} - n_{k-1}}{l - k} = u^{|\mathbf{n} - \boldsymbol{\delta}|} \frac{V(\mathbf{n})}{V(\boldsymbol{\delta})} \qquad \text{for all } u,$$

where $\boldsymbol{\delta} := (0, 1, 2, \ldots, N-1)$. As is well known [17, Chapter I, Equation (5.12)], thanks to Littlewood we have the identity

$$s_{\mathbf{n}}(\mathbf{u}) = \sum_{\mathbf{t}} \mathbf{u}^{\mathbf{t}}, \tag{2.4}$$

a sum of $s_{\mathbf{n}}(\mathbf{1}_N^T) = V(\mathbf{n})/V(\boldsymbol{\delta})$ monomials, where the monomial $\mathbf{u}^{\mathbf{t}} := \prod_{j=1}^N u_j^{t_j}$ has degree $|\mathbf{n} - \boldsymbol{\delta}|$ and the sum is taken over all semistandard Young tableau $\mathbf{t}$ of shape $\mathbf{n} - \boldsymbol{\delta}$.

In particular, if $\mathbf{n}_j$ and $\mathbf{n} = n_0 \mathbf{1}_N^T + \boldsymbol{\delta}$ are as in (1.3) and we let $\mathbf{n}'_j := \mathbf{n}_j - n_0 \mathbf{1}_N^T$ then

$$s_{\mathbf{n}'_j}(\sqrt{\rho}\,\mathbf{1}_N^T)^2 = \rho^{M-n_0-j} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \qquad \text{for any } \rho > 0. \tag{2.5}$$

Furthermore, it may be shown by the hook-content formula [23, Theorem 7.21.2] that

$$\frac{V(\mathbf{n}_j)}{V(\mathbf{n})} = s_{\mathbf{n}_j}(\mathbf{1}_N^T) = s_{\mathbf{n}'_j}(\mathbf{1}_N^T) = \binom{M}{j}\binom{M-j-1}{N-j-1}.$$

Theorem 2.6 ([15])**.** *Let S be a finite set of real numbers of cardinality at least N and suppose*

$$F(x) = \sum_{n \in S} c_n x^n,$$

where each coefficient c_n is real. If $\mathbf{u} \in \mathbb{C}^N$ then

$$\det F[\mathbf{u}\mathbf{u}^*] = \sum_{\mathbf{n} \in S_<^N} |\det \mathbf{u}^{\circ \mathbf{n}}|^2 \prod_{n \in \mathbf{n}} c_n. \tag{2.6}$$

In particular, if the elements of S are non-negative integers then

$$\det F[\mathbf{u}\mathbf{u}^*] = \sum_{\mathbf{n} \in S_<^N} |s_{\mathbf{n}}(\mathbf{u})|^2 |V(\mathbf{u})|^2 \prod_{n \in \mathbf{n}} c_n. \tag{2.7}$$

We state and prove a short lemma before we give the proof of Theorem 2.2.

Lemma 2.7. *Suppose $\mathbf{w} \in \mathbb{C}^N$ has no zero entries. If $B \in \mathcal{P}_N(\mathbb{C})$ is positive definite then so is the Schur product $(\mathbf{w}\mathbf{w}^*) \circ B$.*

Proof. For any vector $\mathbf{v} \neq \mathbf{0}$, we have that $\mathbf{v} \circ \overline{\mathbf{w}} \neq \mathbf{0}$ and therefore

$$\mathbf{v}^*\big((\mathbf{w}\mathbf{w}^*) \circ B\big)\mathbf{v} = (\mathbf{v} \circ \overline{\mathbf{w}})^* B (v \circ \overline{\mathbf{w}}) > 0. \qquad \square$$

Proof of Theorem 2.2. For part (1), we first show that (a) implies (c). Suppose $\mathbf{u} \in \mathbb{C}^N$ has distinct entries and is such that $A \geqslant \mathbf{u}\mathbf{u}^*$. Then $h[\mathbf{u}\mathbf{u}^*]$ is the sum of N rank-one matrices with linearly independent column spaces, since the determinant of the matrix $(u_k^{n_0+l-1})_{k,l=1}^N$ is the product of a Vandermonde determinant and $\prod_{k=1}^N u_k^{n_0}$; recall that we take $0^0 = 1$. Thus, $h[\mathbf{u}\mathbf{u}^*]$ is non-singular and so positive definite. As noted above, entrywise powers of non-negative integers are Loewner monotone on $\mathcal{P}_N$, so $h[A] \geqslant h[\mathbf{u}\mathbf{u}^*]$ and $h[A]$ is also positive definite, by the remark after Proposition 2.4.

Next, we note that (c) implies (b) because the contrapositive is immediate. We now suppose that (b) holds and deduce (a). Let $\mathbf{u}_1^T, \ldots, \mathbf{u}_N^T$ denote the rows of A. As $\mathbb{C}^N$ is not a finite union of proper subspaces, we can choose a vector $\mathbf{v} \in \mathbb{C}^N$ that is not orthogonal to any vector of the form $\mathbf{u}_j - \mathbf{u}_k$ with $j \neq k$ nor any vector $\mathbf{u}_j$ that is non-zero. We set $\mathbf{w} := A\overline{\mathbf{v}}$ and note that $\mathbf{w}$ has distinct entries by the choice of $\mathbf{v}$; moreover, $\mathbf{w}$ has a zero entry if and only if the corresponding row of A is zero. By Proposition 2.4, we have that $A \geqslant t\mathbf{w}\mathbf{w}^*$ for some scalar $t > 0$, so (a) follows by setting $\mathbf{u} := \sqrt{t}\mathbf{w}$.

Finally, that (c) implies (d) follows from the remark after Proposition 2.4 with $C = f[A]$ and $D = h[A]$ when $c' \geqslant 0$, and from Corollary 2.1 and Lemma 2.3 with $C = h[A]$ and $D = \mathcal{C}^{-1} A^{\circ M}$ when $c' < 0$. Conversely, that (d) implies (c) follows from the same remark when $c' \leqslant 0$, while if $c' > 0$, the implication follows from Lemma 2.3 with $C = f[A]$ and $D = (c' + \mathcal{C}^{-1})A^{\circ M}$, together with Corollary 2.1. This concludes the proof of part (1).

To prove part (2), we first show the rank-one case: if $A = \mathbf{u}\mathbf{u}^*$ for some column vector $\mathbf{u} \in \overline{D}(0, \sqrt{\rho})^N$ and A has a row with distinct entries then $\mathbf{u}$ has distinct entries and $g[A]$ is positive definite.

Suppose for contradiction that $\det g[\mathbf{u}\mathbf{u}^*] = 0$, and note that, by specialising (2.7) to the given parameters and using the fact that $s_{\mathbf{n}}(\mathbf{u}) = \prod_{j=1}^N u_j^{n_0}$,

$$\sum_{j=0}^{N-1} \frac{|s_{\mathbf{n}_j}(\mathbf{u})|^2}{c_j} = \mathcal{C} \prod_{j=1}^N |u_j|^{2n_0} = \prod_{j=1}^N |u_j|^{2n_0} \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}'_j}(\sqrt{\rho}\,\mathbf{1}_N^T)^2}{c_j},$$

by (2.5), where the partition $\mathbf{n}_j$ is as in (1.3) and $\mathbf{n}'_j := \mathbf{n}_j - n_0 \mathbf{1}_N^T$.

We note from the definitions that $s_{\mathbf{n}_j}(\mathbf{u}) = s_{\mathbf{n}'_j}(\mathbf{u}) \prod_{j=1}^N u_j^{n_0}$. It now follows from the triangle inequality and the Littlewood identity (2.4) that

$$|s_{\mathbf{n}'_j}(\mathbf{u})|^2 \leqslant s_{\mathbf{n}'_j}(\sqrt{\rho}\,\mathbf{1}_N^T)^2,$$

since $\mathbf{u} \in \overline{D}(0, \sqrt{\rho})^N$, and therefore $|s_{\mathbf{n}'_j}(\mathbf{u})| = s_{\mathbf{n}'_j}(\sqrt{\rho}\,\mathbf{1}_N^T) = \rho^{(M-n_0-j)/2} V(\mathbf{n}_j)/V(\mathbf{n})$ for all j. Another application of the triangle inequality implies that all monomials $\mathbf{u}^{\mathbf{t}}$ in the sum for $s_{\mathbf{n}'_j}(\mathbf{u})$ have modulus $\rho^{(M-n_0-j)/2}$ and so are equal (since the identity $|z_1 + \cdots + z_n| = |z_1| + \cdots + |z_n|$ implies that the non-zero complex numbers $z_1, \ldots, z_n$ have the same argument). Furthermore, as each entry of $\mathbf{u}$ appears in some monomial, none of the entries is zero.

If $M > n_0 + N$ then $u_1^{M-n_0-N} u_2 \cdots u_{N-j} u_k$ is a monomial that occurs in the Littlewood formula for $s_{\mathbf{n}'_j}(\mathbf{u})$ for $k = 1, \ldots, N$, and it follows that $u_1, \ldots, u_N$ are all equal. The edge case $M = n_0 + N$ must be dealt with separately, but in this case $s_{\mathbf{n}'_j}(\mathbf{u})$ is the sum of all monomials made up of $N - j$ distinct entries of $\mathbf{u}$ and the same conclusion holds. This contradicts the assumption that the entries of $\mathbf{u}$ are distinct, showing that $g[\mathbf{u}\mathbf{u}^*]$ is indeed positive definite.

Now suppose A has a row $\mathbf{v}^*$ with distinct entries; in particular, the diagonal entry v' in $\mathbf{v}$ is real and positive. Set $\mathbf{u} := \mathbf{v}/\sqrt{v'}$ and note that $A - \mathbf{u}\mathbf{u}^*$ has a zero row and column. If

$$p_t[B; R, \mathbf{d}] := t(d_0 \mathbf{1}_{N \times N} + d_1 B + \cdots + d_{n-1} B^{\circ(N-1)}) - B^{\circ(N+R)}$$

for any $\mathbf{d} = (d_0, \ldots, d_{N-1})$, then [3, (3.16)] yields the identity

$$\begin{aligned} p_t[A; M-N, \mathbf{c}] &= p_t[\mathbf{u}\mathbf{u}^*; M-N, \mathbf{c}] \\ &\quad + \int_0^1 (A - \mathbf{u}\mathbf{u}^*) \circ M p_{t/M}[\lambda A + (1-\lambda)\mathbf{u}\mathbf{u}^*; M-N, \mathbf{c}'] \, \mathrm{d}\lambda, \end{aligned}$$

where $\mathbf{c}' := (c_1, 2c_2, \ldots, (N-1)c_{N-1})$ and both terms on the right-hand side are positive semidefinite, by [3, (3.7)]. Thus, if $t = \mathcal{C}$ and $g_0(z) := z^{-n_0} g(z)$ then

$$g_0[A] = t^{-1} p_t[A; M-N-n_0, \mathbf{c}] \geqslant t^{-1} p_t[\mathbf{u}\mathbf{u}^*; M-N-n_0, \mathbf{c}] = g_0[\mathbf{u}\mathbf{u}^*],$$

which is positive definite by the previous rank-one case. Thus $g_0[A]$ is positive definite, which completes the proof of part (2) if $n_0 = 0$. Otherwise, $n_0 > 0$ and all the entries of $\mathbf{u}$ are non-zero by hypothesis. In this case, the following calculation implies that the conclusion of part (2) holds:

$$g[A] = A^{\circ n_0} \circ g_0[A] \geqslant (\mathbf{u}^{\circ n_0}(\mathbf{u}^{\circ n_0})^*) \circ g_0[A]$$

and the right-hand side is positive definite by applying Lemma 2.7 with $\mathbf{w} = \mathbf{u}^{\circ n_0}$ and $B = g_0[A]$.

The final assertion is immediate when $N = 1$, so we conclude by showing equality does not hold in (2.1) whenever $N > 1$ and $A \neq \mathbf{0}_{N \times N}$. As A is positive semidefinite, some entry x on the diagonal of A is positive. Suppose $\mathbf{u} \in (0, \sqrt{\rho})^N$ has distinct entries, one of which is $\sqrt{x}$. The matrix $g[\mathbf{u}\mathbf{u}^*]$ is positive definite by part (2), so $g(x) > 0$. Now equality holds in (2.1) if and only if $g[A] = 0$, but this working shows that at least one entry on the main diagonal of $g[A]$ is strictly positive. □

Analogously to Theorem 2.2, one has the following result for the other test sets $\mathcal{P}_\rho$ above.

Theorem 2.8. *Let f and $\mathcal{P}_0$ be as in Definition 1.1(1–3) and suppose $c_0, \ldots, c_{N-1} > 0$ and $c' > -\mathcal{C}^{-1}$, where $\mathcal{C}$ is as in (1.2).*

(1) Let $A \in \mathcal{P}_0$ and suppose $n_0 = 0$ if A has a zero row. The following are equivalent.
  - *(a) There exists a vector $\mathbf{u} \in [0, \sqrt{\rho}]^N$ with distinct entries such that $A \geqslant \mathbf{u}\mathbf{u}^*$ and $\mathbf{u}$ has a zero entry if and only if A has a zero row.*
  - *(b) All of the rows of A are distinct.*
  - *(c) The matrix $h[A]$ is positive definite.*
  - *(d) The inequality (2.1) is strict, that is, $f[A]$ is positive definite.*

  Moreover, (c) is equivalent to (d).

(2) Suppose $A \in \mathcal{P}_0$ has a row with distinct entries and $n_0 = 0$ if any entry in this row is zero. Then $g[A]$ is positive definite.

Furthermore, equality in (2.1) is attained on $\mathcal{P}_0$ if and only if either $N = 1$ and $A = \rho$, or $n_0 > 0$ and $A = \mathbf{0}_{N \times N}$.

This is proved presently.

2.1. **Stronger Schur monotonicity lemmas.** The proof of Theorem 2.8 relies on the following strengthening of the Schur monotonicity lemma above, Theorem 1.4. As above, for any set of real numbers S, denote by $S^N_{\neq}$ the set of all N-tuples of distinct elements of S and by $S^N_{<}$ its subset of N-tuples with increasing entries.

Theorem 2.9 (Schur strict monotonicity lemma 1)**.** *Fix an integer $N \geqslant 1$ and distinct N-tuples $\mathbf{m} = (m_0 < \cdots < m_{N-1})$ and $\mathbf{n} = (n_0 < \cdots < n_{N-1})$ in $\mathbb{R}^N_{<}$ such that $m_j \leqslant n_j$ for all j. The symmetric function*

$$f : (0, \infty)^N_{\neq} \to \mathbb{R}; \ \mathbf{u} \mapsto \frac{\det \mathbf{u}^{\circ \mathbf{n}}}{\det \mathbf{u}^{\circ \mathbf{m}}}$$

is strictly increasing in each coordinate and, for any $\rho \in (0, \infty)$, is bounded above by the constant $\rho^{|\mathbf{n}-\mathbf{m}|/2} V(\mathbf{n})/V(\mathbf{m})$ on $(0, \sqrt{\rho}]^N_{\neq}$. Furthermore, if $m_0 = n_0 = 0$ then f is well defined on $[0, \sqrt{\rho}]^N_{\neq}$ and these two properties hold there.

As announced in Theorem 1.5, an extension of this result holds for Schur polynomials. This will be stated and proved below, after the proof of the present theorem. We state and prove the extended result separately, because the behavior of f on the boundary of the orthant is somewhat delicate.

Proof. We begin by showing the result on $(0,\sqrt{\rho}]^N_{\neq}$ for arbitrary $\rho \in (0,\infty)$. The first step is to prove that f is strictly increasing in each coordinate, say in u_N. If not, then by Theorem 1.4, the function f is constant on $(u_1,\dots,u_{N-1})\times[x,x']$ for some x, $x' \in (0,\sqrt{\rho}]$ with $x' < x$, and we may shrink this interval to ensure that $u_j \notin [x,x']$ for $j \neq N$. The function

$$h : [\log x, \log x'] \to \mathbb{R};\ y \mapsto f(u_1,\dots,u_{N-1},e^y) = \frac{\sum_{j=0}^{N-1} g_j e^{n_j y}}{\sum_{j=0}^{N-1} g'_j e^{m_j y}}$$

is constant, and g_j and g'_j are generalized Vandermonde determinants in u_1, …, u_{N-1} for any j, so are non-zero. Since functions of the form $y \mapsto e^{\lambda y}$ are linearly independent for distinct real λ, this implies that $\mathbf{m} = \mathbf{n}$, contrary to our initial assumption.

Next, we note that any vector in $(0,\sqrt{\rho}]^N_{\neq}$ is coordinatewise bounded above (up to relabeling coordinates) by a vector of the form $\mathbf{v} = \mathbf{v}(\epsilon) := \sqrt{\rho}(1,\epsilon,\dots,\epsilon^{N-1})^T$, where $\epsilon \in (0,1)$. Hence, by 1.4 and (2.3),

$$\frac{\det(\mathbf{u}^{\circ\mathbf{n}})}{\det(\mathbf{u}^{\circ\mathbf{m}})} \leqslant \frac{\det(\mathbf{v}^{\circ\mathbf{n}})}{\det(\mathbf{v}^{\circ\mathbf{m}})} = \rho^{|\mathbf{n}-\mathbf{m}|/2}\frac{V(\epsilon^{\mathbf{n}})}{V(\epsilon^{\mathbf{m}})},$$

where $\epsilon^{\mathbf{n}} := (\epsilon^{n_{i-1}})_{i=1}^N$. It now suffices to show that $V(\epsilon^{\mathbf{n}})/V(\epsilon^{\mathbf{m}})$ is bounded above on $(0,1]$ by $V(\mathbf{n})/V(\mathbf{m})$. As this ratio is non-decreasing in ϵ, by Theorem 1.4, the least upper bound will equal the limit as $\epsilon \to 1^-$, if it exists, but this limit is as claimed, by L'Hôpital's rule. This shows the result on $(0,\sqrt{\rho}]^N_{\neq}$.

We now show that f is well defined and strictly increasing at $\mathbf{u} \in [0,\rho]^N_{\neq}$, where one coordinate of $\mathbf{u}$, say u_1, is zero. Then $m_0 = n_0 = 0$ by assumption, so the matrices $\mathbf{u}^{\circ\mathbf{m}}$ and $\mathbf{u}^{\circ\mathbf{n}}$ both have first row $\mathbf{e}_1 := (1,0,\dots,0)$. Now if $\mathbf{v}_1$ denotes the truncation of the vector $\mathbf{v}$ by removing its first coordinate, then

$$f(\mathbf{u}) = \frac{\det(\mathbf{u}_1^{\circ\mathbf{n}_1})}{\det(\mathbf{u}_1^{\circ\mathbf{m}_1})},$$

by expanding both determinants along their first rows; in particular, $f(\mathbf{u})$ is well defined. As $\mathbf{u}_1 \in (0,\sqrt{\rho}]^{N-1}_{\neq}$, the previous working implies that the right-hand side is strictly increasing in the coordinates of $\mathbf{u}_1$, that is, in all but the first coordinate of $\mathbf{u}$, and has the requisite upper bound.

Finally, say $\nu > 0$ and $\mathbf{v} := \mathbf{u} + \nu\mathbf{e}_1 \in (0,\infty)^N_{\neq}$; we wish to show that $f(\mathbf{v}) > f(\mathbf{u})$. We may assume that $\nu < \min\{u_2,\dots,u_N\}$, by transitivity and the previous working. Hence $\mathbf{v}(t) := \mathbf{u} + t\mathbf{e}_1$ is well defined for any $t \in [0,\mathbf{u}]$ and we see that

$$f(\mathbf{v}) = f\big(\mathbf{v}(\nu)\big) > f\big(\mathbf{v}(\nu/2)\big) > f\big(\mathbf{v}(t)\big) \qquad \text{for any } t \in (0,\nu).$$

Taking the limit as $t \to 0^+$, it follows that

$$f(\mathbf{v}) > \lim_{t\to 0^+} f\big(\mathbf{v}(t)\big) = f\big(\mathbf{v}(0)\big) = f(\mathbf{u}),$$

as desired. ◻

The next result is the analogue of Theorem 2.9 for ratios of Schur polynomials on the positive orthant. Given Theorem 1.4 and the preceding Theorem 2.9, it is natural to ask if strict monotonicity extends to the boundary of the orthant $[0,\infty)^N$. The following remark explains why this cannot happen and why Theorem 2.11 is the best possible result that may be obtained.

Remark 2.10. Here we describe two ways in which the coordinatewise monotonicity of the Schur-polynomial ratio $s_{\mathbf{n}}/s_{\mathbf{m}}$ fails to extend to strict monotonicity on all of $[0,\infty)^N \setminus \{\mathbf{0}\}$.

Suppose $\mathbf{m} \in (\mathbb{Z}_+)^N_<$ is such that $\mathbf{m} - \boldsymbol{\delta}$ has exactly $N-k$ non-zero entries, where $0 \leqslant k \leqslant N$. Then $s_{\mathbf{m}}(\mathbf{u})$ vanishes whenever $u_1 = \cdots = u_{k+1} = 0$, so for every vector $\mathbf{u} \in [0,\infty)^N$ with at least $k+1$ coordinates equal to zero. This is because every semi-standard Young tableau of shape $\mathbf{m} - \boldsymbol{\delta}$ necessarily contains at least one entry in the set $\{1,\ldots,k+1\}$. Thus, the ratio $s_{\mathbf{n}}(\mathbf{u})/s_{\mathbf{m}}(\mathbf{u})$ has domain of definition

$$\mathcal{U}_k := \{\mathbf{u} \in [0,\infty)^N : \text{at most } k \text{ coordinates of } \mathbf{u} \text{ are } 0\}, \tag{2.8}$$

as some of monomials in the Littlewood identity (2.4) must be non-zero when $\mathbf{u} \in \mathcal{U}_k$.

Even restricted to the domain $\mathcal{U}_k$, the function $\mathbf{u} \mapsto s_{\mathbf{n}}(\mathbf{u})/s_{\mathbf{m}}(\mathbf{u})$ need not be strictly increasing in each coordinate. If $\mathbf{n} - \boldsymbol{\delta}$ has exactly l zero entries and $\mathbf{m} - \boldsymbol{\delta}$ has exactly k zero entries, with $l < k$, then $s_{\mathbf{n}}(\mathbf{u})$ vanishes whenever $l+1$ or more coordinates of $\mathbf{u}$ are zero, so $f(\mathbf{u}) = s_{\mathbf{n}}(\mathbf{u})/s_{\mathbf{m}}(\mathbf{u})$ vanishes whenever $\mathbf{u}$ has between $l+1$ and k coordinates equal to 0. In particular, the function f cannot be strictly increasing on the collection of all such vectors.

Given the understanding of obstructions to strict monotonicity afforded by Remark 2.10, we now state and prove the strongest-possible monotonicity result for ratios of Schur polynomials on the closed orthant $[0,\infty)^N$.

Theorem 2.11 (Schur strict monotonicity lemma 2)**.** *Fix an integer $N \geqslant 1$ and distinct N-tuples $\mathbf{m} = (m_0 < \cdots < m_{N-1})$ and $\mathbf{n} = (n_0 < \cdots < n_{N-1})$ in $(\mathbb{Z}_+)^N_<$ such that $m_j \leqslant n_j$ for all j.*

(1) The symmetric function

$$f : (0,\infty)^N \to \mathbb{R}; \ \mathbf{u} \mapsto \frac{s_{\mathbf{n}}(\mathbf{u})}{s_{\mathbf{m}}(\mathbf{u})}$$

is strictly increasing in each coordinate and, for any $\rho \in (0,\infty)$, is bounded above by the constant $\rho^{|\mathbf{n}-\mathbf{m}|/2} V(\mathbf{n})/V(\mathbf{m}) = f(\sqrt{\rho}\mathbf{1}_N^T)$ on $(0,\sqrt{\rho}]^N$.

(2) Suppose that $n_j = j$ for $j = 0, \ldots, k-1$ but $n_k > k$, where $0 \leqslant k \leqslant N$ and the final condition holds vacuously if $k = N$. Then f is non-decreasing in each coordinate on its extended domain of definition $\mathcal{U}_k$ given by (2.8).

(3) Suppose that $m_j = m_j = j$ for $j = 0, \ldots, k-1$ and $m_k, n_k > k$, where $0 \leqslant k \leqslant N$ and the final condition holds vacuously if $k = N$. Then f is strictly increasing in each coordinate on $\mathcal{U}_k$.

Proof. While this result is similar to Theorem 2.9, its proof is slightly different: the first part uses Schur polynomials rather than exponentials, while the other parts use semi-standard Young tableaux.

Given $\mathbf{u} \in (0,\infty)^N_{\neq}$, we can expand both determinants along the Nth row to see that

$$f(\mathbf{u}) = \frac{\det(\mathbf{u}^{\mathbf{n}})}{\det(\mathbf{u}^{\mathbf{m}})} = \frac{\sum_{j=0}^{N-1} (-1)^{N+j+1} \det(\mathbf{u}_0^{\mathbf{n}^{(j)}}) u_N^{n_j}}{\sum_{j=0}^{N-1} (-1)^{N+j+1} \det(\mathbf{u}_0^{\mathbf{m}^{(j)}}) u_N^{m_j}},$$

where $\mathbf{u}_0 := (u_1, \dots, u_{N-1})$, $\mathbf{m}^{(j)}$ equals $\mathbf{m}$ with m_j removed, and similarly for $\mathbf{n}^{(j)}$. Dividing numerator and denominator by the Vandermonde determinant $V(\mathbf{u}_0)$, we see that

$$f(\mathbf{u}) = \frac{\sum_{j=0}^{N-1} (-1)^{N+j+1} s_{\mathbf{n}^{(j)}}(\mathbf{u}_0) u_N^{n_j}}{\sum_{j=0}^{N-1} (-1)^{N+j+1} s_{\mathbf{m}^{(j)}}(\mathbf{u}_0) u_N^{m_j}}. \tag{2.9}$$

As both sides are continuous on $(0,\infty)^N$, the identity (2.9) holds on the entire open orthant.

With (2.9) at hand, we turn to the proof of the theorem.

(1) By symmetry, it suffices to show $f(\mathbf{u})$ is strictly increasing as a function of u_N. If not, by Theorem 1.4 there exists a point $\mathbf{u} \in (0,\infty)^N$ and some $\epsilon > 0$ such that the function $x \mapsto f(\mathbf{u} + x\mathbf{e}_N)$ is constant, say with value c, on $[0,\epsilon]$, where $\mathbf{e}_N := (0, \dots, 0, 1)$. It follows via (2.9) that the function

$$g : x \mapsto \sum_{j=0}^{N-1} (-1)^{N+j+1} (s_{\mathbf{n}^{(j)}}(\mathbf{u}_0) x^{n_j} - c s_{\mathbf{m}^{(j)}}(\mathbf{u}_0) x^{m_j})$$

is identically zero on $[0,\epsilon]$. As g is a non-constant polynomial, since $\mathbf{m} \neq \mathbf{n}$, this yields a contradiction.

(2) Let $\mathbf{u} = (u_1, \dots, u_N)^T \in \mathcal{U}_k$ and suppose without loss of generality that $u_j = 0$ if $j \leqslant l$ and $u_j > 0$ if $j > l$, where $0 \leqslant l \leqslant k$. Given any $i \in \{1, \dots, N\}$ and $t > 0$, we wish to show that $f(\mathbf{u} + t\mathbf{e}_i) \geqslant f(\mathbf{u})$. If ϵ is positive and sufficiently small, we have that

$$\mathbf{u}_\epsilon := (\epsilon, \dots, \epsilon, u_{l+1}, \dots, u_N)^T = \epsilon \sum_{j=1}^{l} \mathbf{e}_j + \mathbf{u} \in (0,\infty)^N.$$

By Theorem 1.4, we know that

$$f(\mathbf{u}_\epsilon + t\mathbf{e}_i) - f(\mathbf{u}_\epsilon) \geqslant 0.$$

We have that $s_{\mathbf{m}}(\mathbf{u}_\epsilon + t\mathbf{e}_i) > 0$ and $s_{\mathbf{m}}(\mathbf{u}_\epsilon) > 0$, and the same holds for $s_{\mathbf{m}}(\mathbf{u} + t\mathbf{e}_i)$ and $s_{\mathbf{m}}(\mathbf{u})$, so we may take $\epsilon \to 0^+$ to obtain the desired inequality.

(3) Let $\mathbf{u}$, k, l, i and t be as for (2). We wish to show that $f(\mathbf{u} + t\mathbf{e}_i) > f(\mathbf{u})$.

We first suppose $i > l$ and so we may take $i = l+1$ by symmetry. We now use the Littlewood identity (2.4). As $u_1 = \cdots = u_l = 0$, the Schur polynomial $s_{\mathbf{n}}(\mathbf{u})$ is obtained by adding monomials corresponding to all semistandard Young tableau of shape $\mathbf{n} - \boldsymbol{\delta}$ that do not contain any of the labels $1, \dots, l$. Hence this sum can be written as a Schur polynomial in the reduced set of variables $\mathbf{u}' := (u_{l+1}, \dots, u_N)^T$ and the Littlewood sum involves tableau of the shape $\mathbf{n}' - \boldsymbol{\delta}$, where

$$\mathbf{n}' := (n_l - l, \dots, n_{N-1} - l).$$

In other words,

$$f(\mathbf{u}) = \frac{s_{\mathbf{n}}(\mathbf{u})}{s_{\mathbf{m}}(\mathbf{u})} = \frac{s_{\mathbf{n}'}(\mathbf{u}')}{s_{\mathbf{m}'}(\mathbf{u}')}$$

and this last ratio is strictly increasing in each of the variables in $\mathbf{u}'$, by part (1). Hence $f(\mathbf{u} + t\mathbf{e}_i) > f(\mathbf{u})$ for any $i > l$.

The remaining case is when $1 \leqslant i \leqslant l$, so by symmetry we may assume $i = l$. We proceed similarly to the previous case, now summing over all semistandard

Young tableaux which do not contain the labels $1, \ldots, l-1$, and form the Schur polynomials $s_{\mathbf{n}''}(u_l, \mathbf{u}')$ and $s_{\mathbf{m}''}(u_l, \mathbf{u}')$, where $\mathbf{u}' \in (0,\infty)^{N-l}$ is as in the previous paragraph,

$$\mathbf{m}'' := (m_{l-1} - l + 1, \ldots, m_{N-1} - l + 1)$$
$$\text{and} \quad \mathbf{n}'' := (n_{l-1} - l + 1, \ldots, n_{N-1} - l + 1).$$

As above, we have that

$$f(\mathbf{u}) = \frac{s_{\mathbf{n}}(\mathbf{u})}{s_{\mathbf{m}}(\mathbf{u})} = \frac{s_{\mathbf{m}''}(u_l, \mathbf{u}')}{s_{\mathbf{n}''}(u_l, \mathbf{u}')}.$$

Hence if the function $x \mapsto f(\mathbf{u} + x\mathbf{e}_l)$ is not strictly monotone on $[0,t]$ then there exist a, $b \in (0,t)$ with $a < b$ such that the function

$$g : [a,b] \to \mathbb{R};\ x \mapsto f(0, \ldots, 0, x, u_{l+1}, \ldots, u_N) = \frac{s_{\mathbf{m}''}(x, \mathbf{u}')}{s_{\mathbf{n}''}(x, \mathbf{u}')}$$

is constant. However this contradicts part (1). $\square$

The following result is used to show that (b) implies (a) in Theorem 2.8(1). The need to ensure the vector $\mathbf{u}$ has non-negative entries means that the elementary argument used in the proof of Theorem 2.2(1) does not translate to this setting.

Theorem 2.12. *Let $A \in \mathcal{P}_N\big([0,\infty)\big)$, where $N \geqslant 1$, and suppose the rows of A are distinct. There exists a vector $\mathbf{u} \in [0,\infty)^N$ with distinct entries such that $A \geqslant \mathbf{u}\mathbf{u}^T$ and $\mathbf{u}$ has a zero entry if and only if A has a zero row.*

The condition that A must have distinct rows in Theorem 2.12 and for corresponding implication in Theorem 2.2(1) is necessary as well as sufficient, as the rank-one case shows. If $A = \mathbf{v}\mathbf{v}^*$ for some $\mathbf{v} \in \mathbb{C}^N$, and $\mathbf{u} \in \mathbb{C}^N$ is such that $A \geqslant \mathbf{u}\mathbf{u}^*$, then $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, by Proposition 2.4. If A has two equal rows, then two coordinates of $\mathbf{v}$ are equal, whence the same holds for $\mathbf{u}$, and so the conclusion of Theorem 2.12 and the implication in Theorem 2.2(1) do not hold.

We note that the full-rank case of Theorem 2.12 is immediate, either by Proposition 2.4 or simply because $A \geqslant \lambda_1 \mathrm{Id}_N \geqslant \lambda_1 \mathbf{u}\mathbf{u}^T$ for any unit vector $\mathbf{u}$, where λ_1 is the smallest eigenvalue of A and Id_N is the $N \times N$ identity matrix. Similarly, the rank-one case is immedate.

Lemma 2.13. *Let A be a real symmetric $N \times N$ matrix, where $N \geqslant 1$, and suppose the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{R}^N$ are such that $A \geqslant \mathbf{u}_j\mathbf{u}_j^T$ for all j. If $\mathbf{u} = \sum_{j=1}^m \lambda_j \mathbf{u}_j$ is an arbitrary convex combination of $\mathbf{u}_1, \ldots, \mathbf{u}_m$, so that $\lambda_j \in [0,1]$ for all j and $\sum_{j=1}^m \lambda_j = 1$, then $A \geqslant \mathbf{u}\mathbf{u}^T$.*

Proof. We recall the following elementary Schur-complement property: for any $\mathbf{v} \in \mathbb{R}^N$ we have the equivalence

$$A \geqslant \mathbf{v}\mathbf{v}^T \quad \iff \quad \begin{pmatrix} A & \mathbf{v} \\ \mathbf{v}^T & 1 \end{pmatrix} \geqslant 0.$$

Replacing $\mathbf{v}$ by $\mathbf{u}_j$ in the right-hand side, multiplying through by λ_j and summing over j gives the result. $\square$

Proof of Theorem 2.12. We have the spectral decomposition $A = \sum_{j=1}^m \mathbf{u}_j \mathbf{u}_j^T$, where the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{R}^N$ are orthogonal and non-zero. We have that $A \geqslant \mathbf{u}_j \mathbf{u}_j^T$ for all j and, by Lemma 2.13, it suffices to show that some convex combination of these vectors has non-negative and distinct entries, with a zero appearing if and only if A has a zero row.

We first note that, for any pair of distinct indices j and k in $\{1, \ldots, N\}$, there exists some eigenvector $\mathbf{u}_i$ whose jth and kth coordinates are distinct. If this does not hold for some such pair then the $\{j,k\} \times \{j,k\}$ principal submatrix of A has the form $\begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix}$ for some $\alpha > 0$. However, if $l \not\in \{j,k\}$ then, up to a simultaneous re-indexing of rows and columns, the $\{j,k,l\} \times \{j,k,l\}$ minor of A is such that

$$0 \leqslant \det \begin{pmatrix} \alpha & \alpha & a_{jl} \\ \alpha & \alpha & a_{kl} \\ a_{jl} & a_{kl} & a_{ll} \end{pmatrix} = -\alpha(a_{jl} - a_{kl})^2 \leqslant 0.$$

From this it follows that $a_{jl} = a_{kl}$ for all $l \not\in \{j,k\}$, which shows that the jth and kth rows of A are equal. This contradiction establishes our first observation.

We next consider the affine map

$$\Psi : \mathbb{R}^{m-1} \to \mathbb{R}^N;\ \mathbf{c} = (c_2, \ldots, c_m) \mapsto \mathbf{u}_1 + \sum_{j=2}^m c_j \mathbf{u}_j.$$

and note that $\Psi(\mathbf{c})$ has distinct coordinates if and only if $p(\mathbf{c}) \neq 0$, where

$$p(\mathbf{c}) := \prod_{1 \leqslant k < l \leqslant N} \big(\Psi(\mathbf{c})_l - \Psi(\mathbf{c})_k\big) = \prod_{1 \leqslant k < l \leqslant N} \Big((\mathbf{u}_1)_l - (\mathbf{u}_1)_k + \sum_{j=2}^m c_j \big((\mathbf{u}_j)_l - (\mathbf{u}_j)_k\big)\Big).$$

Thus, p is a polynomial in $c_2, \ldots, c_m$ that is a product of non-zero factors that are either linear or constant, by the first observation. It follows that $\Psi(\mathbf{c})$ has distinct coordinates for all $\mathbf{c}$ not in $p^{-1}(\{0\})$, which has zero Lebesgue measure.

We now assume that A is irreducible, which implies that A does not have a zero row. By the Perron–Frobenius theorem, we may take $\mathbf{u}_1$ to be the Perron eigenvector, which lies in $(0,\infty)^N$. We can then choose a positive but sufficiently small ϵ so that $\Psi(\mathbf{c})$ has all coordinates positive whenever $\mathbf{c} \in (0,\epsilon)^{m-1}$. Since this set has positive Lebesgue measure, there exists some $\mathbf{c} \in (0,\epsilon)^{m-1} \setminus p^{-1}(\{0\})$ and $\mathbf{u}' = \Psi(\mathbf{c})$ has positive and distinct coordinates. Finally, we let $\mathbf{u} = \beta \mathbf{u}'$, where $\beta = 1/(1 + c_2 + \cdots + c_m)$.

We next suppose that $A = A_1 \oplus A_2$, where A_1 and A_2 have vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ with positive entries such that $A_1 \geqslant \mathbf{u}_1 \mathbf{u}_1^T$ and $A_2 \geqslant \mathbf{u}_2 \mathbf{u}_2^T$. A short calculation shows that

$$\begin{aligned} &\begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix} \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} - \begin{bmatrix} \mu_1 \mathbf{u}_1 \\ \mu_2 \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \mu_1 \mathbf{u}_1^T & \mu_2 \mathbf{u}_2^T \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\ &\qquad = \mathbf{x}_1^T A_1 \mathbf{x}_1 - 2\mu_1^2 (\mathbf{x}_1^T \mathbf{u}_1)^2 + \mathbf{x}_2^T A_2 \mathbf{x}_2 - 2\mu_2^2 (\mathbf{x}_2^T \mathbf{u}_2)^2 + (\mu_1 \mathbf{x}_1^T \mathbf{u}_1 - \mu_2 \mathbf{x}_2^T \mathbf{u}_2)^2, \end{aligned}$$

so any $\mathbf{u}$ of the form $\mu_1 \mathbf{u}_1 \oplus \mu_2 \mathbf{u}_2$, with $\mu_1^2 < 1/2$ and $\mu_2^2 < 1/2$, is such that $A \geqslant \mathbf{u}\mathbf{u}^T$. To ensure that $\mathbf{u}$ has distinct and positive entries, we fix suitable positive μ_1 and take μ_2 positive but sufficiently small to ensure that every entry of $\mu_2 \mathbf{u}_2$ is smaller than every every of $\mu_1 \mathbf{u}_1$. Since A may be written, up to a simultaneous re-indexing of rows and columns, in Frobenius normal form as a block-diagonal sum of irreducible matrices and at most one zero, the result follows. If A has a zero row then zero appears in the appropriate coordinate of $\mathbf{u}$, and otherwise all the entries of $\mathbf{u}$ are positive. $\square$

We now use Theorem 2.9 to show that strict positive definiteness holds generically for polynomial positivity preservers.

Proof of Theorem 2.8. The proof of part (1) is similar to that of the corresponding parts of the proof of Theorem 2.2, with a few minor modifications. To see that (a) implies (c) here, we note first that if A has rank one then we may assume $A = \mathbf{u}\mathbf{u}^*$. As before, the matrix $h[\mathbf{u}\mathbf{u}^*]$ is the sum of N rank-one matrices and their column spaces are spanned by $\{\mathbf{u}^{\circ n_0}, \ldots, \mathbf{u}^{\circ n_{N-1}}\}$. This set is linearly independent, as the generalized Vandermonde determinant of $\mathbf{u}^{\circ \mathbf{n}} = (u_i^{n_{j-1}})_{i,j=1}^N$ is non-zero if $u_1, \ldots, u_N > 0$ and $n_0 < n_1 < \cdots < n_{N-1}$ [10, Example XIII.8.1]. In the case where $u_i = 0$ for some i then the ith row of the matrix $\mathbf{u}^{\circ \mathbf{n}}$ equals $(1, 0, \ldots, 0)$ and expanding the determinant along this row reduces the matter to the former situation. When $A = \mathbf{u}\mathbf{u}^*$ we are now done; otherwise we are in the setting of Definition 1.1(3) and we emply Loewner monotonicity as in the proof of Theorem 2.2.

The arguments to show that (c) implies (b) and that (c) and (d) are equivalent are unchanged and the fact that (b) implies (a) follows immediately from Theorem 2.12.

For part (2), we first suppose as in the proof of Theorem 2.2(2) that $A = \mathbf{u}\mathbf{u}^*$ has rank one, and $\det g[\mathbf{u}\mathbf{u}^*] = 0$. By suitably specializing (2.6), we see that

$$\sum_{j=0}^{N-1} \frac{\det(\mathbf{u}^{\circ \mathbf{n}_j})^2}{c_j} = \mathcal{C} \det(\mathbf{u}^{\circ \mathbf{n}})^2 = \det(\mathbf{u}^{\circ \mathbf{n}})^2 \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_j},$$

where $\mathbf{n}_j$ and $\mathbf{n}$ are as in (1.3). Moreover, by the hypotheses we have $\det(\mathbf{u}^{\circ \mathbf{n}}) \neq 0$. Thus,

$$\sum_{j=0}^{N-1} \frac{\det(\mathbf{u}^{\circ \mathbf{n}_j})^2}{c_j \det(\mathbf{u}^{\circ \mathbf{n}})^2} = \sum_{j=0}^{N-1} \frac{\rho^{|\mathbf{n}_j - \mathbf{n}|} V(\mathbf{n}_j)^2}{c_j V(\mathbf{n})^2}. \tag{2.10}$$

By Theorem 2.9, each summand on the left is strictly less than the corresponding one on the right whenever $\mathbf{u} \in [0, \rho]^N_{\neq}$ and so $g[\mathbf{u}\mathbf{u}^*]$ is positive definite. The remaining case occurs when $\mathbf{u}$ has a zero entry, in which case $n_0 = 0$ and $\mathbf{n}_0 := (n_1 < \cdots < n_{N-1} < M)$ lies in $(0, \infty)^N_<$. Then $\mathbf{u}^{\circ \mathbf{n}_0}$ has a zero row and therefore zero determinant, whereas if $\mathbf{m}$ is such that $m_0 = 0$ then $\det(\mathbf{u}^{\circ \mathbf{m}})^2 = \det(\mathbf{u}_\times^{\circ \mathbf{m}'})^2$, where $\mathbf{u}_\times$ is $\mathbf{u}$ with the zero entry removed, so that $\mathbf{u}_\times \in (0, \sqrt{\rho}]^{N-1}_{\neq}$, and $\mathbf{m}' := (m_1 < \cdots < m_{N-1})$. Hence

$$\frac{\det(\mathbf{u}^{\circ \mathbf{n}_j})^2}{\det(\mathbf{u}^{\circ \mathbf{n}})^2} = \frac{\det(\mathbf{u}_\times^{\circ \mathbf{n}'_j})^2}{\det(\mathbf{u}_\times^{\circ \mathbf{n}'})^2} \leqslant \rho^{|\mathbf{n}'_j - \mathbf{n}'|} \frac{V(\mathbf{n}'_j)^2}{V(\mathbf{n}')^2} \leqslant \rho^{|\mathbf{n}_j - \mathbf{n}|} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2}$$

for $j = 1, \ldots, N-1$. (The final inequality holds because if $m_k \leqslant n_k$ for $k = 0, \ldots, N-1$ and $m_0 = n_0$ then $V(\mathbf{m})/V(\mathbf{m}') \leqslant V(\mathbf{n})/V(\mathbf{n}')$.) Thus the equality (2.10) fails to hold once again and we see that $g[\mathbf{u}\mathbf{u}^*]$ is positive definite.

The proof for general A is identical to that part of the proof of Theorem 2.2(2), and the same holds for the proof of the final part. $\square$

We conclude this section with the following observation.

Remark 2.14. As noted in the introduction, and explained by Loewner (see Horn's thesis [12]), a necessary condition for any smooth function $f : (0, \rho) \to \mathbb{R}$ to preserve positive semidefiniteness when applied entrywise to matrices in $\mathcal{P}_n\big((0,\rho)\big)$ is that f, f', ..., $f^{(n-1)}$ must be non-negative on $(0, \rho)$.

Now a natural question is as follows: is Loewner's necessary condition also sufficient? For power functions of the form $p(x) \equiv x^\alpha$ then this condition is indeed sufficient, as shown by FitzGerald and Horn [9]. However, this necessary condition is not sufficient in general.

From Theorem 1.2 with c_0, $c_1 > 0$, $\mathbf{n} = (0, 1)$ and $M = 2$, we see that the quadratic polynomial $p(x) = c_0 + c_1 x + c' x^2$ preserves positive semidefiniteness on $\mathcal{P}_2\big((0, 1)\big)$ if and only if

$$c' \geqslant \frac{-c_0 c_1}{4c_0 + 2c_1}. \tag{2.11}$$

On the other hand, Loewner's result provides a lower bound for the coefficient c' which can be computed as follows. As p' is non-negative on $[0, 1]$, we have that $2c'x + c_1 \geqslant 0$ for any $x \in [0, 1]$, so $c' \geqslant -c_1/2$. If $x \in [0, 1]$ then this implies that

$$p(x) = c'x^2 + c_1 x + c_0 \geqslant \frac{-c_1}{2} x^2 + c_1 x + c_0 = c_1 x\Big(1 - \frac{x}{2}\Big) + c_0 \geqslant c_0 > 0.$$

(Alternatively, one may observe that p is non-decreasing on $[0, 1]$, since $p'(x) > 0$ for any choice of $x \in (0, 1)$, nd so f is bounded below by $p(0) = c_0$.) Thus, the lower bound on c' to ensure that Loewner's condition holds is $-c_1/2$, which is strictly smaller than the bound in (2.11). Hence Loewner's necessary condition is not sufficient, even for polynomial functions. We thank Siddhartha Sahi for raising this question.

3. Rank properties on strata

Theorem 2.2 provides readily verified criteria to classify when a matrix $A \in \mathcal{P}_N\big(\overline{D}(0, \rho)\big)$ is such that $f[A]$ is non-singular, and also implies that there are at most two choices of A for which $f[A]$ is zero. This section significantly refines both of these results, by provding a method to compute the rank of the matrix $f[A]$. A tool developed in previous work [4, 5], a Schubert cell-type stratification of the cone $\mathcal{P}_N(\mathbb{C})$, turns out to be crucial: the rank of A depends solely on which stratum A lies in. We begin by recalling the relevant notions.

Definition 3.1. Given an integer $N \geqslant 2$, denote by $(\Pi_N, \preccurlyeq)$ the poset of all partitions of the set $\{1, \dots, N\}$, ordered such that $\pi' \preccurlyeq \pi$ if and only if π is a refinement of π': every set in π is a subset of some set in π'.

We let $|\pi|$ denote the number of sets in π and $|I|$ denote the number of elements in a set $I \in \pi$. We insist that $N \geqslant 2$ throughout this section to avoid uninteresting trivialities.

Given non-empty sets I, $J \subseteq \{1, \dots, N\}$ and an $N \times N$ complex matrix A, we let $A_{I\times J}$ denote the $|I| \times |J|$ submatrix of A with row indices in I and column indices in J.

Proposition 3.2 ([5, Propositions 2.4 and 2.6])**.** *Fix an integer $N \geqslant 2$ and a multiplicative subgroup $G \leqslant \mathbb{C}^\times$.*

(1) *For any $N \times N$ complex matrix A, there exists a unique minimal partition $\pi \in \Pi_N$ such that the entries of the submatrix $A_{I\times J}$ lie in a single G-orbit for all I, $J \in \pi$.*

In particular, there exists an $|\pi| \times |\pi|$ complex matrix C such that A is a block matrix with $A_{I\times J} = c_{IJ}\mathbf{1}_{|I|\times|J|}$ for all I, $J \in \pi$. Moreover, A and C have equal rank.

(2) *There is a stratification of the set of $N \times N$ complex matrices,*

$$\mathbb{C}^{N\times N} = \bigsqcup_{\pi\in\Pi_N} \mathcal{S}_\pi^G,$$

where the stratum

$$\mathcal{S}_\pi^G := \{A \in \mathbb{C}^{N\times N} : \pi^G(A) = \pi\}$$

and $\pi^G(A)$ is the partition from (1). The set $\mathcal{S}_\pi^G$ has closure

$$\overline{\mathcal{S}_\pi^G} = \bigsqcup_{\pi'\preccurlyeq\pi} \mathcal{S}_{\pi'}^G \tag{3.1}$$

when $\mathbb{C}^{N\times N}$ is equipped with its usual topology.

Using the above isogenic block stratification, we now refine the results in the preceding section. We let $\pi_\vee := \{\{1\},\ldots,\{N\}\}$ denote the maximum element of the lattice of partitions Π_N and we work henceforth with $\pi^G(A)$ only for the trivial subgroup $G = \{1\}$. To lighten notation, we write $\mathcal{S}_\pi^{\{1\}} = \mathcal{S}_\pi$ and $\pi^{\{1\}}(A) = \pi(A)$.

Theorem 3.3. *Let f be as in Definition 1.1(4), so that n_0 and M are non-negative integers and $n_j = n_0 + j$ for $j = 0, \ldots, N-1$, where $N \geqslant 2$. Suppose $c_0, \ldots, c_{N-1} > 0$ and $c' > -\mathcal{C}^{-1}$, where $\mathcal{C}$ is as in (1.2). Let $A \in \mathcal{P}_N\big(\overline{D}(0,\rho)\big)$, with $n_0 = 0$ if A has a zero row. Then*

$$\operatorname{rank} f[A] = \operatorname{rank} h[A] = |\pi(A)|, \tag{3.2}$$

while $\operatorname{rank} g[A] = |\pi(A)|$ *if*

(a) *$A \notin \mathcal{S}_{\pi_\vee}$ or*

(b) *$A \in \mathcal{S}_{\pi_\vee}$ and A has a row with distinct entries, with $n_0 = 0$ if any entry in this row is zero.*

In particular, for any partition $\pi \in \Pi_N$ and any positive semidefinite matrix $A \in \mathcal{S}_\pi$, both $f[A]$ and $h[A]$ have rank equal to the number of blocks in π (as long as $n_0 = 0$ whenever A has a zero row).

When $N > 2$, the identity matrix is an element of $\mathcal{S}_{\pi_\vee}$ which has no row with distinct entries. It follows that Theorem 3.3(b) is a sufficient but not necessary condition for the rank of $g[A]$ to equal $|\pi(A)|$.

Remark 3.4. Theorem 3.3 is intertwined with Theorem 2.2 in two ways. First, the matrices $f[A]$ and $h[A]$ have rank equal to $|\pi(A)|$, so are never zero. Second, the four equivalent assertions in Theorem 2.2(1) are also equivalent to the following:

(e) The matrix A lies in $\mathcal{S}_{\pi_\vee}$, the top cell of the stratification.

Since $\mathcal{S}_{\pi_\vee}$ is dense in $\mathcal{P}_N$, we see again that $f[A]$ is positive definite for generic A.

The proof of Theorem 3.3 employs the block decomposition of Proposition 3.2, as well as the inflation and compression operators for the entrywise calculus studied elsewhere [5, Section 4], [7]. We begin by recalling these operators and some basic properties.

Definition 3.5 ([5, Definition 4.1])**.** Suppose $\pi = \{I_1, \ldots, I_m\} \in \Pi_N$ for some $N \geqslant 1$. Given $i, j \in \{1, \ldots, m\}$, we let E_{ij} denote the elementary $m \times m$ matrix with (i,j) entry equal to 1 and all other entries 0, and let $\mathbf{1}[I_i \times I_j]$ denote the $N \times N$ matrix with 1 in each entry of the $I_1 \times I_j$ block and 0 elsewhere.

(1) Define the linear *inflation* map

$$\Sigma_\pi^\uparrow : \mathbb{C}^{m\times m} \to \mathbb{C}^{N\times N};\ E_{ij} \mapsto \mathbf{1}[I_i \times I_j] \quad (i, j = 1, \ldots, m)$$

and note that the range of $\Sigma_\pi^\uparrow$ is $\overline{\mathcal{S}_\pi}$.

(2) Define the linear *compression* map

$$\Sigma_\pi^\downarrow : \mathbb{C}^{N\times N} \to \mathbb{C}^{m\times m};\ \Sigma_\pi^\downarrow(A)_{ij} := \frac{1}{|I_i|\,|I_j|} \sum_{p\in I_i, q\in I_j} a_{pq} \quad (i, j = 1, \ldots, m),$$

so that the image $\Sigma_\pi^\downarrow(A) = (b_{ij})_{i,j=1}^m$ is such that b_{ij} is the average of the entries in $A_{I_i\times I_j}$.

The operators $\Sigma_\pi^\uparrow$ and $\Sigma_\pi^\downarrow$ are well behaved with respect to the entrywise calculus:

Theorem 3.6 ([5, Theorem 4.2])**.** *Let $\overline{\mathcal{S}_\pi}$ and $\mathbb{C}^{m\times m}$ each be equipped with the entrywise product, so that the units for this product are $\mathbf{1}_{N\times N}$ and $\mathbf{1}_{m\times m}$, respectively. The maps*

$$\Sigma_\pi^\downarrow : \overline{\mathcal{S}_\pi} \to \mathbb{C}^{m\times m} \qquad \text{and} \qquad \Sigma_\pi^\uparrow : \mathbb{C}^{m\times m} \to \overline{\mathcal{S}_\pi}$$

are mutually inverse, rank-preserving isomorphisms of unital commutative $*$-algebras. Moreover, $A \in \overline{\mathcal{S}_\pi}$ is positive semidefinite if and only if $\Sigma_\pi^\downarrow(A)$ is.

To summarize the preceeding material in plain language, the main picture adapted to the trivial group $G = \{1\}$ is the following: a real symmetric matrix $A = (a_{ij})_{i,j=1}^N$ respects the block structure associated to a partition $\pi = \{I_1, \ldots, I_m\}$ if the entry a_{ij} is independent of $i, j \in I_k$ for some k. The compression map collapses each cell I_k to a single entry, projecting the matrix A to the $m \times m$ matrix with entries given by the constant values along the fibres of the projection map. The reverse inflation map restores the repetitions of matrix entries in A. These are linear, mutually inverse maps that preserve rank, positive semidefiniteness, and the entrywise product.

With these tools at hand, we proceed.

Proof of Theorem 3.3. As f is equal to h when $c' = 0$, we need only consider $f[A]$ and $g[A]$. For convenience, we let $\pi := \pi(A)$.

Suppose $A = a\mathbf{1}_{N\times N}$ for some $a \geqslant 0$. Then $g(a) > 0$ if $a > 0$, by the last part of the proof of Theorem 2.2, and $g(0) = c_0 > 1$ when $A = \mathbf{0}_{N\times N}$. Since $f(a) \geqslant g(a)$, the rank-one case is established.

Next, we note that $\pi = \pi_\vee$ if and only if the rows of A are distinct, so the result follows from Theorem 2.2 in this case.

Otherwise, we suppose that $m := |\pi|$ is strictly between 1 and N. The matrix $g[A]$ is positive semidefinite, by Theorem 1.2, and therefore, if $B := \Sigma_\pi^\downarrow(A)$, so is $g[B] = \Sigma_\pi^\downarrow\big(g[A]\big)$, by Theorem 3.6. If $g[B]$ is positive definite then so is $f[B]$, since $f[B] \geqslant g[B]$, and therefore both of these matrices have rank m. Another application of Theorem 3.6 then gives that the matrices $g[A] = \Sigma_\pi^\uparrow\big(g[B]\big)$ and $f[A] = \Sigma_\pi^\uparrow\big(g[A]\big)$ have rank m, as required.

It thus remains to show that $g[B]$ is positive definite. For this, we will use Lemma 2.3 with $C = h[B]$ and $D = \mathcal{C}_m^{-1} B^{\circ M}$ for a suitable positive scalar $\mathcal{C}_m$. The Schur product theorem gives that C and D are both positive semidefinite. Furthermore, as B has distinct rows and $n_0 = 0$ if B has a zero row, Theorem 2.2(1) gives that

$$C_0 := \sum_{j=0}^{m-1} c_j B^{\circ n_j}$$

is positive definite. Since $h[B] = C \geqslant C_0$, we have that $h[B]$ is positive definite as well. We now let

$$h_m(z) := \sum_{j=0}^{m-1} c_{N-m+j} z^j$$

and let $\mathcal{C}_m$ equal $\mathcal{C}$ as in (1.2) but with N replaced with m, $\mathbf{n} = (0, \ldots, m-1)$, M replaced by $M - n_0 - N + m$ and $\mathbf{c} = (c_{N-m}, \ldots, c_{N-1})$, so that

$$\mathcal{C}_m = \sum_{j=0}^{m-1} \binom{M-N+m}{j}^2 \binom{M-N+m-j-1}{m-j-1}^2 \frac{\rho^{M-n_0-N+m-j}}{c_{N-m+j}}.$$

By Corollary 2.1,

$$h_m[B] \geqslant \mathcal{C}_m^{-1} B^{\circ(M-n_0-N+m)}$$

and therefore, by the Schur product theorem, we have that

$$C = h[B] \geqslant B^{\circ(n_0+N-m)} \circ h_m[B] \geqslant \mathcal{C}_m^{-1} B^{\circ M} = D. \tag{3.3}$$

Moreover, $\mathcal{C}_m < \mathcal{C}_{m+1} \leqslant \mathcal{C}_N$, where the constant $\mathcal{C}_N$ is precisely $\mathcal{C}$ as in (1.2), since

$$\binom{M-N+m+1}{j+1}\binom{M-N+m}{j}^{-1} = \frac{M-N+m+1}{j+1} > 1 \qquad \text{for } j = 0, \ldots, m-1.$$

Hence

$$g[B] = h[B] - \mathcal{C}_N^{-1} B^{\circ M} = C - tD,$$

where $t = \mathcal{C}_m / \mathcal{C}_N \in (0,1)$. By Lemma 2.3, this has the same rank as $C = h[B]$, which was shown above to be positive definite. This completes the proof. □

As in the previous section, there is an analogue of Theorem 3.3 that holds in the other cases set out in Definition 1.1, in the same way that Theorem 2.2 becomes Theorem 2.8.

Theorem 3.7. *Let f and $\mathcal{P}_0$ be as in Definition 1.1(1–3) and suppose $c_0, \ldots, c_{N-1} > 0$ and $c' > -\mathcal{C}^{-1}$, where $\mathcal{C}$ is as in (1.2). Let $A \in \mathcal{P}_0$ and suppose $n_0 = 0$ if A has a zero row. The conclusions of Theorem 3.3 hold once again.*

Proof. The proof proceeds in the same manner as that of Theorem 3.3, with appeals to Theorem 2.2 replaced by employing Theorem 2.8 in its place. As there, it suffices to assume that $m = |\pi|$ is strictly between 1 and N, and show that the positive semidefinite matrix $g[B]$ is in fact positive definite, where $B = \Sigma_\pi^\downarrow(A)$. Here are the steps of the proof, modified to work for Settings (1)–(3) in Definition 1.1.

To see that $C := h[B]$ and $D := \mathcal{C}_m B^{\circ M}$ are positive semidefinite, where $\mathcal{C}_m$ is a positive constant to be determined, we use the result of FitzGerald and Horn [9] that the function $x \mapsto x^\alpha$ acts entrywise to preserve positive semidefiniteness on $N \times N$ real matrices with positive entries whenever $\alpha \in \mathbb{Z}_+ \cup [N-2, \infty)$. As above, the matrix C_0 is positive definite, now by Theorem 2.8(1), and hence so is C.

We now let

$$h_m(z) := \sum_{j=N-m}^{N-1} c_j z^{n_j}$$

and take $\mathcal{C}_m$ to be as in (1.2) with $N = m$, $\mathbf{n} = (n_{N-m}, \ldots, n_{N-1})$, M unchanged and $\mathbf{c} = (c_{N-m}, \ldots, c_{N-1})$. Once again using the result from [9], together with Corolllary 2.1, we have that

$$C := h_N[B] \geqslant h_m[B] \geqslant \mathcal{C}_m^{-1} B^{\circ m} =: D.$$

We now claim that $\mathcal{C}_m < \mathcal{C}_N$; given this, the proof is then completed as for Theorem 3.3.

To show this claim, we note that

$$\mathcal{C}_N = \sum_{j=0}^{N-1} b_j^2 \frac{\rho^{M-n_j}}{c_j} \qquad \text{and} \qquad \mathcal{C}_m = \sum_{j=N-m}^{N-1} a_j^2 \frac{\rho^{M-n_j}}{c_j},$$

where

$$b_j = \prod_{k \in \{0,\ldots,N-1\} \setminus \{j\}} \Big(\frac{M - n_k}{n_j - n_k} \Big)^2 \quad \text{and} \quad a_j = \prod_{k \in \{N-m,\ldots,N-1\} \setminus \{j\}} \Big(\frac{M - n_k}{n_j - n_k} \Big)^2$$

Hence

$$\mathcal{C}_N - \mathcal{C}_m \geqslant \sum_{j=N-m}^{N-1} (b_j^2 - a_j^2) \frac{\rho^{M-n_j}}{c_j},$$

so it suffices to show that $b_j^2 > a_j^2$ for $j \geqslant N - m$. This holds because

$$\frac{b_j^2}{a_j^2} = \prod_{k=0}^{N-m-1} \Big(\frac{M - n_k}{n_j - n_k} \Big)^2 > 1$$

since $M > n_j > n_k$ for $j \geqslant N - m$. $\square$

4. Continuity of the Rayleigh quotient on strata

As well as its relevance for calculating the rank, as seen in Section 3, it was shown in [3] that the constant-block stratification of Proposition 3.2 plays a crucial role in studying the following Rayleigh quotient:

$$R = R(A, \mathbf{u}, \mathbf{c}, M) := \frac{\mathbf{u}^* A^{\circ M} \mathbf{u}}{\mathbf{u}^* (c_0 A^{\circ n_0} + c_1 A^{\circ n_1} + \cdots + c_{N-1} A^{\circ n_{N-1}}) \mathbf{u}}. \tag{4.1}$$

This Rayleigh quotient is connected to the isogenic stratification of the cone $\mathcal{P}_n(\mathbb{C})$, and this theme was developed in [3, Sections 4 and 5] (for consecutive non-negative integer exponents) and later in [15, Section 11] (for more general exponents).

The optimisation of (4.1) gives an alternative approach for establishing Theorem 1.2. Namely, if the coefficients $c_0, \ldots, c_{N-1}$ are positive and the exponents $n_0, \ldots, n_{N-1}$ are non-negative then, given any $A \in \mathcal{P}_N\big((0,\rho)\big)$, or $A \in \mathcal{P}_N(\mathbb{C})$ if the exponents are integral, there exists a constant $\mathcal{C}' \geqslant 0$ such that

$$A^{\circ M} \leqslant \mathcal{C}' \sum_{j=0}^{N-1} c_j A^{\circ n_j} = \mathcal{C}' h[A]$$

The smallest such constant $\mathcal{C}_R = \mathcal{C}_R(A, h, M)$ may be regarded as a Rayleigh quotient, and it was shown in [3, Remark 4.6] and [15, Proposition 11.1] that

$$\mathcal{C}_R = \varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}), \tag{4.2}$$

where $B^{\dagger/2} := (B^\dagger)^{1/2}$ for any square matrix B, with $B^\dagger$ the Moore–Penrose pseudo-inverse of B, and $\varrho(\cdot)$ denotes the spectral radius.

If $A = \mathbf{u}\mathbf{u}^T$ for a vector $\mathbf{u} \in (0,\infty)^N_{\neq}$ then $h[\mathbf{u}\mathbf{u}^T]$ is invertible, since the generalized Vandermonde matrix $\mathbf{u}^{\circ \mathbf{n}}$ is, and

$$\mathcal{C}_R = (\mathbf{u}^{\circ M})^T h[\mathbf{u}\mathbf{u}^T]^{-1} \mathbf{u}^{\circ M} = \sum_{j=0}^{N-1} \frac{(\det \mathbf{u}^{\circ \mathbf{n}_j})^2}{c_j (\det \mathbf{u}^{\circ \mathbf{n}})^2};$$

see [3, Corollary 4.5] and [15, Proposition 11.2]. This explains the connection to the sharp threshold in Theorem 1.2.

We recall from [3, 15] that an alternate approach to proving Theorem 1.2 is to find the maximum of the bound (4.2) over all A in the relevant test set $\mathcal{P}_0$. The difficulty with this approach lies in the fact that the Rayleigh-quotient map is not continuous when crossing strata.

Our focus in this section is on the bound (4.2) for a single matrix A. We are not concerned with the radius ρ that appeared previously and we do not insist that $M > n_{N-1}$, only that $M > n_0 = 0$. In this setting we obtain continuity of the Rayleigh quotient on each individual stratum.

Theorem 4.1. *Let $h(z) = \sum_{j=0}^{N-1} c_j z^{n_j}$, where $N \geqslant 1$, the coefficients $c_0, \ldots, c_{N-1}$ are positive and the exponents $n_0, \ldots, n_{N-1} \in \mathbb{Z}_+ \cup [N-1, \infty)$ are distinct, with $n_0 = 0$. Fix $M > 0$ and let $\mathcal{P}_0 := \mathcal{P}_N(\mathbb{C})$ if the exponents $n_0, \ldots, n_{N-1}$ and M are integers and otherwise let $\mathcal{P}_0 := \mathcal{P}_N\big([0,\infty)\big)$. The map $A \mapsto \mathcal{C}_R(A, h, M)$ is continuous on $\mathcal{P}_0 \cap \mathcal{S}_\pi$ for any partition $\pi \in \Pi_N$.*

The proof employs weighted variants of the inflation and compression operators used in Section 3 that were introduced in [5].

Definition 4.2 ([5])**.** Given a partition $\pi = \{I_1, \ldots, I_m\} \in \Pi_N$, where $N \geqslant 2$, we use the diagonal matrix $D_\pi := \operatorname{diag}\big(|I_1|, \ldots, |I_m|\big)$ to define the linear operators

$$\Theta_\pi^\downarrow : \mathbb{C}^{N \times N} \to \mathbb{C}^{m \times m};\ A \mapsto D_\pi^{1/2} \Sigma_\pi^\downarrow(A) D_\pi^{1/2}$$

$$\text{and} \quad \Theta_\pi^\uparrow : \mathbb{C}^{m \times m} \to \mathbb{C}^{N \times N};\ B \mapsto \Sigma_\pi^\uparrow(D_\pi^{-1/2} B D_\pi^{-1/2}).$$

Just as $\Sigma_\pi^\uparrow$ and $\Sigma_\pi^\downarrow$ work well with the entrywise calculus, the maps $\Theta_\pi^\uparrow$ and $\Theta_\pi^\downarrow$ are well behaved with respect to the functional calculus, as the following result demonstrates.

Theorem 4.3 ([5, Theorem 5.2])**.** *The maps $\Theta_\pi^\downarrow$ and $\Theta_\pi^\uparrow$ are mutually inverse, rank-preserving isomorphisms between the unital $*$-algebras $\overline{\mathcal{S}_\pi}$ and $\mathbb{C}^{m \times m}$ equipped with the* *usual matrix multiplication. Moreover, a matrix $A \in \overline{\mathcal{S}_\pi}$ is positive semidefinite if and only if $\Theta_\pi^\downarrow(A)$ is.*

With these preliminaries at hand, we proceed.

Proof of Theorem 4.1. Suppose $A \in \mathcal{P}_0 \cap \mathcal{S}_\pi$ and let $H := h[A]$ for brevity. As A is positive semidefinite, so is $B = \Sigma_\pi^\downarrow(A)$, which has distinct rows by construction. We have that $h[B]$ has no zero row, since $n_0 = 0$, so either Theorem 2.2 or Theorem 2.8 implies that $\Sigma_\pi^\downarrow(H) = h[B]$ is positive definite, where this identity holds by Theorem 3.6. Hence the matrix $\Sigma_\pi^\downarrow(H)$ has full rank, and therefore so does $\Theta_\pi^\downarrow(H) = \Theta_\pi^\downarrow(\Sigma_\pi^\uparrow(\Sigma_\pi^\downarrow(H)))$, by Theorems 3.6 and 4.3. The matrix $\Theta_\pi^\downarrow(H)$ is therefore invertible, and

$$H^\dagger = \Theta_\pi^\uparrow(\Theta_\pi^\downarrow(H)^\dagger) = \Theta_\pi^\uparrow(\Theta_\pi^\downarrow(H)^{-1})$$

by Theorem 4.3. Hence,

$$H^{\dagger/2} A^{\circ M} H^{\dagger/2} = \Theta_\pi^\uparrow(\Theta_\pi^\downarrow(H)^{-1/2} \Theta_\pi^\downarrow(A^{\circ M}) \Theta_\pi^\downarrow(H)^{-1/2}),$$

and since all the operations $A \mapsto H = h[A]$, $A \mapsto A^{\circ M}$, $B \mapsto B^{-1/2}$, $\Theta_\pi^\downarrow$, $\Theta_\pi^\uparrow$ and $\varrho(\cdot)$ are continuous, this gives the claim. □

We conclude with two questions. A version of the first was originally posed in [3].

Question 4.4. When is the Rayleigh-quotient inequality an equality? More precisely, given $h(z) = \sum_{j=0}^{N-1} c_j z^{n_j}$, where $N \geqslant 1$, the coefficients $c_0, \ldots, c_{N-1}$ are positive and the exponents $n_0 < \cdots < n_{N-1} < M$ lie in $\mathbb{Z}_+ \cup [N-1, \infty)$, when is $A \in \mathcal{P}_N\big([0,\rho]\big)$ such that the inequality

$$\mathcal{C}_R = \varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}) \leqslant \mathcal{C}_V$$

is an equality, where $\mathcal{C}_V = \mathcal{C}$ as in (1.2)? We see from Theorems 2.2(2) and 2.8(2) that equality is not attained if A has a row with distinct entries, so lies in in the top stratum $\mathcal{S}_{\pi_\vee}$ (and $n_0 = 0$ if any entry in this row is zero), since this implies that the matrix $g[A] = h[A] - \mathcal{C}_V^{-1} A^{\circ M}$ is positive definite and $h[A] - \mathcal{C}_R^{-1} A^{\circ M}$ is not, because

$$\mathbf{u}^* h[A]^{1/2} (\mathrm{Id}_N - \mathcal{C}_R^{-1} h[A]^{-1/2} A^{\circ M} h[A]^{-1/2}) h[A]^{1/2} \mathbf{u} = 0$$

if $\mathbf{u} = h[A]^{-1/2}\mathbf{v}$ and $\mathbf{v}$ is an eigenvector corresponding to the maximum eigenvalue of $h[A]^{-1/2} A^{\circ M} h[A]^{-1/2}$.

For our next question, we first present another extension of Theorem 2.2. This result and its proof involve the linear matrix inequality (2.1), in which the matrix $A^{\circ M}$ is bounded above by powers of lower order. When restricted to the closure of a particular stratum, this inequality can be strengthened to involve fewer terms.

Proposition 4.5. *Let the partition $\pi \in \Pi_N$, where $N \geqslant 2$, and suppose π has m blocks, where $m \geqslant 1$. Suppose $c_0, \ldots, c_{m-1}$ are positive and $n_0, \ldots, n_{m-1}$, $M \in \mathbb{Z}_+ \cup [N-1, \infty)$ are distinct, with $n_0 < \cdots < n_{m-1} < M$. Given any $\rho > 0$, we let $\mathcal{P}_0$ equal $\mathcal{P}_N\big(\overline{D}(0,\rho)\big)$ if $n_0, \ldots, n_{N-1}$ and M are integers and $\mathcal{P}_N\big([0,\rho]\big)$ otherwise. We have the bound*

$$A^{\circ M} \leqslant \mathcal{C}_m \sum_{j=0}^{m-1} c_j A^{\circ n_j} \qquad \textit{for all } A \in \mathcal{P}_0 \cap \overline{\mathcal{S}_\pi}, \tag{4.3}$$

where $\mathcal{C}_m$ equals $\mathcal{C}$ as in (1.2) with $\mathbf{c} = (c_0, \ldots, c_{m-1})$ and $\mathbf{n} = (n_0, \ldots, n_{m-1})$. Equality is achieved if and only if either $m = 1$ and $A = \rho \mathbf{1}_{N \times N}$, or $n_0 > 0$ and $A = \mathbf{0}_{N \times N}$.

Furthermore, if $\mathcal{C}_m$ is replaced by any larger constant, and $n_0 = 0$ if $A \in \mathcal{P}_0 \cap \mathcal{S}_\pi$ has a zero row, then the inequality (4.3) is strict for A upon applying $\Sigma_\pi^\downarrow$.

Proof. By Theorem 3.6, the maps $\Sigma_\pi^\uparrow$ and $\Sigma_\pi^\downarrow$ can be used to transfer the setting to either $\mathcal{P}_m\big(\overline{D}(0,\rho)\big)$ or $\mathcal{P}_m\big([0,\rho]\big)$. The assertions then follow directly from their counterparts in Corollary 2.1 and Theorems 2.2 and 2.8; the final statement holds by (1)(d) of each. $\square$

Question 4.6. An explicit expression for the supremum of the function $A \mapsto \mathcal{C}_R(A, h, M)$ on each stratum $\mathcal{S}_\pi \cap \mathcal{P}_0$ is known for $\pi = \pi_\wedge$ [3, Corollary 4.5] and $\pi = \pi_\vee$ [3, 15] since $\mathcal{S}_{\pi_\vee}$ contains all matrices of the form $A = \mathbf{u}\mathbf{u}^T$ where $\mathbf{u} \in (0,\infty)^N$ has distinct coordinates, and so the supremum of $\mathcal{C}_R(A, h, M)$ is at least, so exactly, $\mathcal{C}$ from Theorem 1.2. A natural conjecture, supported by Proposition 4.5, is that the supremum depends only on the number of blocks in the partition π and not on any further data from π.

4.1. **Acknowledgements.** A.B. was partially supported by Lancaster University while the intial phase of this work was carried out.

D.G. was partially supported by a University of Delaware Research Foundation grant, by a Simons Foundation collaboration grant for mathematicians, and by a University of Delaware Research Foundation Strategic Initiative grant.

A.K. was partially supported by the Ramanujan Fellowship SB/S2/RJN-121/2017, MATRICS grant MTR/2017/000295, and SwarnaJayanti Fellowship grants SB/SJF/2019-20/14 and DST/SJF/MS/2019/3 from SERB and DST (Govt. of India), grant F.510/25/CAS-II/2018(SAP-I) from UGC (Govt. of India), a Young Investigator Award from the Infosys Foundation, a Shanti Swarup Bhatnagar Award from CSIR (Govt. of India), and the DST FIST program 2021 [TPN–700661].

M.P was supported by a Simons Foundation collaboration grant for mathematicians.

The authors thank the Institute for Advanced Study, Princeton and the American Institute of Mathematics, Pasadena for their hospitality while this work was concluded.

4.2. **List of symbols.** We collect below some notation used throughout the text.

- $\overline{D}(0,\rho)$ is the closed disc in $\mathbb{C}$ with radius ρ centered at the origin.
- $\mathcal{P}_N^k(I)$ is the set of positive semidefinite $N \times N$ matrices of rank at most k with entries in the set $I \subseteq \mathbb{C}$. Such matrices are necessarily Hermitian.
- $\mathcal{P}_N(I) := \mathcal{P}_N^N(I)$.
- $\mathbf{1}_{N \times N'}$ is the $N \times N'$ matrix with each entry equal to 1.
- $f[A]$ is the matrix obtained by applying the function f to each of the entries of the matrix A.
- $A^{\circ\alpha}$ is the matrix obtained by taking the αth power of each of the entries of the matrix A, whenever this is well defined.
- $\mathbf{u}^\alpha = (u_i^\alpha)_{i=1}^m$ for any real number α and column vector $\mathbf{u} = (u_i)_{i=1}^m$ whenever the entries are well defined.
- $\mathbf{u}^{\circ\mathbf{n}} = (u_i^{n_j})_{i,j=1}^m$ for any column vector $\mathbf{u} = (u_i)_{i=1}^m$ and row vector $\mathbf{n} = (n_1, \ldots, n_m)$ whenever these quantities are well defined.
- $V(\mathbf{u})$ is the Vandermonde determinant of the column vector $\mathbf{u} = (u_i)_{i=1}^m$ or the row vector $\mathbf{u} = (u_1, \ldots u_m)$, so that $V(\mathbf{u}) = \prod_{1 \leqslant k < l \leqslant m} (u_l - u_k)$.

- $S^N_{\neq}$ is the collection of all N-tuples in S with distinct entries and $S^N_<$ the subset of $S^N_{\neq}$ consisting of N-tuples with strictly increasing entries.
- $A^\dagger$ is the Moore–Penrose pseudo-inverse of the matrix A.
- $\varrho(A)$ is the spectral radius of the matrix A.
- $(\Pi_N, \preccurlyeq)$ is the poset of partitions of $\{1, \ldots, N\}$, where $\pi' \preccurlyeq \pi$ if π is a refinement of π', so that every set in π is a subset of some set in π'.
- D_π is the $m \times m$ diagonal matrix with (i, i) entry $|I_i|$, where $\pi = \{I_1, \ldots, I_m\} \in \Pi_N$.
- $\Sigma^\downarrow_\pi$ and $\Sigma^\uparrow_\pi$ are defined in Definition 3.5.
- $\Theta^\downarrow_\pi$ and $\Theta^\uparrow_\pi$ are defined in Definition 4.2.

References

[1] R. Ait-Haddou and M.-L. Mazure. The fundamental blossoming inequality in Chebyshev spaces–I: Applications to Schur functions. *Found. Comput. Math.* 18(1):135–158, 2018.

[2] R. Ait-Haddou, S. Yusuke, and T. Nomura. Chebyshev blossoming in Müntz spaces: Toward shaping with Young diagrams. *J. Comput. Appl. Math.* 247:172–208, 2013.

[3] A. Belton, D. Guillot, A. Khare, and M. Putinar. Matrix positivity preservers in fixed dimension. I. *Adv. Math.* 298:325–368, 2016.

[4] A. Belton, D. Guillot, A. Khare, and M. Putinar. Simultaneous kernels of matrix Hadamard powers. *Linear Algebra Appl.* 576:142–157, 2019.

[5] A. Belton, D. Guillot, A. Khare, and M. Putinar. Matrix compression along isogenic blocks. *Acta Sci. Math. (Szeged)* 88(1-2) – *100th anniversary special volume*, 417–448, 2022.

[6] A. Belton, D. Guillot, A. Khare, and M. Putinar. Moment-sequence transforms. *J. Eur. Math. Soc.* 24(9):3109–3160, 2022.

[7] A. Belton, D. Guillot, A. Khare, and M. Putinar. Negativity-preserving transforms of tuples of symmetric matrices. In preparation.

[8] A. Cuttler, C. Greene, and M. Skandera. Inequalities for symmetric means. *European J. Combin.* 32(6):745–761, 2011.

[9] C.H. FitzGerald and R.A. Horn. On fractional Hadamard powers of positive definite matrices. *J. Math. Anal. Appl.* 61(3):633–642, 1977.

[10] F.R. Gantmacher. *The theory of matrices. Volume 2.* Chelsea Publishing Co., New York, 1959.

[11] F. Hiai. Monotonicity for entrywise functions of matrices. *Linear Algebra Appl.* 431(8):1125–1146, 2009.

[12] R.A. Horn. The theory of infinitely divisible matrices and kernels. *Trans. Amer. Math. Soc.* 136:269–286, 1969.

[13] A. Khare. Smooth entrywise positivity preservers, a Horn–Loewner master theorem, and symmetric function identities. *Trans. Amer. Math. Soc.* 375(3):2217–2236, 2022.

[14] A. Khare. *Matrix analysis and entrywise positivity preservers.* Vol. 471, London Math. Soc. Lecture Note Ser., Cambridge University Press; and Vol. 82, TRIM Series, Hindustan Book Agency; 2022.

[15] A. Khare and T. Tao. On the sign patterns of entrywise positivity preservers in fixed dimension. *Amer. J. Math.* 143(6):1863–1929, 2021.

[16] T. Lam, A.E. Postnikov, and P. Pylyavskyy. Schur positivity and Schur log-concavity. *Amer. J. Math.* 129(6):1611–1622, 2007.

[17] I.G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, New York, second edition, 1995.

[18] C. McSwiggen and J. Novak. Majorization and spherical functions. *Int. Math. Res. Not. IMRN* 2022(5):3977–4000, 2022.

[19] W. Rudin. Positive definite sequences and absolutely monotonic functions. *Duke Math. J* 26(4):617–622, 1959.

[20] I.J. Schoenberg. Positive definite functions on spheres. *Duke Math. J.* 9(1):96–108, 1942.

[21] J. Schur. Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. *J. Reine Angew. Math.* 140:1–28, 1911.

[22] S. Sra. On inequalities for normalized Schur functions. *European J. Combin.* 51:492–494, 2016.
[23] R.P. Stanley. *Enumerative combinatorics. Volume 2.* Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, 1999.

(A. Belton) School of Engineering, Computing and Mathematics, University of Plymouth, Plymouth, UK
Email address: alexander.belton@plymouth.ac.uk

(D. Guillot) University of Delaware, Newark, DE, USA
Email address: dguillot@udel.edu

(A. Khare) Indian Institute of Science; Analysis and Probability Research Group; Bangalore, India
Email address: khare@iisc.ac.in

(M. Putinar) University of California at Santa Barbara, CA, USA and Newcastle University, Newcastle upon Tyne, UK
Email address: mputinar@math.ucsb.edu, mihai.putinar@ncl.ac.uk