THE KHINCHIN–KAHANE INEQUALITY AND BANACH SPACE EMBEDDINGS FOR ABELIAN METRIC GROUPS

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Abstract. The Khinchin–Kahane inequality is a fundamental result in the probability literature, with the most general version to date holding in Banach spaces. Motivated by modern settings and applications, we generalize this inequality to arbitrary abelian metric groups.

To do so, we develop a "transfer principle" that helps carry over questions involving abelian normed metric groups $\mathscr{G}$ and semigroups into the Banach space framework. This allows us to extend the theory of functional analysis to such (semi)groups. As applications: (a) We obtain a sharp version of the Khinchin–Kahane inequality over such groups $\mathscr{G}$, extending the Banach space version. (b) The transfer principle also extends the notion of the expectation to random variables with values in arbitrary abelian normed metric semigroups $\mathscr{G}$. We provide several such applications, including studying the notion of weakly ℓ_p $\mathscr{G}$-valued sequences and related Rademacher series.

On a related note, we also formulate a "general" Lévy inequality, with two features: (i) It subsumes several known variants in the Banach space literature; and (ii) We show the inequality in the minimal framework required to state it: abelian metric groups.

1. Introduction

The Khinchin–Kahane inequality is a classical inequality in the probability literature. It was initially studied by Khinchin [14] in the real case, and later extended by Kahane [10] to normed linear spaces. A detailed history of the inequality can be found in [16]. We begin by presenting a general version of the inequality for Banach spaces, as well as a sharp constant in some cases.

Definition 1.1. A *Rademacher random variable* is a Bernoulli variable that takes values ± 1 with probability $1/2$ each.

Theorem 1.2 (Kahane [10]; Latała and Oleszkiewicz [16])**.** *For all $p, q \in [1, \infty)$, there exists a universal constant $C_{p,q} > 0$ depending only on p, q, such that for all choices of Banach spaces $\mathbb{B}$, finite sets of vectors $x_1, \dots, x_n \in \mathbb{B}$, and independent Rademacher variables $r_1, \dots, r_n$,*

$$\mathbb{E}\left[\left\|\sum_{i=1}^n r_i x_i\right\|^q\right]^{1/q} \leqslant C_{p,q} \cdot \mathbb{E}\left[\left\|\sum_{i=1}^n r_i x_i\right\|^p\right]^{1/p}.$$

If moreover $p = 1 \leqslant q \leqslant 2$, then the constant $C_{1,q} = 2^{1-1/q}$ is optimal.

Notice that to state the theorem, one only requires Rademacher (i.e., random symmetric) sums of vectors. Thus, it is possible to state the result more generally than in a normed linear space:

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in fact, in any abelian group $\mathscr{G}$ equipped with a translation-invariant metric. Now it is natural to ask whether a variant of the Khinchin–Kahane inequality holds in this general (and strictly larger) setting. One of our main results provides a positive answer to this question; see Theorem A.

In working outside the traditional Banach space paradigm, we are motivated by several reasons, both classical and modern. Traditionally, the foundations of probability theory have been systematically and rigorously established in the Banach space setting; see the classic treatise [17] for a compendium of such results. In even greater generality, the study of Fourier analysis and Haar measure for compact abelian groups, as well as of metric group-valued random variables has been carried out in well-known texts including [9, 22]. In this vein, it is of interest to prove stochastic inequalities in the most primitive mathematical framework required to state them. In [11], we showed such a extension of the Hoffmann-Jørgensen inequality for arbitrary metric semigroups, followed by applications and other extensions in [12]. The present paper is in a parallel vein, and achieves such an extension of the Khinchin–Kahane inequality to abelian metric groups.

There are also modern reasons to work with metric groups. In modern-day settings, one often studies random variables with values in permutation groups, or more generally, (connected) abelian and compact Lie groups such as lattices and tori (respectively). Moreover, data can be manifold-valued, living in e.g. real or complex Lie groups rather than in linear spaces. There are other recently studied frameworks arising from the study of large networks, including the space of graphons with the cut-norm [18], or the space of labelled graphs $\mathscr{G}(V)$ with node set V [13]. The latter is an abelian 2-torsion metric group that cannot embed into a Banach space. Thus there is renewed modern interest in studying probability theory outside of the Banach space paradigm. The present paper lies firmly in this setting.

We now state some of the novel contributions of the paper. The first (as discussed above) is to extend the Khinchin–Kahane inequality to abelian metric groups (see Theorem A). Next, note that working with metric groups $\mathscr{G}$ has a fundamental distinction from Banach spaces: the unavailability of an *expectation* function. Thus, another motivation is the question of when such an expectation function can be defined for metric (semi)group-valued random variables. In our second main result, Theorem B, we show that when the metric group $\mathscr{G}$ is *normed* (defined presently), it is possible to define expectations of $\mathscr{G}$-valued random variables. Such an expectation does not necessarily live in the (abelian) group $\mathscr{G}$, but inside its "Banach space closure", a notion that we make precise and study in detail below.

Theorem B has several consequences, including convergence results and inequalities controlling tail behavior. The theorem moreover allows us to define linear functionals, operator spaces, and dual spaces over all abelian normed semigroups $\mathscr{G}$, and therefore extends the powerful theory of functional analysis to all abelian normed semigroups. We also provide several applications of Theorem B, including extending the notion of weakly ℓ_p-sequences to $\mathscr{G}$-valued random variables; as a consequence, several results in the probability literature, including those of Dilworth and Montgomery-Smith [6] (as well as prior results of Talagrand), can be extended to normed abelian metric groups. These applications of our results are described in Section 3.

The above analysis to prove Theorem B prompted us to revisit the Khinchin–Kahane inequality for abelian metric groups $\mathscr{G}$, and to refine it for normed $\mathscr{G}$. Indeed, such a refinement holds for all normed $\mathscr{G}$ with the best possible constants in various regimes for (p, q); see Theorem A. Moreover, we point out natural connections between defining expectations for $\mathscr{G}$-valued random variables, and a related question in geometry on bi-invariant metrics, which may be of independent mathematical interest, and whose answer was not known to experts (it has recently been resolved in a Polymath project) – see Section 3.2. There are other connections to Banach space embeddings of weakly normed groups that we describe in Section 3.

As an additional remark, in the course of proving the Khinchin–Kahane inequality for abelian metric groups (Theorem A), we also provide a two-fold extension of Lévy's inequality; see Theorem 2.12. First, the result simultaneously unifies several existing variants of the inequality in the literature, which to our knowledge had not been consolidated within a common framework. Second, the result is shown to hold in the minimal framework required to state it: for all abelian metric groups.

2. Khinchin–Kahane inequality for abelian metric groups

In this section, we extend the Khinchin–Kahane inequality from Banach spaces to arbitrary abelian metric groups, with a sharp version for groups which are "normed". We also prove a general version of Lévy's inequality, for abelian metric groups.

2.1. Abelian metric groups and the Khinchin–Kahane inequality.

Definition 2.1. A *metric semigroup* is defined to be a semigroup $(\mathscr{G}, \cdot)$ equipped with a metric $d_{\mathscr{G}} : \mathscr{G} \times \mathscr{G} \to [0, \infty)$ that is translation-invariant. In other words,

$$d_{\mathscr{G}}(ac, bc) = d_{\mathscr{G}}(a, b) = d_{\mathscr{G}}(ca, cb), \qquad \forall a, b, c \in \mathscr{G}.$$

Similarly, one defines a *metric monoid* and a *metric group*.

In this paper we deal with metric (semi)groups that are abelian, with the exception of Sections 3.1 and 3.2, and various preliminary results in Section 2.2 that we show hold in even greater generality. Thus, except in these subsections, we will use additive notation unless specified otherwise.

Abelian metric semigroups and groups encompass a large class of examples and spaces studied in modern probability theory. Examples include Euclidean, Banach, and Hilbert spaces, function spaces under suitable metrics (such as L^p-spaces), as well as all connected abelian Lie groups such as circles and tori. Another class of examples consists of discrete abelian semigroups, including all finite groups as well as labelled graph space $\mathscr{G}(V)$ [13]. Certain amenable groups are also abelian metric groups; see Proposition 3.14(4) below. Other examples include abelian, Hausdorff, metrizable, topologically complete groups [15].

We now introduce the following notion that is crucially used throughout the paper.

Definition 2.2. We say that an abelian metric semigroup $(\mathscr{G}, +, d_{\mathscr{G}})$ is *normed* if

$$d_{\mathscr{G}}(g, (n+1)g) = n d_{\mathscr{G}}(g, 2g), \qquad \forall g \in \mathscr{G},\ n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

Notice that if $\mathscr{G}$ is an abelian metric group, then (2.3) implies the following stronger version:

$$d_{\mathscr{G}}(ng, nh) = |n| d_{\mathscr{G}}(g, h), \qquad \forall g, h \in \mathscr{G},\ n \in \mathbb{Z},$$

since $d_{\mathscr{G}}(-g, 0) = d_{\mathscr{G}}(g, 0)$ by Definition 2.1 and symmetry of the metric.

There is extensive literature on the analysis of topological semigroups with translation-invariant metrics and related structures. See [1, 3, 7] and the references therein for more on the subject. These references call any group with a metric (under which the inverse map is an isometry) a "normed" group, while the above definition is termed $\mathbb{N}$-*homogeneity*. However, in Definition 2.2 we instead adopt the notation of [23], and define a norm to be more in the flavor of Banach spaces. The objects in Definition 2.1 will be called metric (semi)groups in this paper.

Note that while normed groups are clearly examples of abelian metric groups, they do not comprise all examples, since normed groups $\mathscr{G}$ are necessarily torsion-free. In light of this discussion, we now state our first main result, the Khinchin–Kahane inequality, for arbitrary abelian metric groups, together with a refinement for normed groups.

Theorem A (Khinchin–Kahane inequality for abelian metric groups). *Given $q \in [1, \infty)$, there exists a universal constant $K_q > 0$ depending only on q such that for all choices of abelian metric groups $\mathscr{G}$, finite sequences of elements $x_1, \dots, x_n \in \mathscr{G}$ (for any $n > 0$), independent Rademacher variables $r_1, \dots, r_n$, and scalar $p \in [1, \infty)$,*

$$\mathbb{E}_\mu \left[d_{\mathscr{G}} \left(0_{\mathscr{G}}, 2^l \sum_{k=1}^n r_k x_k \right)^q \right]^{1/q} \leqslant K_q \cdot \mathbb{E}_\mu \left[d_{\mathscr{G}} \left(0_{\mathscr{G}}, \sum_{k=1}^n r_k x_k \right)^p \right]^{1/p}, \tag{2.4}$$

where $l \in \mathbb{N}$ is such that $2^{l-1} \leqslant q < 2^l$. In fact we may choose $K_q = 64q^2(q/4)^{1/q}$.

If moreover $\mathscr{G}$ is normed*, then one recovers the "usual" form of the Khinchin–Kahane inequality, albeit with a different constant $C_{p,q}$ (which requires fixing $p \in [1,\infty)$ but is universal across all $\mathscr{G}, n, x_k, r_k$ as above):*

$$\mathbb{E}_\mu \left[d_{\mathscr{G}} \left(0_{\mathscr{G}}, \sum_{k=1}^n r_k x_k \right)^q \right]^{1/q} \leqslant C_{p,q} \cdot \mathbb{E}_\mu \left[d_{\mathscr{G}} \left(0_{\mathscr{G}}, \sum_{k=1}^n r_k x_k \right)^p \right]^{1/p}. \qquad (2.5)$$

Moreover, in all regimes for (p,q), the constant $C_{p,q}$ (universal over the category of all normed abelian metric groups) is equal to the universal constant when working only with the sub-category of all Banach spaces. For example, if $p = 1 \leqslant q \leqslant 2$, then one can choose $C_{1,q} = 2^{1-1/q}$, and this constant is best possible for every nontrivial abelian normed group $\mathscr{G}$.

Existing variants in the literature fall under the special case where $\mathscr{G} = \mathbb{B}$ is a Banach space. Note that if $\mathscr{G}$ is not assumed to have a normed structure, then the inequality (2.4) in this more general case does not compare the same quantities as the classical Khinchin–Kahane inequality (2.5) does, and to the best of our knowledge, is a novel inequality that does not follow from the Banach space theory. Also remark for completeness that a separability assumption on $\mathscr{G}$ is not required, since one may restrict to the subgroup generated by $x_1, \dots, x_n$.

Theorem A provides an example of "universal constants" which help compare L^p-norms of sums of independent $\mathscr{G}$-valued variables, across various $p > 0$. This theme is explored in greater detail and generality for abelian metric semigroups in related work [12]. Recall moreover that in the classic paper [16], Latała and Oleszkiewicz had obtained the best such universal constants across all Banach spaces, in the regime $p = 1 \leqslant q \leqslant 2$. Theorem A shows that the same constants work for the Khinchin–Kahane inequality in abelian normed metric groups.

2.2. **Metric semigroups.** We begin by discussing some basic properties of metric semigroups (see Definition 2.1). First note that for a semigroup $\mathscr{G}$ with a bi-invariant metric – for instance if $\mathscr{G}$ is abelian – the following "triangle inequality" is straightforward, and used below without further reference:

$$d_{\mathscr{G}}(y_1 y_2, z_1 z_2) \leqslant d_{\mathscr{G}}(y_1, z_1) + d_{\mathscr{G}}(y_2, z_2) \ \forall y_i, z_i \in \mathscr{G}. \qquad (2.6)$$

We also require the following preliminary result, in this section as well as later.

Proposition 2.7. *Suppose $(\mathscr{G}, d_{\mathscr{G}})$ is a metric semigroup, and $a, b \in \mathscr{G}$. Then*

$$d_{\mathscr{G}}(a, ba) = d_{\mathscr{G}}(b, b^2) = d_{\mathscr{G}}(a, ab) \qquad (2.8)$$

is independent of $a \in \mathscr{G}$. Moreover, $\mathscr{G}$ has at most one idempotent (i.e., $b \in \mathscr{G}$ such that $b^2 = b$). Such an element b is automatically the unique two-sided identity in $\mathscr{G}$, making it a metric monoid.

Proof. Equation (2.8) is immediate using the translation-invariance of $d_{\mathscr{G}}$:

$$d_{\mathscr{G}}(a, ba) = d_{\mathscr{G}}(ba, b^2 a) = d_{\mathscr{G}}(b, b^2) = d_{\mathscr{G}}(ab, ab^2) = d_{\mathscr{G}}(a, ab).$$

Next, if $\mathscr{G}$ has idempotents b, b', then using Equation (2.8),

$$d_{\mathscr{G}}(b, b') = d_{\mathscr{G}}(b^2, bb') = d_{\mathscr{G}}(b^2, b^2 b') = d_{\mathscr{G}}(b, bb') = d_{\mathscr{G}}(b', (b')^2) = 0.$$

Hence $b = b'$. Moreover, given such an idempotent $b \in \mathscr{G}$, compute using Equation (2.8):

$$d_{\mathscr{G}}(a, ba) = d_{\mathscr{G}}(a, ab) = d_{\mathscr{G}}(b, b^2) = 0, \quad \forall a \in \mathscr{G}.$$

Hence b is automatically the unique two-sided identity in $\mathscr{G}$. □

An easy consequence of Proposition 2.7 is the following.

Corollary 2.9. *A set $\mathscr{G}$ is a metric semigroup only if $\mathscr{G}$ is a metric monoid, or the set of non-identity elements in a metric monoid $\mathscr{G}'$. This is if and only if the number of idempotents in $\mathscr{G}$ is one or zero, respectively. Moreover, the metric monoid $\mathscr{G}'$ is (up to a monoid isomorphism) the unique smallest element in the class of metric monoids containing $\mathscr{G}$ as a sub-semigroup. A finite metric semigroup is a metric group.*

Proof. Given any semigroup $\mathscr{G}$ that is not already a monoid, in order to attach an "identity" element $1'$ and obtain a monoid, one defines: $1' \cdot a = a \cdot 1' := a \ \forall a \in \mathscr{G}' := \mathscr{G} \sqcup \{1'\}$. Also extend $d_{\mathscr{G}}$ to $d_{\mathscr{G}'} : \mathscr{G}' \times \mathscr{G}' \to [0, \infty)$ via: $d_{\mathscr{G}'}(1', 1') = 0$ and $d_{\mathscr{G}'}(1', b) = d_{\mathscr{G}'}(b, 1') := d_{\mathscr{G}}(b, b^2)$ for $b \in \mathscr{G}$. Then $\mathscr{G}'$ is a metric monoid. The next assertion now follows from Proposition 2.7. It is clear that the monoid $\mathscr{G}' \supset \mathscr{G}$ is uniquely determined. The final assertion holds since left- and right-multiplication by any $a \in \mathscr{G}$ are bijections. □

Remark 2.10. We will denote the unique metric monoid containing a given metric semigroup $\mathscr{G}$ by $\mathscr{G}' := \mathscr{G} \cup \{1'\}$. Note that the idempotent $1'$ may already be in $\mathscr{G}$, in which case $\mathscr{G} = \mathscr{G}'$. One consequence of Corollary 2.9 is that instead of working with metric semigroups, one can use the associated monoid $\mathscr{G}'$ instead. (In other words, the (non)existence of the identity is not an issue in such cases.) This helps simplify other calculations. For instance, what would usually be a lengthy, inductive computation now becomes much simpler: for non-negative integers k, l, and $z_0, z_1, \ldots, z_{k+l} \in \mathscr{G}$, the triangle inequality (2.6) implies:

$$d_{\mathscr{G}}(z_0 \cdots z_k, z_0 \cdots z_{k+l}) = d_{\mathscr{G}'}(1', \prod_{i=1}^{l} z_{k+i}) \leqslant \sum_{i=1}^{l} d_{\mathscr{G}'}(1', z_{k+i}) = \sum_{i=1}^{l} d_{\mathscr{G}}(z_0, z_0 z_{k+i}).$$

2.3. **A "universal" Lévy inequality and proof of the Khinchin–Kahane inequality.** Next, we define symmetric random variables and show a general version of Lévy's inequality for abelian metric groups.

Definition 2.11. If $(\mathscr{G}, +, d_{\mathscr{G}})$ is an abelian metric group and I is a totally ordered finite set, then a tuple $(X_i)_{i \in I}$ of random variables in $L^0(\Omega, \mathscr{G})$ is *symmetric* if for all finite subsets $J \subset I$ and all functions $\varepsilon : J \to \{\pm 1\}$, the $2^{|J|}$ ordered tuples of variables $(\varepsilon(j) X_j)_{j \in J}$ all have the same joint distribution – i.e., this is independent of ε.

Theorem 2.12 (Lévy's inequality)**.** *Fix an abelian metric group $(\mathscr{G}, 0_{\mathscr{G}}, d_{\mathscr{G}})$, integers $m, n \in \mathbb{N}$, and symmetric random variables $X_1, \ldots, X_n \in L^0(\Omega, \mathscr{G})$. Also fix subsets $B_1, \ldots, B_m \subset \{1, \ldots, n\}$, such that for all $1 \leqslant j \leqslant k \leqslant m$, $B_j \cap B_k$ is either B_j or empty. Set $X_B := \sum_{b \in B} X_b$ for all $B \subset \{1, \ldots, n\}$. If $S_n = X_1 + \cdots + X_n$, then for all $s, t > 0$,*

$$\mathbb{P}_\mu \left(\max_{1 \leqslant k \leqslant m} d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k}) > s + t \right) \leqslant \mathbb{P}_\mu \left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > s \right) + \mathbb{P}_\mu \left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > t \right). \tag{2.13}$$

Note that if $\mathscr{G}$ is a normed linear space and $s = t$, then the left-hand side is concerned with the event that $\max_{1 \leqslant k \leqslant m} \|2X_{B_k}\| > 2t$, which is how the inequality usually appears in the literature.

It is the universal formulation and generalization of the result that is of note here. Indeed, Theorem 2.12 simultaneously strengthens several different variants in the literature, which to our knowledge had not previously been unified. See [17, Proposition 2.3] for two special cases where $\mathscr{G}$ is a Banach space, $s = t$, $m = n$, and $B_k = \{1, \ldots, k\}$ or $B_k = \{k\}$ for all k. Theorem 2.12 also holds for other choices of subsets B_k, e.g. $\{1\}, \{1, 2\}, \{3, 4, 5\}, \{3, 4, 5, 6\}$; or $B_k = \{n - k + 1, \ldots, n\}$ by reversing the order of summation; this last choice is used below. Moreover, Theorem 2.12 extends Lévy's inequality from Banach spaces to all abelian metric groups. We provide a formal proof as it is in a more general setting than what is available in the literature.

Proof of Theorem 2.12. Define the stopping time

$$\tau := \min\{1 \leqslant k \leqslant m : d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k}) > s + t\}. \tag{2.14}$$

Now note that there is a smallest integer $1 \leqslant m_k \leqslant k$ such that $B_{m_k+1}, B_{m_k+2}, \dots, B_{k-1} \subset B_k$. By assumption, $B_1, \dots, B_{m_k}$ are all disjoint from B_k. Thus the event $\tau = k$, which denotes

$$d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_j}) \leqslant s+t < d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k}) \quad \forall 1 \leqslant j \leqslant k-1$$

can be represented also as the event

$$d_{\mathscr{G}}(0_{\mathscr{G}}, -2X_{B_j}) \leqslant s+t \quad \forall 1 \leqslant j \leqslant m_k, \qquad d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_j}) \leqslant s+t < d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k}) \quad \forall m_k < j < k.$$

Thus let $\mathbf{X}_r := X_{\cup_{j=1}^r B_j}$ for all r. Then it follows from the symmetry assumption that

$$\begin{aligned}\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > t, \tau = k\right) &= \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, (-\mathbf{X}_{m_k}) + X_{B_k} - (S_n - \mathbf{X}_{m_k} - X_{B_k})) > t, \tau = k\right)\\ &= \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k} - S_n) > t, \tau = k\right).\end{aligned}$$

We now prove the result. Note by the triangle inequality that

$$d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k}) > s+t \quad \implies \quad d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > s \quad \text{or} \quad d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k} - S_n) > t.$$

Thus by the above analysis,

$$\begin{aligned}&\mathbb{P}_\mu\left(\max_{1 \leqslant k \leqslant m} d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k}) > s+t\right) = \sum_{k=1}^m \mathbb{P}_\mu\left(\tau = k,\ d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k}) > s+t\right)\\ &\leqslant \sum_{k=1}^m \left(\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > s, \tau = k\right) + \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2X_{B_k} - S_n) > t, \tau = k\right)\right)\\ &= \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > s,\ \tau \in [1,m]\right) + \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > t,\ \tau \in [1,m]\right),\end{aligned}$$

and the result follows. ☐

We now prove the Khinchin–Kahane inequality.

Proof of Theorem A. For this proof, fix an abelian metric group $\mathscr{G}$, elements $x_1, \dots, x_n \in \mathscr{G}$, and Rademacher variables $r_1, \dots, r_n$. For ease of exposition we show the result in steps.

Step 1. We claim that the following preliminary result holds:
For all abelian metric groups $\mathscr{G}$ and $\mathscr{G}$-valued Rademacher sums $\sum_{k=1}^n r_k x_k$,

$$\begin{aligned}&\mathbb{P}_\mu\left(d_{\mathscr{G}}\left(0_{\mathscr{G}}, 2\sum_{k=1}^n r_k x_k\right)\right) > s+t+u+v\\ &\leqslant \left(\mathbb{P}_\mu\left(P_n > s\right) + \mathbb{P}_\mu\left(P_n > t\right)\right) \cdot \left(\mathbb{P}_\mu\left(P_n > u\right) + \mathbb{P}_\mu\left(P_n > v\right)\right)\end{aligned} \tag{2.15}$$

for all $s, t, u, v > 0$, where $P_n := d_{\mathscr{G}}\left(0_{\mathscr{G}}, \sum_{k=1}^n r_k x_k\right)$.

Existing variants in the literature are usually special cases with $\mathscr{G} = \mathbb{B}$ a Banach space and $s = t = u = v$. While the proof uses familiar arguments, we include it for the reader's convenience, as it is in somewhat greater generality than can usually be found in the literature.

Define $S_k := \sum_{j=1}^k r_j x_j$ for $1 \leqslant k \leqslant n$. Similar to the proof of Lévy's inequality (Theorem 2.12), define the stopping time $\tau := \min\{1 \leqslant k \leqslant n : d_{\mathscr{G}}(0_{\mathscr{G}}, 2S_k) > s+t\}$. Also recall that $(r_1, \dots, r_n)$ and $(r_1, \dots, r_k, r_k r_{k+1}, \dots, r_k r_n)$ are identically distributed. Therefore (using that

$d_{\mathscr{G}}(0,g) \equiv d_{\mathscr{G}}(0,-g)$),

$$\mathbb{P}_\mu\left(d_{\mathscr{G}}(2S_{k-1},2S_n) > u+v, \tau = k\right) = \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2\sum_{j=k}^n r_j x_j) > u+v, \tau = k\right)$$

$$= \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2\sum_{j=k}^n r_k r_j x_j) > u+v, \tau = k\right)$$

$$= \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2x_k + 2\sum_{j=k+1}^n r_k r_j x_j) > u+v, \tau = k\right)$$

$$= \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2x_k + 2\sum_{j=k+1}^n r_j x_j) > u+v, \tau = k\right) = \mathbb{P}_\mu\left(d_{\mathscr{G}}(2x_k + 2S_n, 2S_k) > u+v, \tau = k\right).$$

The same argument without restricting to the event $\tau = k$ shows that:

$$\mathbb{P}_\mu\left(d_{\mathscr{G}}(2S_{k-1},2S_n) > u+v\right) = \mathbb{P}_\mu\left(d_{\mathscr{G}}(2x_k + 2S_n, 2S_k) > u+v\right).$$

Now note that if $d_{\mathscr{G}}(0_{\mathscr{G}}, 2S_n(\omega)) > s+t+u+v$ and $\tau(\omega) = k$, then $d_{\mathscr{G}}(2S_{k-1}(\omega), 2S_n(\omega)) > u+v$. Since $\tau = k$ and $d_{\mathscr{G}}(2S_k, 2x_k + 2S_n)$ are independent, we compute:

$$\begin{aligned}
&\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2S_n) > s+t+u+v, \tau = k\right) \leqslant \mathbb{P}_\mu\left(d_{\mathscr{G}}(2S_{k-1},2S_n) > u+v, \tau = k\right) \\
&= \mathbb{P}_\mu\left(d_{\mathscr{G}}(2x_k + 2S_n, 2S_k) > u+v\right)\mathbb{P}_\mu\left(\tau = k\right) = \mathbb{P}_\mu\left(d_{\mathscr{G}}(2S_{k-1},2S_n) > u+v\right)\mathbb{P}_\mu\left(\tau = k\right) \\
&\leqslant \mathbb{P}_\mu\left(\tau = k\right)\left(\mathbb{P}_\mu\left(P_n > u\right) + \mathbb{P}_\mu\left(P_n > v\right)\right),
\end{aligned}$$

by using Lévy's inequality (Theorem 2.12) with $m = 1, B_1 = \{k, \dots, n\}, X_l = 2r_l x_l \ \forall l \geqslant k$, and replacing (s,t) by (u,v). Now another application of Lévy's inequality with the same choice of parameters – except with $B_k = \{1, \dots, k\}$ – concludes the proof.

Step 2. We now prove the inequality (2.4) for $p, q \geqslant 1$. Repeatedly applying the inequality (2.15) yields:

$$\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2^l S_n) > 4^l r\right) \leqslant 4^{2^l - 1}\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > r\right)^{2^l}, \qquad \forall l \in \mathbb{N}. \tag{2.16}$$

Set l to be the unique positive integer such that $2^{l-1} \leqslant q < 2^l$, and change variables $t = 4^l r \in (0,\infty)$. Using that $\mathbb{E}_\mu[Z^q] = q\int_0^\infty t^{q-1}\mathbb{P}_\mu\left(Z > t\right)\, dt$ for an L^q random variable $Z \geqslant 0$, we compute:

$$\begin{aligned}
\mathbb{E}_\mu[d_{\mathscr{G}}(0_{\mathscr{G}}, 2^l S_n)^q] &= q\int_0^\infty (4^l r)^{q-1}\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, 2^l S_n) > 4^l r\right) \cdot 4^l\, dr \\
&\leqslant q 4^{lq + 2^l - 1}\int_0^\infty r^{q-1}\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > r\right)^{2^l}\, dr.
\end{aligned}$$

Now $4^{lq} \leqslant (2q)^{2q}$ and $r\mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > r\right) \leqslant \mathbb{E}_\mu[d_{\mathscr{G}}(0_{\mathscr{G}}, S_n)]$ by Markov's inequality. Therefore,

$$\begin{aligned}
\mathbb{E}_\mu[d_{\mathscr{G}}(0_{\mathscr{G}}, 2^l S_n)^q] &\leqslant (2q)^{2q}4^{2q-1}q\int_0^\infty \mathbb{E}_\mu[d_{\mathscr{G}}(0_{\mathscr{G}}, S_n)]^{q-1} \cdot \mathbb{P}_\mu\left(d_{\mathscr{G}}(0_{\mathscr{G}}, S_n) > r\right)\, dr \\
&= \frac{(8q)^{2q+1}}{32}\mathbb{E}_\mu[d_{\mathscr{G}}(0_{\mathscr{G}}, S_n)]^q.
\end{aligned}$$

Taking qth roots and using Hölder's inequality now yields (2.4).

Step 3. Finally, we prove all remaining assertions, assuming that $\mathscr{G}$ is normed. Note that the inequality (2.5) immediately follows with $C_{p,q} = 64q(q/4)^{1/q}$ from (2.4), and in fact with $C_{p,q} = 1$ from Hölder's inequality if $q \leqslant p$.

The assertion about $C_{p,q}$ being the same as for Banach spaces across all regimes for (p,q), will follow from Theorem B. We now provide an alternate argument for the regime $1 = p \leqslant q \leqslant 2$: first claim that if $q = 2$ and $p = 1$, then

$$\mathbb{E}_\mu[P_n^2]^{1/2} \leqslant \sqrt{2}\ \mathbb{E}_\mu[P_n], \tag{2.17}$$

where P_n was defined in Step 1 above. The claim is proved in exactly the same way as [16, Theorem 1]. More precisely, the assumption that $\mathscr{G}$ is normed (together with the triangle inequality (2.6)) is required to prove that (notation as in [16]):

$$\begin{aligned}(n-2)X_\varepsilon = d_{\mathscr{G}}\left(0_{\mathscr{G}}, \sum_{k=1}^n (n-2)\varepsilon_k x_k\right) &= d_{\mathscr{G}}\left(0_{\mathscr{G}}, \sum_{\eta \in \{-1,1\}^n :\ d(\varepsilon,\eta)=1} \sum_{k=1}^n \eta_k x_k\right)\\ &\leqslant \sum_{\eta \in \{-1,1\}^n :\ d(\varepsilon,\eta)=1} X_\eta.\end{aligned}$$

This shows (2) for $q = 2$; now suppose $q \in [1,2]$. Setting $\theta := 2 - 2/q \in [0,1]$; therefore $1/q = \theta \cdot (1/2) + (1-\theta) \cdot 1$. The log-convexity of L^p norms and (2.17) now shows:

$$\mathbb{E}_\mu[P_n^q]^{1/q} \leqslant \mathbb{E}_\mu[P_n^2]^{\theta/2} \mathbb{E}_\mu[P_n]^{1-\theta} \leqslant (\sqrt{2}\ \mathbb{E}_\mu[P_n])^\theta \mathbb{E}_\mu[P_n]^{1-\theta} = 2^{1-1/q}\ \mathbb{E}_\mu[P_n].$$

If $\mathscr{G}$ is not a singleton, then we note that $C_{1,q} = 2^{1-1/q}$ is the best possible constant by considering $n = 2$ and $x_1 = x_2 \neq 0_{\mathscr{G}}$. □

3. Abelian normed semigroups, expectations, and universal envelopes

In this section we formulate and prove our second main result, which in particular provides a more conceptual reason why the sharp constants in Theorem A(2) for abelian normed metric groups $\mathscr{G}$ are precisely the ones obtained by Latała and Oleszkiewicz in [16]. As we will see below, this is intimately connected with extending the notion of Bochner integration and expectations to $\mathscr{G}$-valued random variables.

Given Theorem A, it is natural to explore further the consequences of a normed structure on an abelian metric group $\mathscr{G}$. Specifically, we focus on the following embedding questions:

(1) Does every abelian normed metric (semi)group $\mathscr{G}$ embed into a normed linear space?
(2) Is it possible to construct the smallest such Banach space?

As we explain below, the first of these questions has been answered in [7] for $\mathscr{G}$ a group. However, to our knowledge a minimal "enveloping" Banach space was not constructed to date. Thus our goals in this section are two-fold: first, to construct such a minimal Banach space – for all semigroups, not just groups (thereby also answering (1) for semigroups); and second, to explain why the optimal constants in Theorem A(2) are the same for abelian normed groups and normed linear spaces.

We begin with the former, and prove that every abelian normed metric semigroup isometrically embeds into a "smallest" Banach space (which is essentially unique). Our proof is constructive and shows a stronger result: $\mathscr{G}$ in fact embeds into a "smallest" normed monoid, which embeds into a "smallest" normed group; and similarly, every abelian normed group embeds into a "smallest" Banach space. Note that the first of these steps was shown in Corollary 2.9. Our next main result (or more precisely, its proof) shows that this phenomenon occurs when extending at every stage: from abelian monoids to groups, to torsion-free divisible groups, to linear spaces.

Theorem B (Transfer principle)**.** *Every (separable) abelian normed metric semigroup $\mathscr{G}$ canonically and isometrically embeds into a "smallest" (separable) Banach space $\mathbb{B}(\mathscr{G})$. In particular, the theory of Bochner integration extends to all such semigroups $\mathscr{G}$.*

Remark 3.1. While the results in this paper are formulated and proved only for abelian groups, surprisingly this abelian hypothesis is *not* required in Theorem B. See Section 3.2 – specifically, Theorem 3.20.

Before proving Theorem B, we discuss some of its consequences and applications:

Example 3.2. An immediate consequence is that the final step in the proof of Theorem A directly follows from the analogous results in Banach spaces [16].

Example 3.3. More generally, Theorem B provides a transfer principle to translate problems from abelian normed metric semigroups to Banach spaces. For instance, Lévy's equivalences between modes of stochastic convergence of sums of independent $\mathscr{G}$-valued random variables immediately follow from their Banach space counterparts, e.g. [17, Theorem 2.4].

Example 3.4. A third – and more challenging – application of Theorem B is to extend to normed $\mathscr{G}$ the main result of [6], which provides universal constants that occur in bounding vector-valued Rademacher series. We now extend this theorem to arbitrary normed $\mathscr{G}$ (and the $K^w_{1,2}$ in the statement of the next result will be explained following Corollary 3.9). Note that such an extension result is not immediate as one has to first understand better the notion of "linear functionals" on $\mathscr{G}$. This is carried out below; in what follows, $\|g\|$ denotes $d_{\mathscr{G}}(0, g)$.

Theorem 3.5. *Fix an i.i.d. sequence of Rademacher variables $\varepsilon_n \sim \mathrm{Unif}\{-1, 1\}$. Then there exists an absolute constant $c > 0$ such that for all choices of (a) separable abelian normed metric semigroups $\mathscr{G}$, (b) points $g_n \in \mathscr{G}$ such that the Rademacher series $X := \sum_n \varepsilon_n g_n$ is almost surely convergent, and (c) scalars $t > 0$, we have:*

$$\mathbb{P}_\mu \left(\|X\| > 2\mathbb{E}\|X\| + 6K^w_{1,2}((g_n), t)\right) \leqslant 4e^{-t^2/8}, \tag{3.6}$$

$$\mathbb{P}_\mu \left(\|X\| > \frac{1}{2}\mathbb{E}\|X\| + cK^w_{1,2}((g_n), t)\right) \geqslant ce^{-t^2/c}. \tag{3.7}$$

Observe that the results of Talagrand [26] that are cited in [6] also extend to abelian normed semigroups, as does the observation in [6, Lemma 2]:

Proposition 3.8. *Every separable abelian normed semigroup embeds in ℓ_∞.*

Furthermore, the various applications of (the version of) Theorem 3.5 in [6] also hold in abelian normed semigroups. These include the following "semigroup-valued" precise form of the Khinchin–Kahane inequality, in a sense bringing us back full circle to Theorem A.

Corollary 3.9 (cf. [6, Corollary 3])**.** *As above, let $X := \sum_n \varepsilon_n g_n$ be an almost surely convergent Rademacher series in a separable abelian normed metric semigroup $\mathscr{G}$. Then for $p \in [1, \infty)$, there is a constant $c > 0$ such that:*

$$\frac{1}{c}\mathbb{E}[\|X\|^p]^{1/p} \leqslant \mathbb{E}\|X\| + K^w_{1,2}((g_n), \sqrt{p}) \leqslant c\mathbb{E}[\|X\|^p]^{1/p},$$

and the implied constant c is absolute.

We now explain the preceding theorem (and hence its corollary), and in particular, why these results are not direct applications of the transfer princple in Theorem B. In Theorem 3.5 and Corollary 3.9, the constant $K^w_{1,2}((g_n), t)$ was used for scalars $t > 0$. In the original setting of [6], defining this constant involves Banach space analysis and weakly ℓ_p sequences. We now extend this definition to all abelian normed semigroups $\mathscr{G}$. For $p \in [1, \infty)$, we say a sequence of points $(g_n)_n$ in $\mathscr{G}$ is *weakly ℓ_p* if $(g^*(g_n))_n$ is ℓ_p for every $g^* \in \mathscr{G}^*$, where $\mathscr{G}^*$ denotes the set of additive Lipschitz real-valued maps on $\mathscr{G}$. Note, this differs from the Banach space definition, which would require running over all functionals in $\mathbb{B}(\mathscr{G})^*$ (or the dual space to a larger Banach space), via Theorem B.

Now define for a weakly ℓ_2 sequence (g_n) and a scalar sequence $(a_n) \in \ell_2$:

$$K_{1,2}((a_n), t) := \inf\{\|(a_{1,n})\|_1 + t\|(a_{2,n})\|_2 : a_n = a_{1,n} + a_{2,n}\ \forall n,\ (a_{j,n})_n \in \ell_j \text{ for } j = 1, 2\},$$

$$K^w_{1,2}((g_n), t) := \sup\{K_{1,2}((g^*(g_n))_n, t) : g^* \in \mathscr{G}^*, \|g^*\| \leqslant 1\}.$$

Then the key observation is that the computation of $K^w_{1,2}$ using $\mathscr{G}^*$ exactly matches the Banach space version that uses $\mathbb{B}(\mathscr{G})^*$ (and hence the results of [6] extend to abelian normed semigroups), because of the following result.

Proposition 3.10. *Suppose $\mathscr{G}$ is a metric semigroup. Let $\mathscr{G}^*$ denote the set of additive Lipschitz real-valued maps on $\mathscr{G}$. Then $\mathscr{G}^*$ is a Banach space, which coincides with the dual space construction if $\mathscr{G}$ is a Banach space. More generally if $\mathscr{G}$ is a normed semigroup, then $\mathscr{G}^* \simeq \mathbb{B}(\mathscr{G})^*$.*

As this paper focusses on probability inequalities, we defer the proof of Proposition 3.10 to the appendix, for the interested reader – see Proposition A.7. In particular, as noted in [6], the assignment $t \mapsto K^w_{1,2}((g_n), t)$ is Lipschitz with Lipschitz constant at most

$$\ell^w_2((g_n)) := \sup_{\|g^*\| \leqslant 1} \|(g^*(g_n))\|_2$$

(where g^* runs over $\mathscr{G}^*$), and Theorem 3.5 holds over all abelian normed metric semigroups.

Example 3.11. As additional consequences of our "transfer principle" in Theorem B, the main results in [19, 20] immediately extend to arbitrary abelian normed semigroups.

Finally, we return to Theorem B and provide a proof.

Proof of Theorem B. We will use additive notation throughout this proof as $\mathscr{G}$ is abelian. The proof is constructive, and carried out in stages; however, an outline is in the following equation:

$$\mathscr{G}_{\mathbb{N}} := \mathscr{G} \hookrightarrow \mathscr{G}_{\mathbb{N}\cup\{0\}} := \mathscr{G}' \hookrightarrow \mathscr{G}_{\mathbb{Z}} := \mathbb{Z} \otimes_{\mathbb{N}\cup\{0\}} \mathscr{G}_{\mathbb{N}\cup\{0\}} \hookrightarrow \mathscr{G}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{G}_{\mathbb{Z}} \hookrightarrow \mathbb{B}(\mathscr{G}) := \overline{\mathscr{G}_{\mathbb{Q}}}. \tag{3.12}$$

We now explain these steps one by one.

(1) Embed the semigroup into a metric monoid $\mathscr{G}'$ via Corollary 2.9. We label $\mathscr{G}_{\mathbb{N}} := \mathscr{G}$ and $\mathscr{G}_{\mathbb{N}\cup\{0\}} := \mathscr{G}'$ to denote that $\mathscr{G}, \mathscr{G}'$ are "modules" over $\mathbb{N}, \mathbb{N}\cup\{0\}$ respectively.

(2) It is easily shown that $\mathscr{G}_{\mathbb{N}}$ and hence $\mathscr{G}_{\mathbb{N}\cup\{0\}}$ is cancellative. Therefore the monoid $\mathscr{G}_{\mathbb{N}\cup\{0\}}$ embeds into its Grothendieck group $\mathscr{G}_{\mathbb{Z}}$ (which is a $\mathbb{Z}$-module) by attaching additive inverses and quotienting by an equivalence relation. Extend the metric $d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}$ to $\mathscr{G}_{\mathbb{Z}}$ via: $d_{\mathscr{G}_{\mathbb{Z}}}(p-q, r-s) := d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}(p+s, q+r)$, for all $p,q,r,s \in \mathscr{G}_{\mathbb{N}\cup\{0\}}$. Then $(\mathscr{G}_{\mathbb{Z}}, 0_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}, d_{\mathscr{G}_{\mathbb{Z}}})$ is an abelian metric group and $\mathscr{G}_{\mathbb{N}} \hookrightarrow \mathscr{G}_{\mathbb{N}\cup\{0\}} \hookrightarrow \mathscr{G}_{\mathbb{Z}}$ are isometric (hence injective) semigroup/monoid homomorphisms. $\mathscr{G}_{\mathbb{Z}}$ is also normed since for all $n \in \mathbb{Z}$ and all $p, q \in \mathscr{G}_{\mathbb{N}\cup\{0\}}$,

$$d_{\mathscr{G}_{\mathbb{Z}}}(0_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}, n(p-q)) = d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}(|n|q, |n|p) = |n| d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}(q,p) = |n| d_{\mathscr{G}_{\mathbb{Z}}}(0_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}, p-q).$$

(3) Note that $\mathscr{G}_{\mathbb{Z}}$ is a torsion-free $\mathbb{Z}$-module because if $ng = 0_{\mathscr{G}_{\mathbb{Z}}}$ for $n \in \mathbb{Z} \setminus \{0\}$ and $g \in \mathscr{G}_{\mathbb{Z}}$, then the preceding equation implies that $g = 0_{\mathscr{G}_{\mathbb{Z}}}$. Now define $\mathscr{G}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{G}_{\mathbb{Z}}$; thus $\mathscr{G}_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space (i.e., a torsion-free divisible group), and $\mathscr{G}_{\mathbb{Z}}$ embeds into $\mathscr{G}_{\mathbb{Q}}$. Moreover for every $g \in \mathscr{G}_{\mathbb{Q}}$ there exists $n_g \in \mathbb{N}$ such that $n_g g \in \mathscr{G}_{\mathbb{Z}}$. Now define $d_{\mathscr{G}_{\mathbb{Q}}}$ on $\mathscr{G}_{\mathbb{Q}}^2$ via:

$$d_{\mathscr{G}_{\mathbb{Q}}}(g,h) := \frac{1}{n_g n_h} d_{\mathscr{G}_{\mathbb{Z}}}(n_h(n_g g), n_g(n_h h)).$$

It is not hard to check that $d_{\mathscr{G}_{\mathbb{Q}}}$ is well-defined and induces a "$\mathbb{Q}$-norm" on $\mathscr{G}_{\mathbb{Q}}$ that extends $d_{\mathscr{G}_{\mathbb{Z}}}$ on $\mathscr{G}_{\mathbb{Z}}$. In particular, it induces a translation-invariant metric on $\mathscr{G}_{\mathbb{Q}}$, so that we have embedded the normed semigroup $\mathscr{G}_{\mathbb{N}}$ isometrically into a "$\mathbb{Q}$-normed" $\mathbb{Q}$-vector space.

(4) Define $\mathbb{B}(\mathscr{G})$ to be the set of equivalence classes of $d_{\mathscr{G}_{\mathbb{Q}}}$-Cauchy sequences (i.e., the topological completion) of $\mathscr{G}_{\mathbb{Q}}$. One shows using algebraic and topological arguments that $\mathbb{B}(\mathscr{G})$ is an abelian group and $\mathscr{G}_{\mathbb{Q}}$ embeds isometrically into $\mathbb{B}(\mathscr{G})$. Moreover, if $x \in \mathbb{R}$ and $(g_n)_n$ is Cauchy in $\mathbb{B}(\mathscr{G})$, then choose any sequence $x_n \in \mathbb{Q}$ converging to x, and define $x \cdot [(g_n)_n] := [(x_n g_n)_n]$. It is easy to verify that $(x_n g_n)_n$ is also a Cauchy sequence in $\mathscr{G}_{\mathbb{Q}}$, and the resulting operation makes $\mathbb{B}(\mathscr{G})$ into an $\mathbb{R}$-vector space.

Now define $d_{\mathbb{B}(\mathscr{G})}([(g_n)_n],[(h_n)_n]) := \lim_{n\to\infty} d_{\mathscr{G}_{\mathbb{Q}}}(g_n,h_n)$ (this exists and is well-defined by applying topological arguments). It is easily verified that $d_{\mathbb{B}(\mathscr{G})}$ induces a norm on $\mathbb{B}(\mathscr{G})$, making $\mathbb{B}(\mathscr{G})$ a complete normed linear space, and proving (3.12).

To conclude the proof, observe that if any of the steps starts with a separable metric space, then the subsequent constructions also yield separable metric spaces. The final assertion about extending Bochner integration to $\mathscr{G}$ now follows; note the Bochner integral (or expectation) of $\mathscr{G}$-valued random variables now lives in $\mathbb{B}(\mathscr{G}_{\mathbb{Q}})$ and not necessarily in $\mathscr{G}$. □

Remark 3.13. Note that (as discussed immediately following the statement of Theorem B,) while the final step (Step 3) in the proof of Theorem A follows immediately from Theorem B for normed groups, the first two steps, which formed the technical heart of the proof of Theorem A, hold in greater generality in arbitrary abelian metric semigroups. Thus the Khinchin–Kahane inequality (2.4) holds for instance in all abelian Lie groups, as well as in finite abelian groups, which necessarily cannot be normed.

3.1. **Banach space embeddings.** We now discuss Theorem B vis-a-vis the question of embedding a given topological group into a Banach space. The theorem says that for a metric (semi)group $(\mathscr{G}, d_{\mathscr{G}})$, the assumption of being abelian and normed is sufficient to embed $\mathscr{G}$ into a Banach space. Clearly, the assumption is also necessary. The next result provides additional equivalent conditions when $\mathscr{G}$ is a group, and also relates it to results in the literature.

Proposition 3.14. *Suppose $\mathscr{G}$ is a topological group, with a continuous map $\|\cdot\| : \mathscr{G} \to [0,\infty)$ satisfying: (a) $\|g\| = 0$ if and only if $g = 1_{\mathscr{G}}$; (b) $\|g^{-1}\| = \|g\|$ for all $g \in \mathscr{G}$; and (c) the triangle inequality holds: $\|gh\| \leqslant \|g\| + \|h\|$ for $g, h \in \mathscr{G}$. Then the following are equivalent:*

(1) *There exists a Banach space $\mathbb{B}$ and a group homomorphism $: \mathscr{G} \to (\mathbb{B}, +)$ that is an isometric embedding.*
(2) *$\mathscr{G}$ is abelian and $d_{\mathscr{G}}(g,h) := \|g^{-1}h\|$ is a translation-invariant metric for which $\mathscr{G}$ is normed.*
(3) *$\mathscr{G}$ is $\{2\}$-normed (see Lemma 3.16 below) and is* weakly commutative*, i.e., for all $g, h \in \mathscr{G}$ there exists $n = n(g,h) \in \mathbb{N}$ such that $(gh)^{2^n} = g^{2^n} h^{2^n}$.*
(4) *$\mathscr{G}$ is $\{2\}$-normed and amenable.*

In fact there is a fifth (*a priori* weaker, yet) equivalent condition – that $\mathscr{G}$ is $\{2\}$-normed *without* additional restrictions – which we explain in Theorem 3.20.

In this connection, the following result shows that the "normed" property of a translation-invariant metric on a semigroup:

$$d_{\mathscr{G}}(z_0, z_0^{n+1}) = n\, d_{\mathscr{G}}(z_0, z_0^2), \qquad \forall z_0 \in \mathscr{G},\ n \in \mathbb{N} \cup \{0\},$$

already follows from – hence is equivalent to – the "doubling" property of being $\{2\}$-*normed*, and without requiring the semigroup to be abelian:

$$d_{\mathscr{G}}(z_0, z_0^3) = 2 d_{\mathscr{G}}(z_0, z_0^2), \qquad \forall z_0 \in \mathscr{G}. \tag{3.15}$$

More generally, we have:

Lemma 3.16. *Fix a metric semigroup $(\mathscr{G}, d_{\mathscr{G}})$. Given a subset $J \subset \mathbb{N}$, we say that $(\mathscr{G}, d_{\mathscr{G}})$ is* J-normed *if*

$$d_{\mathscr{G}}(z_0, z_0^{n+1}) = n d_{\mathscr{G}}(z_0, z_0^2), \qquad \forall z_0 \in \mathscr{G}, n \in J. \tag{3.17}$$

Now the following are equivalent.

(1) *$\mathscr{G}$ is J-normed for some nonempty subset $J \subset \mathbb{N}$, $J \neq \{1\}$.*
(2) *$\mathscr{G}$ is $\mathbb{N}$-normed.*

Lemma 3.16 is similar to [7, Lemma 1], which was stated with part (1) involving $J = \{2\}$ (which is precisely (3.15)). For the reader's convenience, we include a proof.

Proof. Using Remark 2.10, we work in the metric monoid $\mathscr{G}'$ containing $\mathscr{G}$. Then $\mathscr{G}$ is J-normed if and only if so is $\mathscr{G}'$; moreover, in $\mathscr{G}'$ the property of being J-normed reads: $d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0^n) = n d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0)$ for all $n \in J$ and $z_0 \in \mathscr{G}$. Now clearly (2) $\implies$ (1). Conversely, suppose (1) holds for $J \supset \{n\}$, with $n > 1$. Then it immediately follows that $d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0^{n^k}) = n^k d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0)$ for all $k \in \mathbb{N}$. Now given $m \in \mathbb{N}$, choose $k \in \mathbb{N}$ such that $n^{k-1} \leqslant m < n^k$; then,

$$\begin{aligned} n^k d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0) = d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0^{n^k}) &\leqslant d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0^m) + d_{\mathscr{G}'}(z_0^m, z_0^{n^k}) \\ &\leqslant d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0^m) + (n^k - m) d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0). \end{aligned}$$

It follows that $m d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0) \leqslant d_{\mathscr{G}'}(1_{\mathscr{G}'}, z_0^m)$. The reverse inequality follows by the triangle inequality in $\mathscr{G}'$. This shows (2) and concludes the proof. □

Proof of Proposition 3.14. That (1) $\implies$ (2) $\implies$ (3) is immediate. That (3) or (4) implies (1) follows from [3, Proposition 4.12] via [7, Corollary 1]. This is a constructive proof, and the formula for the Banach space in question is discussed presently. Finally, that (1) $\implies$ (4) follows since every abelian group is amenable (see [5] for more on amenable groups). □

Remark 3.18. We now discuss Theorem B, with its constructive proof, vis-a-vis Proposition 3.14. The latter result shows that topological groups with *a priori* less structure also embed into Banach spaces, although the two sets of structures turn out to be equivalent. As the proof of [3, Proposition 4.12] is constructive as well, it is natural to ask if the Banach spaces constructed in the two results agree. This turns out not to be the case, as we now explain. More precisely, the Banach space $\mathbb{B}$ constructed in [3, Proposition 4.12] turns out to be the "double-dual construction"

$$\mathbb{B} := \mathrm{Hom}_{gp,bdd}(\mathscr{G}, \mathbb{R})^*,$$

which is the dual space to the set of real-valued bounded group maps : $\mathscr{G} \to \mathbb{R}$. Thus, if $\mathscr{G}$ is an infinite-dimensional Banach space, then the double-dual construction $\mathscr{G}^{**}$ is strictly larger than $\mathscr{G}$. On the other hand, the constructive proof of our result, Theorem B, yields the "minimum" Banach space containing $\mathscr{G}$, which is precisely $\mathscr{G}$. Thus Theorem B provides a sharpening of Proposition 3.14.

We end with a few remarks. First, we point out that each step in (3.12) is canonical, in the sense that it uses only the given information without any additional structure. The natural way to encode this information is via category theory. In other words, every further step/extension in (3.12) is the smallest possible – hence *universal* – "enveloping" object in some category. For the interested reader, we defer these categorical discussions to Appendix A.

Notice also that given Corollary 2.9, it is natural to ask in the non-abelian situation if every (cancellative) metric semigroup embeds into a metric group. This question is harder to tackle; see [4, Chapter 1] for a sufficient condition involving right reversibility.

3.2. **Non-abelian normed groups.** We end this section with a geometric question: Do non-commutative normed metric groups exist? In other words, find an example of a non-abelian topological group $\mathscr{G}$ with a bi-invariant metric $d_{\mathscr{G}}$, such that $d_{\mathscr{G}}(1_{\mathscr{G}}, g^n) = |n| d_{\mathscr{G}}(1, g)$ for all $g \in \mathscr{G}$ and $n \in \mathbb{Z}$. To our knowledge (and that of experts including [8, 25] and others), the answer to this question was not known until recent work [21], whose main result we now describe.

As a possible approach to answering the aforementioned question, a first step is to ask if certain prototypical examples of non-commutative groups with a bi-invariant metric are normed. This is now shown to be false for a well-studied example:

Lemma 3.19. *Let $\mathscr{G} = F_2$ denote the free group on generators a, b, say. Let $d_{\mathscr{G}}$ denote the bi-invariant word metric $d_{\mathscr{G}}$ in the generators $a^{\pm 1}, b^{\pm 1}$ and their conjugates. Then $(\mathscr{G}, d_{\mathscr{G}})$ is not normed.*

Note that we work with $d_{\mathscr{G}}$ as opposed to the usual word metric in the four semigroup generators $a^{\pm 1}, b^{\pm 1}$ of $\mathscr{G}$. The metric $d_{\mathscr{G}}$ and related structures have been studied in many papers; we cite [2] and the references therein.

Proof. We compute:

$$[a,b]^3 = aba^{-1} \cdot b^{-1}ab \cdot a^{-1}b^{-1}a \cdot ba^{-1}b^{-1}.$$

Computing the word lengths, the right-hand side yields at most 4, while $l_{\mathscr{G}}([a,b]) \neq 1$. Therefore,

$$l_{\mathscr{G}}([a,b]^3) \leqslant 4 < 6 \leqslant 3 l_{\mathscr{G}}([a,b]).$$

It follows that $(\mathscr{G}, d_{\mathscr{G}})$ is not normed. □

We conclude with a solution to the above question, obtained by the first author in recent joint work [21] with T. Fritz, S. Gadgil, P. Nielsen, L. Silberman, and T. Tao. It turns out that non-commutative normed metric groups do not exist! Namely:

Theorem 3.20 ([21])**.** *Given a group $\mathscr{G}$, the following are equivalent:*

1. *$\mathscr{G}$ is a metric group (with a bi-invariant metric) that is 2-normed (equivalently, normed).*
2. *$\mathscr{G}$ is abelian and torsion-free.*
3. *$\mathscr{G}$ is an additive subgroup of (i.e., embeds isometrically and additively into) a Banach space.*

This yields a novel characterization – from analysis – of a fundamental class of algebraic objects: abelian torsion-free groups. More strongly, the paper [21] classifies left-invariant normed pseudometrics on any group $\mathscr{G}$; every such map is obtained via a group homomorphism from $\mathscr{G}$ to a Banach space $\mathbb{B}$, followed by taking the norm in $\mathbb{B}$. In particular, the commutator must map to zero, whence no such norm map can exist if $\mathscr{G}$ is non-abelian.

We mention two consequences. First, Theorem B also holds for normed metric groups, as remarked following that result. Note however that this can fail if "group" is replaced by "semigroup" or even "monoid", since non-abelian (free) monoids with norms – i.e., homogeneous length functions – indeed exist. See [21] for details. A second consequence is that the four assertions in Proposition 3.14 are further equivalent to an *a priori* weaker assertion than (4): namely, that $\mathscr{G}$ is $\{2\}$-normed.

Appendix A. Categories of normed metric modules

We now construct "dual spaces" to abelian normed metric groups, as promised in the discussion following Theorem 3.5. This construction is similar – with very small adjustments – for abelian normed metric semigroups and their refinements: (i) semigroups, (ii) monoids, (iii) groups, (iv) torsion-free divisible groups, (v) real vector spaces, and (vi) Banach spaces. In order to explore all of these constructions systematically, we use the language of category theory. We will show in Proposition A.7 below that "dual space constructions" are covariant endofunctors – and more generally, so are spaces of linear Lipschitz operators.

Using categories has additional advantages. Recall that the proof of Theorem B showed that every abelian normed semigroup (respectively, group) embeds into a smallest abelian normed group (respectively, Banach space). We now make these statements precise using category theory. Briefly, we will show in a unified way that the above constructions are instances of "universal objects", and provide examples of pairs of adjoint "induction-restriction functors".

To explore the aforementioned constructions in full detail, we first propose a unifying framework in which to simultaneously study abelian normed metric semigroups of types (i)–(vi) above: normed metric modules.

Definition A.1. Suppose a subset $S \subset \mathbb{R}$ is closed under addition and multiplication.

(1) An *S-module* is defined to be an abelian semigroup $(G,+)$ together with an action map $\cdot : S \times G \to G$, satisfying the following properties for $s, s' \in S$ and $g, g' \in G$:[1]

$$s \cdot (g+g') = s \cdot g + s \cdot g', \quad (s+s') \cdot g = (s \cdot g) + (s' \cdot g), \quad (ss') \cdot g = s \cdot (s' \cdot g), \quad 1 \cdot g = g \text{ if } 1 \in S.$$

(2) A *metric S-module* is an S-module $(G,+)$ together with a translation-invariant metric d_G. We say $(G,+,d_G)$ is *normed* if $d_G(s \cdot g, s \cdot g') = |s| d_G(g,g')$ for all $s \in S$ and $g, g' \in G$.
(3) Let $\mathscr{C}_S$ denote the category whose objects are normed metric S-modules G_S, and morphisms are S-module maps that are moreover Lipschitz. For each such morphism $\varphi : G_S \to G'_S$, define $\|\varphi\|$ to be the smallest constant $K \geqslant 0$ such that $\|\varphi(g)\| \leqslant K\|g\|$ for all $g \in G_S$. Also denote by $\overline{\mathscr{C}}_S$ the full subcategory of all objects in $\mathscr{C}_S$ that are complete metric spaces.

Now $\mathbb{N}$-modules are semigroups and $(\mathbb{N} \cup \{0\})$-modules are monoids. Using this notation, Theorem B discusses the objects in the categories $\mathscr{C}_S$ for $S = \mathbb{N}, \mathbb{N} \cup \{0\}, \mathbb{Z}, \mathbb{Q}$, as well as $\overline{\mathscr{C}}_{\mathbb{R}}$, the category of Banach spaces and bounded operators. Note that we did not discuss normed linear spaces in Theorem B, i.e., the category $\mathscr{C}_{\mathbb{R}}$; however, it is natural to ask if there exists a similarly "universal" normed linear space containing an abelian normed metric group. In our next result we provide a positive answer to this question, again using categorical methods. Thus, we show that the constructions in (3.12) possess functorial properties and therefore are universal in the above categories.

Theorem A.2. *Suppose each of S, T, U is either $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$, with $S \subset T$ or $S \supset T$. Suppose also that G_S is an object of $\mathscr{C}_S$. Now define*

$$\mathscr{G}_T(G_S) := \begin{cases} G_S \text{ (viewed as an object of } \mathscr{C}_T), & \text{if } S \supset T; \\ \text{the unique object of } \mathscr{C}_T \text{ defined as in (3.12)}, & \text{if } S = \mathbb{N}, \mathbb{N} \cup \{0\},\ T \supset S; \\ T \otimes_S G_S, & \text{if } \mathbb{Z} \subset S \subset T. \end{cases} \tag{A.3}$$

(1) *$\mathscr{G}_T(G_S)$ is an object of $\mathscr{C}_S \cap \mathscr{C}_T$ with the following universal property: given an object G_T in $\mathscr{C}_S \cap \mathscr{C}_T$, together with a morphism $\iota : G_S \to G_T$ in $\mathscr{C}_S$, ι extends via the unique isometric monomorphism $G_S \hookrightarrow \mathscr{G}_T(G_S)$ to a unique morphism $\iota_T : \mathscr{G}_T(G_S) \to G_T$ in $\mathscr{C}_T$.*
(2) *In particular, $(\mathscr{G}_T(G_S), \iota_T)$ is unique up to a unique isomorphism in $\mathscr{C}_T$.*
(3) *Given G_S, define $\overline{\mathscr{G}}_T(G_S)$ to be the Cauchy completion of $\mathscr{G}_T(G_S)$ (as a metric space). Then $\overline{\mathscr{G}}_T(G_S)$ is an object of $\overline{\mathscr{C}}_T$ and satisfies the same properties as in the previous parts.*
(4) *Suppose $\mathbb{N} \subset S \subset T \subset U \subset \mathbb{R}$, with S, T, U of the form $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$. For all objects G_S in $\mathscr{G}_S$, there exist unique isomorphisms:*

$$\mathscr{G}_U(\mathscr{G}_T(G_S)) \cong \mathscr{G}_U(G_S), \qquad \overline{\mathscr{G}}_U(\mathscr{G}_T(G_S)) \cong \overline{\mathscr{G}}_U(\overline{\mathscr{G}}_T(G_S)) \cong \overline{\mathscr{G}}_U(G_S).$$

(5) *The following are equivalent for a unital subring $S \subset \mathbb{R}$:*
  (a) *S is dense in $\mathbb{R}$.*
  (b) *$\overline{G_S} = \overline{\mathscr{G}}_T(G_S) = \mathbb{B}(G_S)$ for all objects G_S of $\mathscr{C}_S$ and all subrings $S \subset T \subset \mathbb{R}$.*

For the above reason, if $S \subset T$ or $S \supset T$ then we call $\mathscr{G}_T(G_S), \overline{\mathscr{G}}_T(G_S)$ the *universal envelopes* of G_S in $\mathscr{C}_T$ and $\overline{\mathscr{C}}_T$ respectively. Note that such "minimal envelopes" are ubiquitous in mathematics; examples include the universal enveloping algebra of a Lie algebra, the convex hull of a set (in a real vector space), and the σ-algebra generated by a set of subsets. Also observe that $\mathbb{B}(G_S)$ is the completion of the smallest normed linear space containing G_S, for all $S \supset \mathbb{Q}$ and objects G_S in $\mathscr{C}_S$.

Proof of Theorem A.2. The proof involves (sometimes standard) category-theoretic arguments, and is included for the convenience of the reader.

[1]Note that if $0 \in S$ then G is necessarily a monoid.

(1) The first part is immediate if $S \supset T$; we now show it assuming that $S \subset T$. Given an object G_S in $\mathscr{C}_S$, note $\mathscr{G}_T(G_S) \subset \mathbb{B}(G_S)$. This immediately shows $\mathscr{G}_T(G_S)$ is an object of $\mathscr{C}_T$. Now given a morphism $\iota : G_S \to G_T$ in $\mathscr{C}_S$, if $S = \mathbb{N}$ then first define $\iota_T(0_{\mathscr{G}_T(G_S)}) := 0_{G_T}$. If $S = \mathbb{N} \cup \{0\}$ then define $\iota_T(-g) := -\iota(g)$ for $g \in G_S$. Finally, if S is a unital subring of $\mathbb{R}$ and $x := \sum_{j=1}^n t_j g_j \in T \otimes_S G_S$ (with $g_j \in G_S \ \forall i$), then define $\iota_T(x) := \sum_{j=1}^n t_j \iota(g_j)$. These conditions are necessary to extend ι to ι_T; moreover, it is not hard to show using Theorem B that they are also sufficient to uniquely extend ι to ι_T. Also using Theorem B, one verifies that ι_T is Lipschitz, with $\|\iota_T\| = \|\iota\|$.

(2) This is a standard categorical consequence of universality.

(3) This part is obvious for $S \supset T$, so suppose $S \subset T$, $G_S \in \mathscr{C}_S$. Given $\iota : G_S \to G_T$ with $G_T \in \mathscr{C}_S \cap \overline{\mathscr{C}}_T$, by (1) ι extends uniquely to $\iota_T : \mathscr{G}_T(G_S) \to G_T$, which in turn extends uniquely to $\overline{\iota_T} : \overline{\mathscr{G}}_T(G_S) \to G_T$ by uniform continuity. Now verify $\overline{\iota_T}$ is a morphism in $\overline{\mathscr{C}}_T$, with $\|\overline{\iota_T}\| = \|\iota_T\|$.

(4) This part is standard from above using universal properties, and is omitted for brevity.

(5) First if S is not dense in $\mathbb{R}$, i.e. $S = \mathbb{Z}$, then choose $G_S = \mathbb{Z}$. Now $\overline{G_S} = \mathbb{Z} \neq \mathbb{R} = \mathbb{B}(G_S)$, whence (b) implies (a). Conversely, suppose (a) holds and G_S is in $\mathscr{C}_S$. Repeat the construction in step (4) of the proof of Theorem B, to show the embedding : $G_S \hookrightarrow \mathbb{B}(G_S)$ uniquely extends to an isometric isomorphism : $\overline{G_S} \to \mathbb{B}(G_S)$ of Banach spaces.

Finally, given $S \subset T \subset \mathbb{R}$, note that $\mathscr{G}_T(G_S) = T \otimes_S G_S \subset \mathbb{R} \otimes_S G_S \subset \mathbb{B}(G_S)$. Hence by universality of completions, $\overline{\mathscr{G}}_T(G_S) \subset \mathbb{B}(G_S)$. Moreover, by the previous paragraph $\overline{\mathscr{G}}_T(G_S)$ is a Banach space containing G_S. This shows the reverse inclusion. □

Having discussed *universality*, we now study *functoriality*. The following result shows that the assignments $\mathscr{G}_S$ provide examples of induction and restriction functors.

Theorem A.4. *Suppose each of $S \subsetneq T$ is either $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$.*

(1) *Then $\mathscr{G}_S : \mathscr{C}_T \to \mathscr{C}_S$ is a covariant "restriction" (of scalars) functor which is fully faithful but not essentially surjective. If S is a ring then $\mathscr{G}_S$ is faithfully exact.*

(2) *Moreover, $\mathscr{G}_T : \mathscr{C}_S \to \mathscr{C}_T$ is a covariant "extension" (of scalars) functor which is faithful and essentially surjective but not full. If S is a ring, then $\mathscr{G}_T$ is additive, right-exact, and left adjoint to $\mathscr{G}_S$.*

(3) *If S is dense in $\mathbb{R}$, then $\mathscr{G}_S, \mathscr{G}_T$ yield an equivalence of categories : $\overline{\mathscr{C}}_S \leftrightarrow \overline{\mathscr{C}}_T$.*

In other words, the module-theoretic correspondence involving extension-restriction of scalars also holds for the categories $\mathscr{C}_S, \overline{\mathscr{C}}_S$ of normed metric modules.

Proof. Assume henceforth that G_S, G'_S are objects in $\mathscr{C}_S$, and G_T, G'_T are objects in $\mathscr{C}_T$.

(1) It is immediate that $\mathscr{G}_S : \mathscr{C}_T \to \mathscr{C}_S$ is a faithful, covariant functor. It is not essentially surjective because $S \subsetneq T$ is not a T-module. We now show $\mathscr{G}_S$ is full – in fact we show more strongly that all S-module maps are in fact T-linear. Note, every S-module map between objects G_T, G'_T in $\mathscr{C}_T$ gives rise to a unique $\mathbb{Z}$-module map between them. Given such a map φ, we only use the continuity and additivity of φ to show that φ is in fact T-linear. Thus, fix $g \in G_T$ and consider the function $f : T \to G'_T$ given by $f(t) := \varphi(tg)$. Clearly f is continuous and additive, so given a sequence of rationals $m_k/n_k \to t$, we compute:

$$0 \leftarrow f(m_k - tn_k) = m_k f(1) - n_k f(t) = m_k \varphi(g) - n_k \varphi(tg).$$

It follows that $\varphi(tg) = t\varphi(g)$, showing that φ is in fact T-linear and hence $\mathscr{G}_S$ is full. Finally if S is a ring, the restriction functor $\mathscr{G}_S$ is easily seen to be faithfully exact (i.e., it takes a short sequence to a short exact sequence if and only if the short sequence is exact).

(2) That $\mathscr{G}_T : \mathscr{C}_S \to \mathscr{C}_T$ is a faithful, covariant functor is trivial. It is also essentially surjective because $G_T \cong \mathscr{G}_T(\mathscr{G}_S(G_T))$ for all objects G_T in $\mathscr{C}_T$. Now fix $t_0 \in T \setminus S$. To show that $\mathscr{G}_T$ is not full, set $G_S = G'_S := S$ and define $\varphi_T : \mathscr{G}_T(G_S) = T \to \mathscr{G}_T(G'_S) = T$ via: $\varphi_T(t) = t_0 t$.

Then there does not exist a map $\varphi_S : G_S = S \to G'_S = S$ such that $\varphi_T = \mathscr{G}_T(\varphi_S)$. The assertions in the case when S is a ring are also standard.

(3) This part follows from straightforward verifications using the last part of Theorem A.2. $\square$

Remark A.5. The above results continue to hold upon replacing the categories $\mathscr{C}_S, \overline{\mathscr{C}}_S$ by the larger categories with the same objects, but where the morphisms are allowed to be uniformly continuous rather than Lipschitz.

We now construct dual spaces, as promised in the discussion following Theorem 3.5 above. More generally, we will study the structure of the spaces $\mathrm{Hom}_{\mathscr{C}_T}(\mathscr{G}_T(G_S), G_T)$ for $S \subset T$. We begin with an elementary observation, which helps define norms of Lipschitz maps.

Lemma A.6. *Suppose S is either $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$. Fix a morphism $\varphi : G_S \to G'_S$ in $\mathscr{C}_S$, and consider the following assertions:*

(1) *φ is Lipschitz on G_S.*
(2) *φ is (uniformly) continuous.*
(3) *(If $0 \in S$:) φ is continuous at 0.*

Then (2) and (3) are equivalent, and implied by (1). The converse holds if and only if S is dense in $\mathbb{R}$.

Proof. We only show the very last assertion, as the rest is standard. If $S = \mathbb{Z}$ then consider $G_S = G'_S$ to be the functions from $\mathbb{N}$ to S with finite support. Let $\{e_n : n \in \mathbb{N}\}$ denote the "standard basis" of G_S, and define $\varphi(e_n) := n e_n$. Then φ is continuous but not Lipschitz. Conversely, suppose S is dense in $\mathbb{R}$, and $\|\varphi\| = \infty$. Then there exist $g_n \in G_S$ such that $\|\varphi(g_n)\| > 2n\|g_n\|$ for all n. Choose $s_n \in (n, 2n)$ such that $(s_n\|g_n\|)^{-1} \in S$. Then $\varphi(h_n) > 1\ \forall n$, where $h_n := (s_n\|g_n\|)^{-1} g_n \in G_S$. Since $h_n \to 0$, it follows φ is not continuous at 0. $\square$

Proposition A.7. *Suppose $S \subset T$ are both of the form $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$, and $G_S \in \mathscr{C}_S, G'_T \in \mathscr{C}_T$. Identifying G'_T with $\mathscr{G}_S(G'_T)$, the set $\mathrm{Hom}_{\mathscr{C}_S}(G_S, G'_T)$ is itself an object of $\mathscr{C}_T$. It is moreover an object of $\overline{\mathscr{C}}_T$ (i.e., complete) for all $G_S \in \mathscr{C}_S$, if and only if G'_T is complete.*

In particular for $T = \mathbb{R}$, the above construction yields a Banach space of "linear functionals", which we called the *dual space* G_S^* above. More generally, the assignment $\mathrm{Hom}_{\mathscr{C}_S}(G_S, -)$ defines a covariant additive functor : $\mathscr{C}_S \to \mathscr{C}_T$ and : $\overline{\mathscr{C}}_S \to \overline{\mathscr{C}}_T$. This result (together with Lemma A.6) explains why we chose the category $\mathscr{C}_S$ to have linear morphisms that were also Lipschitz, and not merely uniformly continuous.

Proof. We only sketch why if G'_T is complete, then so is $H := \mathrm{Hom}_{\mathscr{C}_S}(G_S, G'_T)$ for any fixed G_S. Suppose $\varphi_n \in H$ is a Cauchy sequence. Then so is $\varphi_n(g)$ for any $g \in G_S$, whence one defines $\varphi : G_S \to G'_T$ via: $\varphi(g) := \lim_n \varphi_n(g)$. One checks φ is S-linear. Moreover $\|\varphi\| \leqslant \sup_n \|\varphi_n\| < \infty$, whence $\varphi \in H$. A standard argument now shows $d_H(\varphi_n, \varphi) := \|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. $\square$

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